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CONTROL CO-DESIGN UNDER UNCERTAINTIES: FORMULATIONS

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ABSTRACT

This article explores various uncertain control co-design (UCCD) problem formulations. While previous work offers formulations that are method-dependent and limited to only a handful of uncertainties (often from one discipline), effective application of UCCD to real-world dynamic systems requires a thorough understanding of uncertainties and how their impact can be captured. Since the first step is defining the UCCD problem of interest, this article aims at addressing some of the limitations of the literature by identifying possible sources of uncertainties in a general UCCD context and then formalizing ways in which their impact is captured through problem formulation alone (without having to immediately resort to solution strategies). We first develop a universal UCCD formulation and discuss its fundamental elements. Issues such as the treatment of the objective function, the challenge of the analysis-type equality constraints, and various formulations for inequality constraints are discussed. Then, more specialized problem formulations such as the risk-neutral and risk-averse stochastic, worst-case robust, probabilistic robust, fuzzy expected value, and possibilistic chance-constrained UCCD formulations are presented. Key concepts from these formulations, along with insights from closely-related fields, such as robust and stochastic control theory, are discussed, and future research directions are identified.

Keywords: control co-design; dynamics; uncertainty; stochastic programming; fuzzy programming; robust optimization

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1 INTRODUCTION

With the ever-growing complexity and integrated nature of dynamic engineering systems, the need for effective control co-design (CCD) strategies, i.e., integrated consideration of the physical and control system design, is ever present [1,2]. When investigating a CCD problem, it is often the case that some of its elements (such as inputs, model parameters, and/or some aspects of system dynamics) are inherently uncertain or not entirely known. In this paper, we refer to both of these characteristics as uncertainty. Overlooking the impact of uncertainties in CCD may result in solutions that are no longer effective in realistic scenarios.

These uncertainties may stem from multiple sources and affect various elements of the CCD activity, including:

- The noise acting through the control channel transforms the deterministic control trajectories into stochastic ones
- Plant optimization variables may be uncertain due to imperfect manufacturing processes, measurement errors, and mass production of components
- Uncertain problem data, such as wind speeds, wave energy densities, earthquake loads, and material properties may also affect various elements of the problem
- Fidelity of the dynamic model (i.e., unmodeled or neglected dynamics)

All of these uncertainties may propagate through the dynamic system and transform the states into uncertain trajectories. Such uncertainties transform the CCD problem into an *uncertain control co-design* (UCCD) problem. Even before attempting to solve such problems, a necessary step is to identify ways in which the impact of such uncertainties can be mathematically captured in an optimization formulation context. Therefore, it is critical to establish and understand various possible UCCD problem formulations.

This paper aims to identify the sources of uncertainties and formalize their inclusion in various UCCD formulations. This contribution is motivated by the fact that, currently, uncertainty quantification is reasonably well understood in specific control and plant design optimization communities [3–7]. However, current UCCD studies in the literature generally suffer from the lack of a holistic view towards uncertainties, focusing on specific uncertainties, often motivated by a particular solution technique [8–12]. Therefore, the distinction between various UCCD problem formulations is rarely discussed.

In this article, we present an initial effort at a universal UCCD problem formulation. Various problem elements including optimization variables, the objective function, equality and inequality constraints, and relevant concepts such as risk are discussed. Next, the article transitions towards more specialized formulations that are motivated by concepts from stochastic programming [13, 14], robust optimization [15–17], and fuzzy programming [18–20]. These formulations may provide the necessary framework for the development and widespread adoption of UCCD formulations in order to meet the evermore increasing demands on performance, robustness, and reliability of real-world dynamic systems.

The remainder of this article is organized in the following manner: Sec. 2 describes the deterministic CCD problem formulation and discusses various representations of uncertainty; Sec. 3 provides a mathematical foundation for a general UCCD problem formulation; Sec. 4 describes some of the specialized UCCD formulations that are inspired by stochastic, robust, and fuzzy programming frameworks, including the risk neutral stochastic UCCD, stochastic chance-constrained UCCD, worst-case robust UCCD, probabilistic robust UCCD, and possibilistic chance-constrained UCCD; and Sec. 5 discusses more specialized versions of the previously defined formulations and provides additional insights, considerations, and formulations from the theory of robust and stochastic control theory. Finally, Sec. 6 presents the conclusions.

2 Uncertainty Representations in UCCD

In this section, the deterministic CCD, which is a special case of UCCD formulation is introduced. For mathematical clarity, we define sets associated with both time-dependent and time-independent deterministic and uncertain variables. This section also introduces three distinct ways to represent uncertainties in UCCD context: stochastic, deterministic, and possibilistic.

2.1 Deterministic CCD

We begin by introducing the nominal continuous-time, deterministic, all-at-once (AAO), simultaneous, CCD problem:

$$\begin{aligned}
\underset{\boldsymbol{u},\boldsymbol{\xi},\boldsymbol{p}}{\text{minimize:}} \quad & o = \int_{t_0}^{t_f} \ell(t,\boldsymbol{u},\boldsymbol{\xi},\boldsymbol{p},\boldsymbol{d})\,\mathrm{d}t + m(\boldsymbol{p},\boldsymbol{\xi}_0,\boldsymbol{\xi}_f,\boldsymbol{d}) && \text{(1a)}\\
\text{subject to:} \quad & \boldsymbol{g}(t,\boldsymbol{u},\boldsymbol{\xi},\boldsymbol{p},\boldsymbol{\xi}_0,\boldsymbol{\xi}_f,\boldsymbol{d}) \le \boldsymbol{0} && \text{(1b)}\\
& \boldsymbol{h}(t,\boldsymbol{u},\boldsymbol{\xi},\boldsymbol{p},\boldsymbol{\xi}_0,\boldsymbol{\xi}_f,\boldsymbol{d}) = \boldsymbol{0} && \text{(1c)}\\
& \dot{\boldsymbol{\xi}} - \boldsymbol{f}(t,\boldsymbol{u},\boldsymbol{\xi},\boldsymbol{p},\boldsymbol{\xi}_0,\boldsymbol{\xi}_f,\boldsymbol{d}) = \boldsymbol{0} && \text{(1d)}\\
\text{where:} \quad & \boldsymbol{\xi}(t_0) = \boldsymbol{\xi}_0,\ \boldsymbol{\xi}(t_f) = \boldsymbol{\xi}_f,\ \boldsymbol{u}(t) = \boldsymbol{u},\ \boldsymbol{\xi}(t) = \boldsymbol{\xi} && \text{(1e)}\\
& \boldsymbol{d}(t) = \boldsymbol{d}
\end{aligned}$$

where $t \in [t_0, t_f]$ is the time horizon, $\{\boldsymbol{u},\boldsymbol{\xi},\boldsymbol{p}\}$ are the collection of optimization variables including the open-loop control trajectories $\boldsymbol{u}(t) \in \mathbb{R}^{n_u}$, state trajectories $\boldsymbol{\xi}(t) \in \mathbb{R}^{n_s}$, and the vector of time-independent optimization variables $\boldsymbol{p} \in \mathbb{R}^{n_p}$, respectively. Note that $\boldsymbol{p}$ may entail plant optimization variables $\boldsymbol{p}_p$, and/or time-independent control optimization variables [21, 22] (i.e., gains $\boldsymbol{p}_c$, such that $\boldsymbol{p} = [\boldsymbol{p}_p, \boldsymbol{p}_c]$). The objective function $o(\cdot)$ is composed of the Lagrange term $\ell(\cdot)$ and the Mayer term $m(\cdot)$. The vectors of inequality and equality constraints are described by $\boldsymbol{g}(\cdot)$ and $\boldsymbol{h}(\cdot)$, respectively. The transition or derivative function $\boldsymbol{f}(\cdot)$ describes the evolution of the system through time in terms of a set of ordinary differential equations (ODEs). All of the data associated with the problem formulation is represented through $\boldsymbol{d} \in \mathbb{R}^{n_d}$. This data, which may be time-dependent or time-independent, includes information such as problem constants, environmental signals, initial/final times, etc.

In the remainder of this article, we assume that constraints associated with the initial and final conditions $\{\boldsymbol{\xi}_0, \boldsymbol{\xi}_f\}$ are already included in $\boldsymbol{h}(\cdot)$ or $\boldsymbol{g}(\cdot)$. In addition, we will often drop the explicit dependence on t from time-dependent quantities such as control and state trajectories, as well as the problem data. For more details on deterministic CCD, the readers are referred to Refs. [2, 23].

2.2 Representation of Uncertainties

The first step in accounting for uncertainties in a UCCD problem is the representation of input and model uncertainties. In the risk assessment context, these uncertainties are either aleatory (irreducible) or epistemic (reducible) [24]. Aleatory uncertainty is associated with the inherent irregularity of the phenomenon, while epistemic uncertainty is associated with the lack of knowledge. Accordingly, acquiring more knowledge cannot reduce aleatory uncertainties, but it can reduce epistemic uncertainties. In fact, epistemic uncertainty captures the analyst's confidence in the model by quantifying their degree of belief in how well the model represents the reality [25]. As an example, consider the uncertainty in plant optimization variables due to imperfect manufacturing processes. This uncertainty is intrinsically aleatory or irreducible because acquiring more knowledge cannot reduce this uncertainty (no two plants are identical). However, the uncertainty in the true probability distribution of plant optimization variables can be reduced by acquiring more knowledge (observations). Therefore, this is an epistemic-type uncertainty. Another example of aleatory uncertainty is randomness in material properties or flipping a biased coin. However, our belief in the probabilistic and distributional information of such a phenomenon is epistemic.

Conventionally, these two types of uncertainty are segregated in a nested algorithm, with aleatory analysis in the inner loop and epistemic analysis on the outer loop [26]. While this allows for the simple separation and tracking of each type of uncertainty, a uniform treatment of aleatory and epistemic uncertainties has been implemented in the literature [27] and assumed in this article. It is important to note that information scarcity on epistemic uncertainties may render the output probabilistic information impractical. Therefore, when complete distributional information is available, it should be integrated into the UCCD problem. However, in the case of incomplete and limited information, methods associated with epistemic uncertainties, such as fuzzy programming, are generally preferred.

Elements in a UCCD problem formulation may be deterministic or uncertain. In this article, the notation $\tilde{\bullet}$ is used to distinguish uncertain quantities from deterministic ones. Stochastic processes are distinguished with a time argument $\tilde{\bullet}(t)$. To better distinguish between these quantities in the future sections, we first define four general types of variables along with their associated sets. Any arbitrary, time-independent deterministic variable x is defined in the set: $\mathcal{V}_d$, that is

$$\mathcal{V}_d := \{x \mid x \in \mathcal{V}_d\} \tag{2}$$

As an example, $\mathcal{V}_d$ may be the set of real numbers $\mathbb{R}$, or natural numbers $\mathbb{N}$, or integers $\mathbb{Z}$, etc. Figure 1a shows an arbitrary value belonging to $\mathcal{V}_d$.

The set associated with a time-dependent deterministic variable (i.e., a trajectory) is defined as:

$$\mathcal{T}_d(t) := \{x(t) \mid t \in [t_0, t_f],\ x(t) \in \mathcal{V}_d\} \tag{3}$$

In other words, at every point in time, the trajectory is defined within a deterministic set. Figure 1b shows an arbitrary value belonging to $\mathcal{T}_d(t)$.

For an arbitrary uncertain variable $\tilde{x}$, the sampling space is defined as:

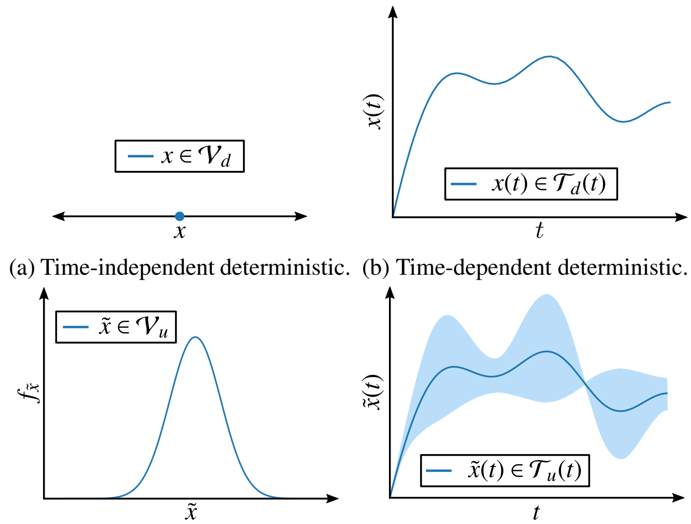
$$\mathcal{V}_u := \{\tilde{x} \mid \tilde{x} \in \mathcal{V}_u\} \tag{4}$$

Note that while the term sampling space implies uncertainty, it does not have any implications on probability. Therefore, $\mathcal{V}_u$ may be an uncertainty set with or without a probability measure. Figure 1c shows an arbitrary time-independent uncertainty set with a Gaussian distribution.

Finally, for an arbitrary uncertain trajectory, the sampling space is defined as:

$$\mathcal{T}_u(t) := \{\tilde{x}(t) \mid t \in [t_0, t_f],\ \tilde{x}(t) \in \mathcal{V}_u\} \tag{5}$$

Similarly, $\mathcal{T}_u(t)$ makes no assumptions regarding the probability measure. Figure 1d shows an arbitrary time-dependent uncertain trajectory along with its associated sampling space. Any uncertain variable belonging to $\mathcal{V}_u$ and $\mathcal{T}_u(t)$ may be represented in three ways: (i) probabilistic, (ii) deterministic, and (iii) possibilistic [15]. In this article, we use these representations to develop various UCCD formulations. These specialized formulations are outlined in Fig. 3, categorized by their uncertainty representation.



(a) Time-independent deterministic. (b) Time-dependent deterministic. (c) Time-independent uncertain. (d) Time-dependent uncertain.

FIGURE 1: Illustration of sets associated with time-independent and time-dependent deterministic and uncertain variables.

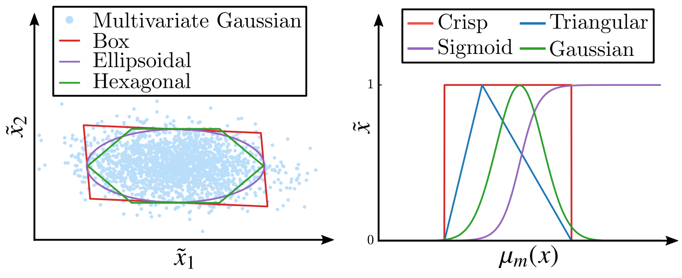
Stochastic (Probabilistic). In the stochastic representation of uncertainties (also known as probabilistic), it is assumed that the associated probability distribution is known or it can be estimated. Therefore, if $\mathcal{V}_u$ and/or $\mathcal{T}_u(t)$ is endowed with a probability measure, uncertainties can be described probabilistically. For an arbitrary, time-independent, continuous uncertain variable $\tilde{x}$, the stochastic set is defined as:

$$\mathcal{X}_{stc} := \{(\tilde{x}, F_{\tilde{x}}(x)) \mid \tilde{x} \in \mathcal{V}_u,\ F_{\tilde{x}}(x) = \mathbb{P}[\tilde{x} \le x] \in [0,1]\} \tag{6}$$

where the subscript stc stands for stochastic, $\mathcal{X}_{stc}$ is the probabilistic set characterized by $F_{\tilde{x}}(x)$, which is the distribution function of $\tilde{x}$, and x is a realization. The probabilistic set for a time-dependent uncertain variable $\tilde{x}(t)$ is described as:

$$\mathcal{X}_{stc}(t) := \{\tilde{x}(t) \mid t \in [t_0, t_f],\ \tilde{x}(t) \in \mathcal{X}_{stc}\} \tag{7}$$

An example of the probabilistic representation of uncertainties is the assumption of Gaussian distribution for uncertainties in plant optimization variables. Samples of a multivariate Gaussian distribution are shown in Fig. 2a. This description of uncertainties motivates stochastic UCCD formulations.



(a) (b)

FIGURE 2: Various representations of uncertainties: (a) comparison of samples of Gaussian multivariate distribution for random variables $\tilde{x}_1$ and $\tilde{x}_2$ to the box, ellipsoidal, and hexagonal uncertainty sets and, (b) several examples of fuzzy set membership functions.

Deterministic. In the deterministic representation of uncertainties, no probability measure is available and uncertainties are assumed to belong to a crisp deterministic set. For an arbitrary, time-independent uncertain variable $\tilde{x}$, the deterministic representation of uncertainties entails a membership function that assigns one to all members and zero to all non-members:

$$\mathcal{X}_{det} := \{(\tilde{x}, \mu_{m,det}(x)) \mid \tilde{x} \in \mathcal{V}_u,\ \mu_{m,det}(x) \in \{0,1\}\} \quad (8)$$

where $\mathcal{X}_{det}$ is the deterministic set characterized by its associated membership function $\mu_{m,det}(x)$. For a time-dependent uncertain variable $\tilde{x}(t)$, the deterministic representation is described as:

$$\mathcal{X}_{det}(t) := \{\tilde{x}(t) \mid t \in [t_0, t_f],\ \tilde{x}(t) \in \mathcal{X}_{det}\} \quad (9)$$

This representation is traditionally inspired by the presence of bounded uncertainties in the problem formulation. Figure 2a compares samples from an arbitrary multivariate Gaussian distribution to the deterministic representation associated with box, ellipsoidal, and hexagonal uncertainty sets. Among these uncertainty sets, the box and hexagonal uncertainty sets are convex polytopes. Under special conditions, when uncertainties or the matrix of the state system is restricted to a polytope, the number of function evaluations for uncertainty propagation may be reduced.

Possibilistic. Uncertainty representations discussed so far are based on some available information, i.e., the known (or estimated) probability distribution function or geometry and size of the uncertainty set. However, when too little is known about the uncertainty, one might utilize descriptive (and often vague) language (also known as linguistic variables) to express the desired or expected events. This information is interpreted by an expert in the field and is best represented through a fuzzy set, which is a class with a continuum of grades of membership.

For an arbitrary, time-independent uncertain variable $\tilde{x}$, the fuzzy set is defined as:

$$\mathcal{X}_{fuzzy} := \{(\tilde{x}, \mu_{m,fuzzy}(x) \mid \tilde{x} \in \mathcal{V}_u,\ \mu_{m,fuzzy}(x) \in [0,1]\} \quad (10)$$

where $\mathcal{X}_{fuzzy}$ is the fuzzy set characterized by its associated membership function $\mu_{m,fuzzy}(x)$. This membership function practically quantifies the degree of membership of an element, or the possibility that an element belongs to the set—leading to concepts from possibility theory [18, 19]. For a time-dependent variable $\tilde{x}(t)$, the fuzzy set is defined as:

$$\mathcal{X}_{fuzzy}(t) := \{\tilde{x} \mid t \in [t_0, t_f],\ \tilde{x}(t) \in \mathcal{X}_{fuzzy}\} \quad (11)$$

Figure 2b compares the membership function of a crisp uncertainty set to that of a triangular, sigmoid, and Gaussian fuzzy membership functions.

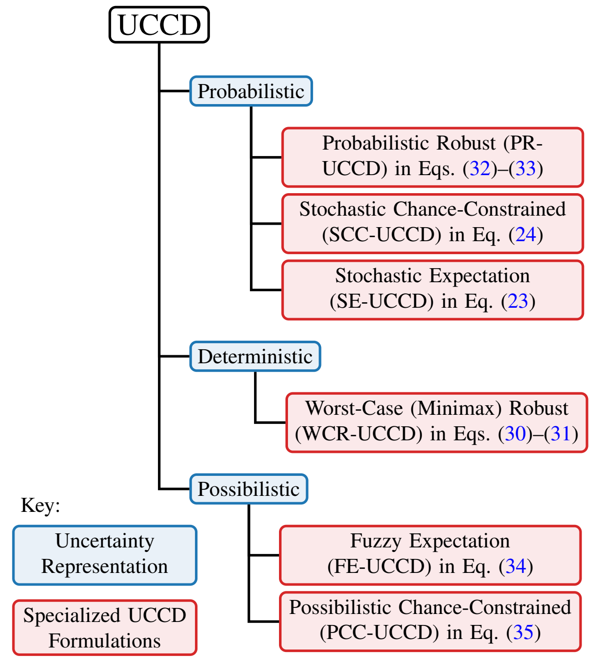


FIGURE 3: Specialized UCCD formulations based on the uncertainty representation.

2.3 Other Considerations

In general, it is natural to assume that in an arbitrary UCCD problem, uncertainties are represented based on the availability of information. The choice of uncertainty representation, to some degree, informs the associated class of formulation. Despite that, the decision-making process may entail other factors that ultimately demand an alternative choice of uncertainty representation. For instance, the risk associated with specific performance criteria may be so critical that no constraint violation can be tolerated. In this case, even if the distributional information is available, a worst-case robust formulation may be more practical.

A general UCCD problem may entail known uncertainties requiring one or more of the aforementioned representations.

Therefore, comprehensive treatment of uncertainties in UCCD problems requires the development of hybrid methods that are adept at integrating, combining, and interpreting all of such known uncertainties. These methods are generally referred to as hybrid programming [20] and have not yet been investigated for UCCD problems. It is also important to note that many real-world systems may also entail some unknown unknowns. These are uncertainties that we don't know we don't know. Unknown unknowns will most likely be present in UCCD formulations and require additional protective measures [25]. In this article, we only focus on known unknowns.

3 MATHEMATICAL FOUNDATIONS FOR UCCD FORMULATIONS

In this section, we start by introducing a universal UCCD problem formulation using concepts from probability theory. Defining this formulation in the probability space is without any loss of generality because specialized forms of this formulation can be derived through the appropriate selection of the objective function and constraints. This is specifically evident for crisp uncertainty sets as the associated expectation of the objective function and constraints reduce to deterministic quantities. For fuzzy uncertainties, several formulations become viable, such as deterministic (crisp) formulation [28], expected value [29, 30], optimistic/pessimistic, and credibility measures [20]. Due to the general correlation between operators in the probability and fuzzy space, specialized problem formulations in the fuzzy space can also be derived from the proposed formulation.

Preliminaries. The stochastic modeling of any arbitrary vector $\boldsymbol{x} \in \mathbb{R}^{n_x}$ consists in introducing a sampling space Θ (such that any element of Θ is a combination of causes that affect the state of $\boldsymbol{x}$), and then endowing it with an event space $\mathcal{F}$, and a probability measure $\mathbb{P}$, which results in the probability space $(\Theta, \mathcal{F}, \mathbb{P})$ [31]. A stochastic variable $\tilde{\boldsymbol{x}} = (\tilde{x}_1, \ldots, \tilde{x}_{n_x})$ defined on $(\Theta, \mathcal{F}, \mathbb{P})$ and endowed with a measurable space is then a mapping from Θ to $\mathbb{R}^{n_x}$ such that $\tilde{\boldsymbol{x}} \in \mathcal{X}_{stc}$. A stochastic process $\tilde{\boldsymbol{x}}(t) \in \mathcal{X}_{stc}(t)$, is defined on the probability space and has values in $\mathbb{R}^{n_x}$. $\tilde{\boldsymbol{x}}(t)$ is indexed by any finite or infinite subset T and is a mapping $t \mapsto \tilde{\boldsymbol{x}}(t)$ from $T \times (\Theta, \mathcal{F}, \mathbb{P})$ into $L^0(\Theta, \mathbb{R}^{n_x})$. Here, $L^0(\Theta, \mathbb{R}^{n_x})$ is the vector space of all $\mathbb{R}^{n_x}$-valued random variables defined on $(\Theta, \mathcal{F}, \mathbb{P})$. For any fixed $\theta \in \Theta$, the mapping $t \mapsto \tilde{\boldsymbol{x}}(t, \theta)$ is a trajectory or a sample path. For an arbitrary stochastic variable $\tilde{\boldsymbol{x}}$, $\boldsymbol{x}_\mu$ is the mean value and $\boldsymbol{x}_\sigma$ is the standard deviation. In addition, $\mathbb{P}[\cdot]$ is the probability measure and $\mathbb{E}[\cdot]$ is the expected value operator. For an arbitrary function of random variables, $o(\tilde{\boldsymbol{x}})$, its expected value is defined as $\mathbb{E}[o(\tilde{\boldsymbol{x}})] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} o(\boldsymbol{x}) f_{\tilde{\boldsymbol{x}}}(\boldsymbol{x}) dx_1 \cdots dx_{n_x}$ for a continuous random vector and $\mathbb{E}[o(\tilde{\boldsymbol{x}})] = \sum_{\boldsymbol{x}_1} \cdots \sum_{\boldsymbol{x}_{n_x}} o(\boldsymbol{x}) p_{\tilde{\boldsymbol{x}}}(\boldsymbol{x})$ for a discrete random vector. In these definitions, $f_{\tilde{\boldsymbol{x}}}(\boldsymbol{x})$ and $p_{\tilde{\boldsymbol{x}}}(\boldsymbol{x})$ are the probability distribution functions and mass functions, respectively. We use $\bar{\bullet}(\cdot)$ to indicate an arbitrary function composition of $\bullet(\cdot)$. For example, $\bar{o}(\cdot)$ describes a function of the original objective function $o(\cdot)$ such that $\bar{o} = y \circ o(\cdot) = y(o(\cdot))$, where $y(\cdot)$ is an explicit or implicit function. With these definitions, the universal UCCD problem formulation can be introduced.

3.1 A Universal UCCD Formulation

A universal, all-at-once (AAO), continuous-time, simultaneous UCCD problem can be formulated as:

$$\begin{aligned}
\underset{\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{p}}}{\text{minimize:}} \quad & \mathbb{E}\left[\bar{o}(t, \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{d}})\right] && \text{(12a)} \\
\text{subject to:} \quad & \mathbb{E}\left[\bar{\boldsymbol{g}}(t, \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{d}})\right] \leq \boldsymbol{0} && \text{(12b)} \\
& \boldsymbol{h}(t, \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{d}}) = \boldsymbol{0} && \text{(12c)} \\
& \dot{\tilde{\boldsymbol{\xi}}}(t) - \boldsymbol{f}(t, \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{d}}) = \boldsymbol{0} && \text{(12d)} \\
\text{where:} \quad & \tilde{\boldsymbol{u}}(t) = \tilde{\boldsymbol{u}},\ \tilde{\boldsymbol{\xi}}(t) = \tilde{\boldsymbol{\xi}},\ \tilde{\boldsymbol{d}}(t) = \tilde{\boldsymbol{d}} && \text{(12e)}
\end{aligned}$$

In this equation, the expectation of the composite function $\bar{o}(\cdot)$ (i.e., a function of the original objective $o(\cdot)$) is optimized over the set of optimization variables $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{p}})$, and is subject to the expectation of the composite function $\bar{\boldsymbol{g}}(\cdot)$ (i.e., a function of the original inequality constraint vector $\boldsymbol{g}(\cdot)$), analysis-type equality constraints $\boldsymbol{h}(\cdot)$, and uncertain dynamic system equality constraints in Eq. (12d). Note that $\mathbb{E}[\bar{o}(\cdot)]$ and $\mathbb{E}[\bar{\boldsymbol{g}}(\cdot)]$ refer to any of the variations that will be discussed in Sec. 3.6 (such as the nominal, worst-case, expected value, etc.). In addition, when uncertainties are not present, $\bar{o}(\cdot)$ and $\bar{\boldsymbol{g}}(\cdot)$ will be reduced to their original deterministic forms $o(\cdot)$ and $\boldsymbol{g}(\cdot)$, respectively.

This formulation includes the vector of uncertain control processes $\tilde{\boldsymbol{u}}(t) \subset \mathcal{U}(t) \in \mathcal{T}_u(t)$, uncertain state processes $\tilde{\boldsymbol{\xi}}(t) \subset \mathcal{S}(t) \in \mathcal{T}_u(t)$, time-independent uncertain optimization variables $\tilde{\boldsymbol{p}} \subset \mathcal{P} \in \mathcal{V}_u$, time-dependent uncertain problem data $\tilde{\boldsymbol{d}}(t) \subset \mathcal{D}(t) \in \mathcal{T}_u(t)$, and time-independent uncertain problem data $\tilde{\boldsymbol{d}} \subset \mathcal{D} \in \mathcal{V}_u$. Not that $\mathcal{D}(t)$ may entail some noise or disturbances. In these definitions, $\mathcal{U}(t)$ is a non-empty set of continuous-time, admissible stochastic control processes, $\mathcal{S}(t)$ is the non-empty set of stochastic state processes, $\mathcal{P}$ is the set of stochastic optimization parameters, and $\mathcal{D}$ and $\mathcal{D}(t)$ are the sets associated with stochastic problem data. In the remainder of this article, we use $\mathcal{D}$ to refer to both time-dependent and time-independent problem data.

The proposed UCCD formulation is infinite-dimensional in time and uncertainty dimension. We can draw an analogy between the infinite-dimensional time vector and the infinite-dimensional uncertainty vector. To transcribe Eq. (12) in time, numerical methods such as direct transcription have been implemented [2, 32–35]. Similarly, different uncertainty propagation techniques, such as Monte Carlo simulation (MCS), as well as special interpretations, such as worst-case, have been proposed to parameterize the uncertain dimensions. In this article, we discuss some of these formulations and special considerations, but leave the discussion on methods to our future work. This is because depending on the selected methodology, the original UCCD problem formulation might require slight modifications, especially in the decision space. This becomes more evident if we consider a UCCD problem with only plant uncertainty. For this problem, a simple MCS approach requires the satisfaction of dynamic equations for all MCS samples, while a most-probable-point (MPP) approach requires the satisfaction of dynamic equations for only the mean-values and the MPPs [8]. Through this, we develop standard, method-independent formulations for general UCCD problems.

We emphasize that describing optimization variables $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{p}})$ in the uncertain space is to avoid introducing any unnecessary assumptions/structure at this point. Furthermore, this description should not imply that the designer has complete control over all uncertainties; instead, it suggests that the decision space may entail elements associated with uncertainties. In other words, these uncertain vectors entail some deterministic part the designer/optimizer has decision-making power over. This deterministic part may be associated with mean-values, parameters of an entire distribution, shape or geometry of the deterministic uncertainty set, or parameters of the fuzzy membership function.

3.2 Uncertainties in Optimization Variables

Since Eq. (12) formulates the UCCD problem in the presence of a broad range of uncertainties, it is critical to understand where uncertainties in optimization variables $[\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{p}}]$ originate from and how they affect various elements of the problem.

3.2.1 Control Trajectories. In Eq. (12), control trajectories are modeled as stochastic processes because $\tilde{\boldsymbol{d}}$ may entail noise elements (induced by factors such as electrical noise, actuator imprecision, etc.) that directly affect control signals. This, in the control community, is referred to as matching (or lumped) uncertainties because uncertainties act on the system through the same channels as the control input. If uncertainties do not act through the control channel, they are called mismatched uncertainties [5]. Therefore, the above formulation entails both matched and mismatched uncertainties. However, it is possible to model the control input deterministically since possible disturbances on the control can be modeled in dynamics as multiplicative noise [36]. Note that "closing the control loop" with feedback controller architectures in a UCCD problem may also transform the control trajectories into stochastic quantities.

3.2.2 State Trajectories. In Eq. (12), state trajectories are uncertain due to a variety of reasons. The uncertainties from $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{d}})$ may propagate through the dynamic system and transform them into stochastic processes. Note that the resulting stochastic systems are not necessarily the same as the classical stochastic differential equations where the inputs are some idealized processes, such as Wiener or Poisson [37]. The vector of problem data, $\tilde{\boldsymbol{d}}$, may entail some information about uncertain initial/final conditions. In addition, $\tilde{\boldsymbol{d}}$ entails some noise elements that may enter the state equation in a linear or nonlinear manner. This noise may be stationary or non-stationary, exogenous (independent of decisions), or dependent on states and controls. As an example of the dependence of noise on states and controls, consider a system that starts to witness more chaotic changes after it is steered through the control command to a specific state. However, note that this dependency is already captured through the dynamic model in Eq. (12d). Note that $\tilde{\boldsymbol{\xi}}$ may also entail variables that are being controlled, or parameters of a distribution (such as mean and variance) describing the time-evolution of uncertainties in the system. The distributional (or set) information of these parameters, however, is specified and already included in the vector of uncertain problem data $\tilde{\boldsymbol{d}}$. Also, note that the effects of unmodeled, mismodeled, and neglected dynamics can be captured in Eq. (12) [5].

3.2.3 Time-independent Optimization Variables. The vector of time-independent optimization variables may also be uncertain due to factors such as imperfect manufacturing processes, plant measurement errors, or mass production of plants. Therefore, $\tilde{\boldsymbol{p}}$ is modeled as a random variable whose distributional (or set) information is known. This uncertainty will be propagated through state equations, transforming all of its associated parameter-dependent functions and variables into uncertain quantities. In addition, for free-final-time UCCD problems, uncertainties may transform t_f into an uncertain variable, requiring a transformation similar to the one described in Ref. [22].

3.3 Risk in UCCD Formulations

In a UCCD formulation, uncertainties must be represented in a way that their impact on decision-making is completely captured. This brings us to the notion of risk, which is a fundamental element of any uncertain problem. In general, risk measures can be qualitative or quantitative [38]. In a qualitative risk measure, the amount by which a threshold is surpassed does not matter. An example of qualitative risk measures are failures that result in the loss of life. In quantitative risk measures, on the other hand, it is important to know the extent to which the threshold is violated. For example, a quantitative risk measure may be associated with the energy consumption of a vehicle following a reference trajectory. When the energy consumption exceeds the threshold, it is important to know by how much. This type of risk measure can be dealt with by introducing a penalty term or constraining the amount of extra energy. In general, due to mathematical difficulties associated with probabilistic constraints, it is recommended to use probabilistic descriptions only for qualitative failure problems. Other risk measures, such as conditional value at risk that offer more desirable mathematical properties (such as convexity), are more suitable for quantitative constraint problems [38, 39].

The notion of risk is so central in decision-making under uncertainty that it is used to classify various problem classes based on the designer's attitude towards risk. These include risk-neutral, risk-averse, risk-aware, and risk-sensitive problem formulations. It is the designer's understanding of the risk associated with uncertainties in an arbitrary problem that determines the associated risk attitude in that formulation. The focus of this article is mainly on risk-neutral and risk-averse UCCD formulations.

3.4 Decision Space in UCCD Problems

CCD is an attractive approach because it simultaneously explores the plant and control design spaces to improve the dynamic system's performance [2]. When uncertainties are present, it is imperative to maintain this advantage by introducing balanced UCCD formulations in which the whole space of optimization variables is leveraged in response to uncertainties. Therefore, in a balanced formulation, uncertain control and state trajectories, as well as the vector of time-independent optimization variables, must be utilized to achieve a system-level, integrated solution that is less costly and more robust/reliable.

This discussion may also be extended to the methodological distinction between simulation-based methods (such as single shooting and multiple shooting) and simultaneous or direct-transcription (DT) methods [33, 35]. In simulation-based methods, state variables are treated as dependent variables, while in DT, state variables are treated as optimization variables. Because in the latter approach, the optimization and simulation are performed concurrently, these approaches are referred to as AAO. Consequently, the set of decision variables in a simulation-based formulation only entails physical plant and control variables, while the AAO problem formulation also entails state variables. Therefore, we must facilitate the adoption of algorithms and solution methodologies that enable the simultaneous exploitation of the design space of all decision variables, including $\boldsymbol{u}$, $\boldsymbol{\xi}$ (in AAO formulations), and $\boldsymbol{p}$ (i.e., plant optimization vector and gains).

3.5 Objective Function

While some of the elements of a UCCD problem require specific treatment in the presence of uncertainties, an important principle to remember is that there's no conceptual distinction between the treatment of an objective function and inequality constraints [40]. This statement is without any loss of generality because, for any uncertain UCCD problem, the uncertain objective function may be transferred to the vector of inequality constraints through the addition of a new decision variable. This form is referred to as the epigraph representation of the objective function and allows us to deal with all of the complications resulting from uncertainties separately within inequality constraints. Depending on the problem structure and the extent to which uncertainties affect various elements of the formulation, one may decide to keep or transfer the objective function. The computational efficiency and resulting implications of such decisions on various classes of UCCD problems remain to be investigated.

3.6 Inequality Constraints

The formulation presented in Eq. (12) allows us to select $\bar{o}(\cdot)$ and $\bar{\boldsymbol{g}}(\cdot)$ in order to formulate various forms of objective functions and constraints. In this section, these formulations are described only for the uncertain vector of inequality constraints $\boldsymbol{g}(\cdot)$. However, the same principles can be applied to formulate the objective function per the discussion in Sec. 3.5.

3.6.1 Nominal. In this formulation, uncertain quantities are prescribed and evaluated at their nominal (deterministic) values. This concept, which is referred to as guessing the future [40], attempts to estimate the unknown information for uncertain quantities. As an example, instead of creating a probabilistic model for wind velocity at a given altitude, one may use a fixed, nominal input to evaluate and solve the problem. This estimate, however, does not capture the impacts of uncertainties and makes no practical provisions for the risk associated with such uncertainties. Recalling that the expected value of a deterministic term is a deterministic quantity, Eq. (12b) can be formulated by selecting a nominal value for uncertain factors:

$$\mathbb{E}\left[\bar{g}_i(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}})\right] = g_i(t,\boldsymbol{u}_N,\boldsymbol{\xi}_N,\boldsymbol{p}_N,\boldsymbol{d}_N) \leq 0 \qquad (13)$$

where $\bullet_N$ refers to the nominal values of uncertain quantities in the ith inequality constraint.

3.6.2 Worst-Case. When uncertainties are represented as (bounded) crisp sets, it is generally desired to solve the UCCD problem such that the resulting solution is feasible for *all* realizations of randomness within the uncertainty set. This goal is equivalent to minimizing the worst-case response in the presence of the uncertainty set. As an example, consider the design of an automotive brake system subject to uncertainties from the road surface, velocity, temperature, etc. For such a design problem, it is imperative that the brake system is capable of bringing the vehicle to a halt within a reasonable amount of time, under any circumstances. If the bounds on uncertainties are known, one can minimize the worst combination of uncertainties in order to make sure that the brake system performs well for all other cases.

In the worst-case description, the ith inequality constraint can then be represented as

$$\mathbb{E}\left[\bar{g}_i(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}})\right] = \underset{\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}}{\text{maximize:}} \quad g_i(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) \leq 0 \qquad (14)$$

This formulation is also referred to as the min − max or minimax and is one of the most common interpretations of robustness in the literature [10, 41]. We note that for UCCD problems, this maximization problem must be solved subject to analysis-type system equality constraints, which will be discussed in more details in Sec. 4.3.

3.6.3 Expected Value. One of the most common probabilistic descriptions of uncertain inequality constraints is to utilize their corresponding average values [14,36,42,43]. In the stochastic programming community, this formulation is known as the expected value model. This description, however, does not hedge against the risks associated with constraint violation. Therefore, the expected value model is more suitable for objective function descriptions, or risk-neutral formulations. As an example, the expected value model may be used to maximize the average energy production of a wind farm. The expected value model for the ith constraint is described as:

$$\mathbb{E}\left[\bar{g}_i(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}})\right] = g_{\mu,i}(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) \leq 0 \tag{15}$$

3.6.4 Long-Run Expected Value. The long-run expected average [44, 45], which is also known as the infinite-horizon expected average is important in applications where the horizon is infinite and it is desired to minimize the cost per unit time or satisfy some constraints over this infinite horizon. Similar to the expected value model, the long-run expected value is most suitable for the description of the objective function, or risk-neutral formulations. As an example, this model may be used to describe the objective of minimizing the long-run average cost in a stochastic manufacturing system [46]. While infinite-horizon problems may take different forms, here, we introduce the formulation with a discounted cost:

$$\mathbb{E}\left[\bar{g}_i(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}})\right] = \limsup_{t\to\infty} \mathbb{E}\left[g_i(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}},\gamma)\right] \leq 0 \tag{16}$$

where $\gamma \geq 0$ is a discount parameter and lim sup is used to highlight that it is not known whether the limit exists. The discount rate is included to emphasize short-term rewards versus rewards that might be obtained in distant future.

This formulation is generally used to describe the long-run average cost (objective function), but it can be transferred to the vector of inequality constraints.

3.6.5 Higher-Order Moments. Sometimes, it is desired to limit the higher-order moments of an uncertain inequality constraint. For instance, it might be desired to limit the standard deviation (or variance) of one of the performance criteria, such as ride comfort, in an automotive active suspension design. This can be accomplished by defining:

$$\mathbb{E}\left[\bar{g}_i(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}})\right] = \sqrt{\mathbb{E}[g_i(\cdot)^2] - g_{\mu,i}(\cdot)^2} = g_{i,\sigma}(\cdot) \leq \sigma_{a,i} \tag{17}$$

where $g_{i,\sigma}$ refers to the standard deviation of the constraint and $\sigma_{a,i}$ is the allowable standard deviation associated with the ith constraint. This description is generally accompanied by the expectation or the nominal value of the constraint (or objective function) [9, 13, 47–49].

3.6.6 Probabilistic Chance-Constrained. For some problems, it is desirable to express and satisfy constraints in terms of the probability of an event. For example, the probability that a constraint associated with stress or defection on a part is satisfied within a given threshold. This can be done by defining the ith constraint in terms of an indicator function of an arbitrary event E:

$$\mathbb{I}_E(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) = \begin{cases} 1 \text{ if } \{\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}\} \in E \\ 0 \text{ if } \{\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}\} \notin E \end{cases} \tag{18}$$

Then, the probability can be defined through the expectation of the indicator function:

$$\mathbb{E}\left[\bar{g}_i(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}})\right] = \mathbb{E}\left[\mathbb{I}_E(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}})\right] = \mathbb{P}[E] \tag{19}$$

This formulation is the basis for the well-known chance-constraint programs and has resulted in wide range of methods that attempt to handle uncertain constraints reliably by prescribing a target failure probability $\mathbb{P}_{f,i}$ for the ith constraint [8, 11, 42], such that:

$$\mathbb{P}[g_i(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) \geq 0] \leq \mathbb{P}_{f,i} \tag{20}$$

3.6.7 Probabilistic System Chance-Constrained. Alternative chance-constrained formulations can also be developed in which the emphasis is on the system performance. One common approach is to consider a series configuration, in which the failure of any component results in the failure of the entire system. As an example, in an electric vehicle, the failure of the electric motor(s) or the battery package result in the failure of the entire system. Therefore, from the reliability point of view, these components are in series. Since UCCD problems are highly multidisciplinary, this idea may be slightly modified and extended such that the violation of any probabilistic inequality constraint results in the failure of the entire system. The probabilistic system chance-constrained formulation is then written as [4]:

$$\mathbb{P}_{sys} = \mathbb{P}\left[\bigcup_{i=i}^{n_g} g_i(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) \geq 0\right] \leq \mathbb{P}_{f,sys} \tag{21}$$

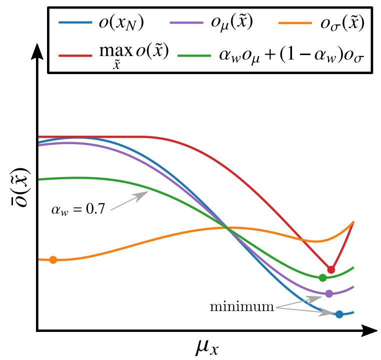
where $\mathbb{P}_{f,sys}$ is the system failure probability. This formulation ensures an overall system reliability.

3.6.8 Possibilistic Chance-Constrained. When uncertainties are defined through fuzzy variables/processes, equivalent chance-constrained formulations may be developed in the possibility space. As an example, when little information in known about uncertainties in the vehicle side-impact performance problem, one may formulate a chance-constraint such that the possibility of failure is below a given threshold. The associated possibility-based constraint can be written as:

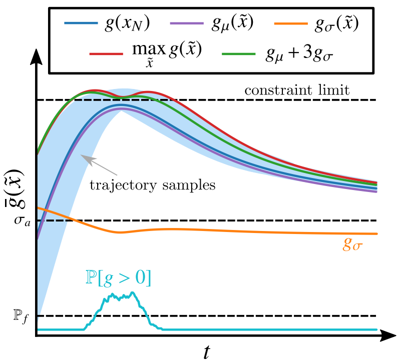
$$\mathbb{POS}[g_i(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) \geq 0] \leq \mathbb{POS}_{f,i} \tag{22}$$

where $\mathbb{POS}[\cdot]$ is the possibility measure defined on a proper possibility space, and $\mathbb{POS}_{f,i}$ is the failure possibility for ith constraint. For the sake of brevity, in this article, we avoid a detailed mathematical description of the possibility space and refer the readers to Refs. [29, 30, 50] for further discussion.

The formulations introduced above are among the common descriptions of uncertain inequality constraints (and objective functions). Other variations exist that generally attempt to address some of the shortcomings of these formulations. For ex-



(a) Objective representations.



(b) Constraint representations.

FIGURE 4: Various representations of: (a) an arbitrary objective function, and (b) an arbitrary path constraint, under uncertainties.

ample, multiple formulations have been developed to address the issue of the conservativeness of the minimax approach [51]. Figure 4 illustrates some of these representations for an arbitrary objective function and inequality path constraint.

3.7 Equality Constraints

In the presence of uncertainties, equality constraints are divided into two categories [52,53]: (i) those that must be strictly satisfied regardless of uncertainties (Type I), and (ii) those that cannot be strictly satisfied due to uncertainties (Type II).

Type I equality constraints, which are also referred to as analysis-type constraints, generally describe the laws of nature or dynamics of the system, such as Eqs. (12c) and (12d). Therefore, for the problem to be physically meaningful, these constraints must be strictly satisfied at all parameterized points along the uncertain dimension. These constitute all points at which the problem will be evaluated, such as samples generated through MCS, expected values of optimization variables, most-probable-points in reliability-based design optimization approaches, or collocation grids in generalized polynomial chaos expansion.

For an example of Type II equality constraints, assume that the sum of two length dimensions is required to be a constant value. If both of these quantities are uncertain, this condition cannot be strictly satisfied. Rather, the constraint may be relaxed or satisfied at its expected value while its standard deviation is minimized. In this article, we assume that all Type II equality constraints are already relaxed and included in the vector of inequality constraints in Eq. (12b).

A fundamental step in formulating the general UCCD problem is to identify the sources of uncertainties that affect ordinary differential equations (ODEs). When the source of uncertainty is some white-noise, idealized process, such as Wiener and Poisson processes, the resulting differential equations are termed stochastic differential equations (SDEs) [54]. As an example, the motion of electrons in a conductor can be modeled through the Wiener process. SDEs have been studied extensively and generally require methods based on Itô and Stratonovich calculus [55]. However, for general engineering applications, modeling disturbances as an idealized process is not always sufficient. Therefore, in this article, we focus on the case where the disturbance vector is a generalized process. For fuzzy uncertainties, a natural way to model uncertainty propagation in the dynamic system is through fuzzy differential equations (FDEs) [56–58].

4 SPECIALIZED FORMULATIONS

Based on the previous discussion, it is evident that both uncertainties and problem elements can be represented in different ways—resulting in multiple interpretations of uncertainties with distinct implications on the integrated UCCD solution. Therefore, it is necessary to formalize some of these interpretations through appropriate UCCD formulations.

4.1 Stochastic in Expectation (SE-UCCD)

Stochastic programming assumes that the probability distributions of the uncertain factors are known. In these situations, constraints can be modeled in different ways, such as almost surely, in expectation, or in probability [38]. Constraints that are described as "almost surely" (or "a.s") must be satisfied with the probability of one. All Type I equality constraints described in Sec. 3.7 are a.s. constraints. According to Sec. 3.6.3, a risk-neutral UCCD problem can be formulated by using the expectation of the objective function and inequality constraints:

$$\underset{\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}}}{\text{minimize:}} \quad o_\mu(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) \tag{23a}$$

$$\text{subject to:} \quad \boldsymbol{g}_\mu(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) \leq \boldsymbol{0} \tag{23b}$$

$$\boldsymbol{h}(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) = \boldsymbol{0} \quad \text{(a.s)} \tag{23c}$$

$$\dot{\tilde{\boldsymbol{\xi}}}(t) - \boldsymbol{f}(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) = \boldsymbol{0} \quad \text{(a.s)} \tag{23d}$$

$$\text{Eq. (12e)} \tag{23e}$$

Note that in this formulation $(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{d}}) \in \mathcal{X}_{stc}(t)$ and $\tilde{\boldsymbol{p}} \in \mathcal{X}_{stc}$, and Eqs. (23c) and (23d) are satisfied almost surely. Also, satisfaction of inequality constraints in expectation points to the risk-neutral nature of this formulation. A lot of real-world CCD problems, however, require an explicit risk measures for safety and functionality.

4.2 Stochastic Chance-Constrained (SCC-UCCD)

Problems with probabilistic inequality constraints are generally referred to as chance-constrained programming. They are ubiquitous in various research fields, such as reliability-based design optimization (RBDO) and trajectory optimization. Recently, novel UCCD formulations based on RBDO have been developed in Refs. [8, 11]. Here we introduce a more general chance-constrained formulation referred to as stochastic chance-constrained UCCD. The problem formulation is described as:

$$\underset{\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}}}{\text{minimize:}} \quad o_{\mu}(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) \tag{24a}$$

$$\text{subject to:} \quad \mathbb{P}[g_i(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}})>0] \le \mathbb{P}_{f,i} \quad i=1,\dots,n_g \tag{24b}$$

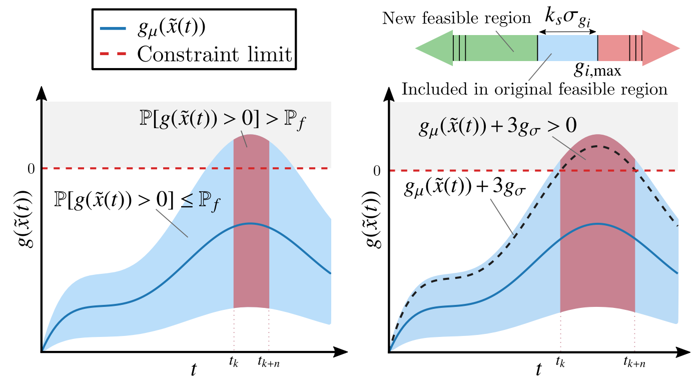
$$\text{Eqs. (23c)–(23e)} \tag{24c}$$

Again, in this formulation we have $(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{d}}) \in \mathcal{X}_{stc}(t)$ and $\tilde{\boldsymbol{p}} \in \mathcal{X}_{stc}$. Equations (23c) and (23d) are satisfied almost surely, and the probabilistic representation of inequality constraints ensures that they are satisfied with a given target reliability of $1-\mathbb{P}_{f,i}$. The stochastic interpretation of path constraints is further illustrated in Fig. 5a. In this figure, blue areas have failure probabilities that do not exceed $\mathbb{P}_f$, while red regions violate the constraint with probabilities higher than $\mathbb{P}_f$. When used only with open-loop control, the above formulation may lead to conservative trajectories. This is because, in practice, feedback controllers are often implemented for such systems and have the capacity to compensate for some of these uncertainties. However, when only open-loop control is considered, Eq. (24) often neglects the possible role of feedback controller at the time of implementation [59]. Therefore, closing the control loop in such UCCD problems may entail improvements in performance and cost.

A major challenge associated with the stochastic formulations in Eqs. (23) and (24) is that obtaining distributional information about the uncertain factors is not always viable. In addition, even if this information can be estimated, the resulting stochastic formulation is generally computationally intractable [60]. The first challenge is generally addressed by using concepts from robust optimization, which is discussed next.

4.3 Worst-Case Robust (WCR-UCCD)

Robustness in UCCD is motivated by the fact that when a solution to a deterministic CCD problem exhibits large sensitivities to perturbations in problem parameters, it becomes highly infeasible and impractical. This issue has been traditionally addressed by robust control, as well as robust design optimization communities in disparate efforts. However, to utilize the full synergistic performance potential of UCCD, both plant design and control system domains must be explored simultaneously in a balanced way. While robust UCCD has only been investigated in a handful of studies [9, 10, 49], there's a need for precise formulations and interpretations of robustness in UCCD problems. In this section, we first describe robustness and its associated worst-case realization and then introduce the WCR-UCCD formulation.



(a) Stochastic Chance-Constraint. (b) Probabilistic Robust.

FIGURE 5: Illustration of uncertain probabilistic constraints (a) Stochastic path constraint with prescribed failure probability of $\mathbb{P}_f$ and, (b) probabilistic robust constraint interpretation with constraint shift index $k_s = 3$.

4.3.1 Robust Interpretation.

In its most common interpretation, a solution is robust if it remains feasible for all of the realizations of uncertainty within the uncertainty set. The major assumption in the classical worst-case robust approach is that uncertainties belong to a (typically compact) set. Therefore, uncertainties will be unknown, but bounded. The robust counterpart (RC) of UCCD problem (which utilizes the epigraph representation of the objective function introduced in Sec. 3.5) can then be formulated as:

$$\underset{\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},v}{\text{minimize:}} \quad v \tag{25a}$$

$$\text{subject to:} \quad \left.\begin{array}{l} \boldsymbol{g}(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) \le \boldsymbol{0} \\ o(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) - v \le 0 \\ \boldsymbol{h}(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) = \boldsymbol{0} \\ \dot{\tilde{\boldsymbol{\xi}}}(t) - \boldsymbol{f}(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) = \boldsymbol{0} \end{array}\right\} \begin{array}{l} \forall\, \tilde{\boldsymbol{u}} \in \mathcal{U}(t),\ \forall\, \tilde{\boldsymbol{p}} \in \mathcal{P} \\ \forall\, \tilde{\boldsymbol{d}} \in \mathcal{D} \end{array}$$

$$\text{Eq. (12e)} \tag{25b}$$

where $(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{d}}) \in \mathcal{X}_{det}(t)$ and $\tilde{\boldsymbol{p}} \in \mathcal{X}_{det}$, i.e., the deterministic representation for uncercainty is utilized. In addition, all the constraints in Eq. (25) are hard constraints that must be satisfied for all possible realizations of uncertainty. In the remainder of this section, we assume that the new deterministic optimization variable v is included in $\tilde{\boldsymbol{p}}$, and the new inequality constraint is included in $\boldsymbol{g}(\cdot)$.

If Eq. (25) is to be satisfied for every realization of the uncertainties, then the union of their associated uncertainty sets must

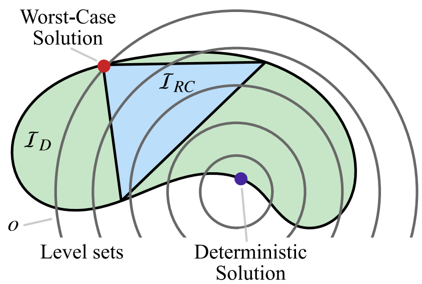


FIGURE 6: Illustration of the worst-case solution in context of the constraint feasible space and uncertainty sets.

be within the feasible space. Mathematically, this can be described as:

$$\{\mathcal{U}(t) \cup \mathcal{P} \cup \mathcal{D}\} = \mathcal{I}_{RC} \subseteq \mathcal{I}_D \tag{26}$$

$$\mathcal{I}_D := \left\{(\boldsymbol{u},\boldsymbol{p},\boldsymbol{d}) \mid \boldsymbol{g}(\cdot) \leq \boldsymbol{0}, \boldsymbol{h}(\cdot) = \boldsymbol{0}, \dot{\boldsymbol{\xi}} = \boldsymbol{f}(\cdot)\right\} \tag{27}$$

where $\mathcal{I}_D$ is the constraint feasible space, and $\mathcal{I}_{RC}$ is the union of $\mathcal{U}(t)$, $\mathcal{P}$, and $\mathcal{D}$, conceptually illustrated in Fig. 6. This definition, which is required for the solution of Eq. (25), sheds some light on some of the considerations in constructing uncertainty sets for practical robust implementations.

4.3.2 Worst-Case Robust Interpretation.

Equation (25) is a semi-infinite problem, where there's a finite number of decision variables and an infinite number of constraints. Generally, this RC problem is large, intractable, and difficult to solve. For instance, the RC of a linear optimization problem is typically a nonlinear optimization problem. Despite such difficulties, the robust interpretation offers a certain relative simplicity and computational viability compared to other interpretations, making it a valuable tool in understanding and addressing uncertainties in many engineering problems, including UCCD.

One approach to deal with this semi-infinite problem is to replace the infinite uncertainty set with a finite subset or a sequence of successively refined grids [61]. A more constructive approach, however, is to replace semi-infinite constraints with the solution of the constraint maximization problem. To understand this idea, we draw an analogy from the game theory literature. Assume that the optimizer has a natural adversarial opponent [62, 63]. Therefore, for every decision the optimizer makes, the adversarial opponent makes a decision (over uncertainties) to disturb constraints as strongly as possible. This notion leads to the realization of worst-case uncertainties and, consequently, the concept of min − max, or minimax robust formulation, which was briefly introduced in Eq. (14).

4.3.3 WCR-UCCD Formulation.

To adopt the WCR interpretation for UCCD, we need to differentiate between the decision space of the optimizer and the decision space of the adverse player. In addition to the model equations (such as system dynamics), the adverse player is restricted in its decisions to uncertainty sets of the form:

$$\mathcal{R}_q := \left\{\tilde{\boldsymbol{q}} \mid \|\boldsymbol{\delta}_q\| \leq 1\right\} \quad \text{where} \quad \boldsymbol{\delta}_q = \tilde{\boldsymbol{q}} - \hat{\boldsymbol{q}} \tag{28}$$

where $\boldsymbol{\delta}$ are bounded uncertainties, $\hat{\boldsymbol{q}}$ are the nominal values that are either decided by the optimizer (for $\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{p}}$) or prescribed in problem data, and $\boldsymbol{q}$ is:

$$\boldsymbol{q}^T := [u_1, \ldots, u_{n_u}, p_1, \ldots, p_{n_p}, d_1, \ldots, d_{n_d}] \tag{29}$$

or a column vector composed of the concatenation of all uncertain quantities. In this definition, $\|\cdot\|$ is a suitable applied norm that may be defined based on ℓ_p-norm, D-norm, conditional value at risk (CVaR)-norm, etc. [64]. The resulting uncertainty sets may have different shapes and geometries, such as box, ellipsoidal, polyhedron, etc. As an example, a simple bounded uncertainty in the space of plant optimization variables can be written as $\tilde{\boldsymbol{p}}_p = \hat{\boldsymbol{p}}_p + \boldsymbol{\delta}_p$, where $\hat{\boldsymbol{p}}_p$ is the vector of nominal plant variables (decided by the optimizer), and $\tilde{\boldsymbol{p}}_p$ is the uncertain plant optimization vector decided by the adverse player. Note $\boldsymbol{\delta}_p$, along with other factors that dictate the geometry and size of the uncertainty set, are already prescribed in $\tilde{\boldsymbol{d}}$.

The WCR-UCCD problem is now formulated such that the deterministic objective function v is minimized over the set of optimizations variables $[\hat{\boldsymbol{u}}, \hat{\boldsymbol{p}}]$, subject to constraint maximization problems:

$$\underset{\hat{\boldsymbol{u}},\hat{\boldsymbol{p}}}{\text{minimize:}} \quad v \tag{30a}$$

$$\text{subject to:} \quad \Phi_i(t, \hat{\boldsymbol{u}}, \tilde{\boldsymbol{\xi}}, \hat{\boldsymbol{p}}, \tilde{\boldsymbol{d}}) \leq 0 \quad \text{for} \quad i = 1, \ldots, n_g \tag{30b}$$

$$\boldsymbol{\psi}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{p}}) \leq \boldsymbol{0} \tag{30c}$$

where $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{p}}$ are inputs to the inner-loop optimization problem for all n_g inequality constraints. $\boldsymbol{\psi}(\cdot)$ are optional additional feasibility constraints, similar to the one used in Ref. [23]. The inner-loop maximization problem $\Phi_i(\cdot)$ is:

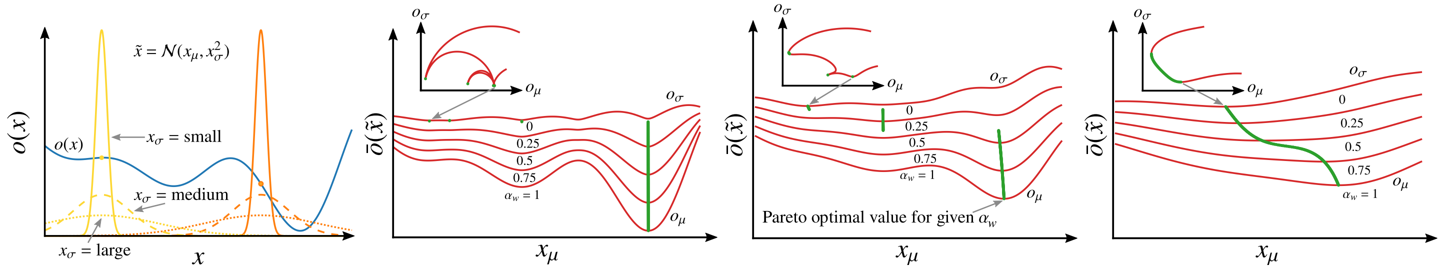
$$\underset{\boldsymbol{u},\boldsymbol{\xi},\boldsymbol{p},\boldsymbol{d}}{\text{maximize:}} \quad g_i(t, \boldsymbol{u}, \boldsymbol{\xi}, \boldsymbol{p}, \boldsymbol{d}) \tag{31a}$$

$$\text{subject to:} \quad \left.\begin{array}{l} \boldsymbol{h}(t, \boldsymbol{u}, \boldsymbol{\xi}, \boldsymbol{p}, \boldsymbol{d}) = \boldsymbol{0} \\ \dot{\boldsymbol{\xi}}(t) - \boldsymbol{f}(t, \boldsymbol{u}, \boldsymbol{\xi}, \boldsymbol{p}, \boldsymbol{d}) = \boldsymbol{0} \\ \boldsymbol{u} \in \mathcal{R}_u(t), \boldsymbol{p} \in \mathcal{R}_p \\ \boldsymbol{d} \in \mathcal{R}_{\boldsymbol{d}}(t) \end{array}\right\} \boldsymbol{C}(\cdot) \leq \boldsymbol{0} \tag{31b}$$

$$\text{where:} \quad \boldsymbol{u}(t, \hat{\boldsymbol{u}}) = \boldsymbol{u}, \; \boldsymbol{p}(\hat{\boldsymbol{p}}) = \boldsymbol{p}, \; \boldsymbol{d}(t) = \boldsymbol{d} \tag{31c}$$

where the $(\boldsymbol{u}, \boldsymbol{\xi}, \boldsymbol{d}) \in \mathcal{X}_{det}(t)$ and $\boldsymbol{p} \in \mathcal{X}_{det}$ each are a realization of uncertainties belonging to their associated sets. This inner-loop optimization problem attempts to maximize g_i by selecting the worst-case combination of uncertainties, subject to all of the Type I equality constraints and the norm-bound definition in the uncertainty sets. The feasibility sets associated with the inner-loop and outer-loop problem structure require special considerations similar to the ones described in Ref. [23].

In the above formulations, often, the decision-maker has the



(a) Deterministic $o(x)$. (b) Small uncertainty. (c) Medium uncertainty. (d) Large uncertainty.

FIGURE 7: Illustration of a Pareto trade-offs in the probabilistic robust objective function for several different levels of uncertainty (x_σ) of a single uncertain variable $\tilde{x} = \mathcal{N}(x_\mu, x_\sigma^2)$.

advantage of leveraging the size and structure of the uncertainty set to benefit from tractability, and other desirable properties of the problem structure, such as convexity [16, 17, 65, 66]. The choice of polyhedral, ellipsoidal, and box uncertainty sets, which are equivalent to ℓ_1, ℓ_2, and ℓ_∞-norm induced sets, are common in the literature [64]. Note that if the size of the selected set compared to the reality of the uncertain phenomenon is too large or too small, it might result in a solution that is too conservative or high-risk, respectively. To address this issue, one may attempt to optimally leverage the uncertainty set's size, shape, and structure to obtain a meaningful solution. This requires that the uncertainty sets are treated as additional optimization variables, leading to the concept of adjustable uncertainty sets as described in Refs. [41, 67].

4.4 Probabilistic Robust (PR-UCCD)

The robust approach introduced in the previous section is based on a worst-case philosophy that often results in conservative solutions with poor performance. If we assume that the decision-maker has some knowledge about the probabilistic behavior of uncertainties, an alternative interpretation of robustness can be used. In this interpretation, robustness is defined as the reduced sensitivity of the objective function and constraints to variations in uncertain quantities. Robustness measures commonly used with this interpretation are the expectancy and dispersion, which were described in Secs. 3.6.3 and 3.6.5, respectively, and are commonly used together in a multiobjective optimization problem to find a compromise solution. Thus, methods from robust multiobjective optimization are generally used with such formulations [68]. The PR-UCCD problem can be written as:

$$\underset{\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}}}{\text{minimize:}} \quad \alpha_w o_\mu(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) + (1-\alpha_w) o_\sigma(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) \tag{32a}$$

$$\text{subject to:} \quad \boldsymbol{g}_\mu(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) + k_s \boldsymbol{g}_\sigma(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) \leq \boldsymbol{0} \tag{32b}$$

$$\text{Eqs. (23c)–(23e)} \tag{32c}$$

where $(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{d}}) \in \mathcal{X}_{stc}(t)$ and $\tilde{\boldsymbol{p}} \in \mathcal{X}_{stc}$. In addition, α_w and $(1-\alpha_w)$ are weights associated with the multiobjective optimization problem. In the above formulation, a constraint shift index k_s, selected by the designer, is used to reduce the feasibility region of constraints. This approach practically moves the optimal solution away from constraint boundaries but does not always offer a probabilistic interpretation. In that sense, a PR-UCCD formulation is still a risk-averse approach. Alternatively, the problem can be formulated as:

$$\underset{\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}}}{\text{minimize:}} \quad \alpha_w o_\mu(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) + (1-\alpha_w) o_\sigma(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) \tag{33a}$$

$$\text{subject to:} \quad \boldsymbol{g}_\mu(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) \leq \boldsymbol{0} \tag{33b}$$

$$\boldsymbol{g}_\sigma(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) - \boldsymbol{\sigma}_a \leq \boldsymbol{0} \tag{33c}$$

$$\text{Eqs. (23c)–(23e)} \tag{33d}$$

where σ_a is the allowable standard deviation for $\boldsymbol{g}(\cdot)$. In this formulation, the uncertain inequality constraints are satisfied at their expected value, and their corresponding standard deviation (or variance) is below the allowable limit.

Probabilistic robust path constraints are further illustrated in Fig. 5b. In the top part of the illustration demonstrates the reduced feasible space for constraints with simple bounds, while the bottom shows the $3g_\sigma$ bound for an arbitrary path constraint. For an arbitrary objective function, the probabilistic robust interpretation, along with the Pareto optimal front between the expectancy and dispersion terms for notional small, medium, and large uncertainties are presented in Fig. 7.

4.5 Fuzzy Expected Value (FE-UCCD)

When uncertainties in UCCD are represented as fuzzy variables and processes, the UCCD problem can be formulated using a fuzzy expected-value model. The challenge is to choose optimization variables such that the objective function, which is related to some fuzzy processes (through fuzzy differential equations), is optimized. Here, we use the expected-value model [30, 50]:

$$\underset{\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}}}{\text{minimize:}} \quad \mathbb{E}[o(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}})] \tag{34a}$$

$$\text{subject to:} \quad \text{Eqs. (12b-12e)} \tag{34b}$$

where we note that in this formulation $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{d}}) \in \mathcal{X}_{fuzzy}(t)$ and $\tilde{\boldsymbol{p}} \in \mathcal{X}_{fuzzy}$ and $\dot{\tilde{\boldsymbol{\xi}}} - \boldsymbol{f}(\cdot) = 0$ are fuzzy differential equations.

4.6 Possibilistic Chance-Constrained (PCC-UCCD)

To offer protection against risk, a possibility measure may be used. This ensures that fuzzy constraints hold within a given confidence threshold [50]. The possibility-based chance-constrained UCCD formulation can be written as:

$$\underset{\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{p}}}{\text{minimize:}} \quad \mathbb{E}[o(t, \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{d}})] \tag{35a}$$

$$\text{subject to:} \quad \mathbb{POS}[g_i(t, \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{d}}) > 0] \leq \mathbb{POS}_{f,i} \tag{35b}$$

$$\text{Eqs. (12c-12e)} \tag{35c}$$

where $\mathbb{POS}_{f,i}$ is the target possibility of failure of constraint i, and $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{d}}) \in \mathcal{X}_{fuzzy}(t)$ and $\tilde{\boldsymbol{p}} \in \mathcal{X}_{fuzzy}$ and $\dot{\tilde{\boldsymbol{\xi}}} - \boldsymbol{f}(\cdot) = 0$ are fuzzy differential equations.

5 DISCUSSION

With the various formulations now defined, we discuss several of them in more detail, focusing on their connections and existing research.

5.1 Linking Stochastic and Robust Formulations

Different forms of uncertainty representation lead to different interpretations and, therefore, problem formulations. Specifically, in SCC-UCCD, it is assumed that the probability distribution of uncertainties is known, or it can be estimated. In contrast, the WCR-UCCD assumes that uncertainties belong to a crisp set and no probabilistic information is available. Therefore, while the SCC-UCCD gives a probabilistic measure to quantify the risks associated with constraint violation, the robust UCCD cannot offer such a measure. Nevertheless, strict satisfaction of (infinitely many) hard constraints in WCR-UCCD in Eq. (25) (when an appropriately sized/shaped uncertainty set is selected) is equivalent (in the limit) to the satisfaction of probabilistic constraints in SCC-UCCD with an infinitesimally small failure probability.

In addition, in modern robust approaches, the size and geometry of the uncertainty sets may be leveraged to adjust the associated risk. For instance, increasing the size of the uncertainty set in the WCR-UCCD increases the number of constraints that need to be satisfied in Eq. (25), which is equivalent to reducing the probability of failure $\mathbb{P}_f$ in SCC-UCCD formulations. Finally, Refs. [15, 17] offer probabilistic interpretations of robust formulations, which practically bridge the gap between the minimax interpretation of robust formulations and the probabilistic interpretation of stochastic chance-constrained problems. This interpretation leads to the notion of probabilistic guarantees for robust optimization problems and seeks to connect robust feasibility to the probability of feasibility. Consequently, even when the underlying distribution is known, benefits from the tractability of robust formulations may compel one to use such probabilistic guarantees in robust formulations instead of using stochastic ones. Such probabilistic guarantees may be computed a priori as a function of the structure and size of the uncertainty set and lead to the notion of a budget of uncertainty [17].

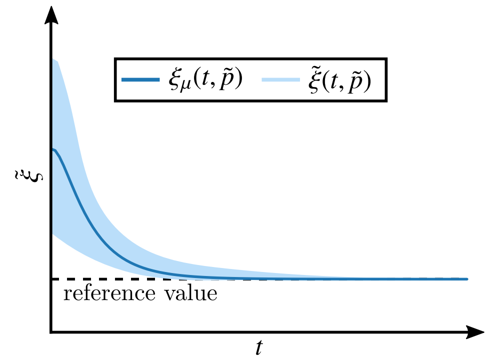


FIGURE 8: Reference tracking of a stable control system with several uncertainties using an infinite-horizon linear quadratic regulator (LQR).

5.2 Insights From Robust Control Theory

Robust control theory is involved with the analysis and synthesis of controllers that can mitigate the impact of uncertainties on performance specifications and stability. In classical control theory, these performance specifications are described through frequency or time domain measures. Various tools such as gain and phase margins [69], disk margins [70], H_2, H_∞, and μ-synthesis [6] have been developed to address uncertainty-related challenges.

The development of robust control theory has been largely dependent upon the benefits of feedback control. First, it should be emphasized that the universal formulation introduced in Eq. (12) may entail control gains $\boldsymbol{p}_c$ that are used to establish a feedback control. In addition, while closed-loop control plays an essential role in mitigating the impact of some uncertainties in UCCD problems, these uncertainties still affect the dynamic system behavior and the overall system performance. As shown in Fig. 8, a notional infinite-horizon linear quadratic regulator (LQR) (which is an optimal controller for its associated cost function) reduces uncertainty in the system response over time to the reference value, assuming stability under the uncertainties.

5.3 Formulations from Stochastic Control Theory

In stochastic control theory, idealized processes such as stationary, normal, Markov, second-order, and Wiener are used to characterize the distribution of stochastic processes. Many of the disturbances affecting the control system can be modeled by processes generated from Wiener processes [7]. While we previously assumed that the noise vector is included in the vector of problem data $\tilde{\boldsymbol{d}}$, to keep the notation consistent with stochastic control theory, here, we use $\tilde{\boldsymbol{w}}$ to describe an n_w-dimensional standard Brownian motion defined on a complete probability space. The nonlinear stochastic system model can be described as:

$$\begin{cases} d\tilde{\boldsymbol{\xi}}(t) = \boldsymbol{f}(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}})\,dt + \boldsymbol{b}(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}})\,d\tilde{\boldsymbol{w}}(t) \\ \text{where: } \tilde{\boldsymbol{u}}(t) = \tilde{\boldsymbol{u}},\ \tilde{\boldsymbol{\xi}}(t) = \tilde{\boldsymbol{\xi}},\ \tilde{\boldsymbol{\xi}}(t_0) = \boldsymbol{\xi}_0 \end{cases} \tag{36}$$

where the $\boldsymbol{f}(\cdot)$ and $\boldsymbol{b}(\cdot)$ are maps that are commonly referred to as the drift and diffusion terms, respectively [71]. Because standard Brownian motion is not differentiable, its associated integral form is commonly used instead and requires Iˆto, Stratonovich, or backward integral approaches.

A special case of Eq. (36) is when the dynamics are linear and the objective function is quadratic in $(\tilde{\boldsymbol{\xi}}(t),\tilde{\boldsymbol{u}}(t))$. This problem, referred to as a stochastic linear-quadratic problem (SLQ-UCCD), is significant because the optimal control law can be synthesized into a feedback form of the optimal state, and the corresponding proportional coefficients may be specified through the associated Ricatti equation. This unique control law is a combination of the Kalman filter and LQR. Additionally, we note that for linear systems with additive white noise, several tools become available. For example, using linear filters such as Wiener filter in the frequency domain and Kalman filters in the state-space domain, one can separate the noise from the signal of interest by minimizing the mean-square error [72]. Finally, there are other cases studied when the state equation is linear [71, 73, 74].

5.4 Open-loop Control Structure Under Uncertainties

There is an essential question on the role of optimal control trajectories in the open-loop formulation of UCCD problems. In response to uncertainties, one may use an open-loop single-control (OLSC) or an open-loop multiple-control (OLMC) structure. OLSC is structured to find a single control command, which is often used for reference tracking applications, while OLMC elicits a range of optimal control responses based on the realization of uncertainties. Distinctions between the two structures are best manifested when solving boundary-value UCCD problems. This is because, unlike OLMC, the single control command in OLSC cannot satisfy all of the prescribed initial and terminal boundary conditions in the presence of uncertainties.

This issue has been dealt with in two different ways in the literature: (i) relaxing the prescribed terminal boundary conditions [8, 49], or (ii) minimizing the variance of the terminal state in a multi-objective optimization problem [75, 76]. These remedies enable a solution to the OLSC-UCCD problem, but they have limitations because they do not enforce the terminal boundary conditions. This caveat is problematic because relaxing the boundary conditions is not practically viable for many real-world applications. Therefore, OLSC should be used selectively in the appropriate context.

On the other hand, OLMC is based on the idea that uncertainty realizations should elicit a distinct optimal control response from the UCCD problem (which has conceptual similarities to how closed-loop systems respond). Because each distinct optimal control response is only associated with a specific uncertainty realization, all the initial and terminal boundary conditions may be satisfied in this control structure. Through this, OLMC provides additional insights into the uncertainty-informed limits of the system performance. Therefore, OLMC is suitable during early-stage design where plant and control spaces are being explored, not only for optimality but also for reliability, robustness, or any other risk measures. OLSC and OLMC structures are compared in Ref. [77].

5.5 Stochastic and Robust Model Predictive Control

While all formulations introduced in this article consider a single-horizon UCCD problem, model predictive control (MPC) solves a sequence of such problems to find a cost-minimizing control action for a relatively short horizon in the future. For online implementations, this controller has the advantage of the current state information to predict state trajectories that emanate from the current state. The issue of uncertainties considered with robust and stochastic MPC [10, 78, 79].

6 CONCLUSION

With all the recent advances and applications of (deterministic) control co-design, significant work is still needed to handle uncertainty when developing effective combined plant and control solutions. Investigating the current state-of-the-art for uncertain control co-design (UCCD), we have identified several significant limitations. Generally, the scope of uncertainties is limited to a single discipline (often either with a plant or control or even solution method emphasis). Additionally, different interpretations and representations of uncertainty affect different problem elements, including the objective function, equality/inequality constraints, and optimization variables.

To start to address these shortcomings, this article discussed a broad range of relevant uncertainties and the multitude of ways to handle UCCD problem elements. The discussion naturally led to six specialized UCCD problem formulations, including stochastic in expectation, stochastic chance-constrained, worst-case (minimax) robust, probabilistic robust, fuzzy expectation, and possibilistic chance-constrained. These formulations are not disconnected; the link between minimax robust and stochastic chance-constrained UCCD was also discussed.

Overall, this article aims at providing a concrete framework to discussing and representing uncertainties in UCCD, providing a foundation for additional advances, both in theory and applications of UCCD. Understanding how to represent and interpret a domain's uncertainties is one of the first challenges. A natural next step is to investigate methods and solution strategies corresponding to these formulations, seeking to balance various design goals and computational expense.

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