SECOND ORDER FORMULAS

Problem:
Given the differential equation:

\[
\frac{dy}{dx} = f(x, y) \quad (1) \\
y(x_i) = y_i \quad (2)
\]

and

\[
y_{i+1} = y_i + [ak_1 + bk_2]h \quad (4) \\
k_1 = f(x_i, y_i) \quad (5) \\
k_2 = f(x_i + mh, y_i + k_1 mh) \quad (6)
\]

we seek to determine \(a, b,\) and \(m\) so that the expression for \(y_{i+1}\) agrees with the Taylor series expansion up to and including \(h^2\) terms.

Solution:
Expand \(k_1\) and \(k_2\) in a Taylor series about \(x_i\) and \(y_i\) to give:

\[
k_i = f(x_i, y_i) = f_i \quad (7)
\]

and

\[
k_2 = f + \frac{\partial f}{\partial x} mh + \frac{\partial f}{\partial y} k_1 mh + O(h^2) \\
= f + \frac{\partial f}{\partial x} mh + \frac{\partial f}{\partial y} f mh + O(h^2) \\
= f + (f_x + f_y f) mh + O(h^2) \quad (8)
\]
where $f$ and its derivatives are evaluated at $(x_i, y_i)$.

Substitution now gives:

$$y_{i+1} = y_i f h + b f h + b (f_x + f_y f) m h^2 + O(h^3)$$

$$y_{i+1} = y_i + (a + b) f h + m b (f_x + f_y f) h^2 + O(h^3) \quad (9)$$

which agrees with the Taylor series up to $h^2$ if:

$$(a + b) = 1 \quad (10)$$

$$mb = 1/2 \quad (11)$$

Because the above two equations contain three parameters, there are an infinite number of ways to satisfy them, including:

$m = 1, a = b = 1/2$ for modified Euler’s method

$m = 1/2, a = 0, b = 1$ for improved Euler’s method
FOURTH ORDER FORMULAS

Problem:
Given the following formula:

\[
y_{i+1} = y_i + [a k_1 + b k_2 + c k_3 + d k_4] h
\]

\[
k_1 = f(x_i, y_i)
\]
\[
k_2 = f(x_i + mh, y_i + k_1 mh)
\]
\[
k_3 = f(x_i + nh, y_i + k_2 nh)
\]
\[
k_4 = f(x_i + ph, y_i + k_3 ph)
\]

we seek values for \(a, b, c, d, m, n,\) and \(p\) so that \(y_{i+1}\) will agree with the Taylor series expansion up to and including terms of \(h^4\).

After a great deal of algebra, the following equations can be found:

\[
a + b + c + d = 1
\]
\[
b m + cn + dp = 1/2
\]
\[
b m^2 + cn^2 + dp^2 = 1/3
\]
\[
b m^3 + cn^3 + dp^3 = 1/4
\]
\[
c mn + dnp = 1/6
\]
\[
c mn^2 + dnp^2 = 1/8
\]
\[
c m^2 n + d n^2 p = 1/12
\]
\[
d m np = 1/24
\]

There are seven equations and eight parameters. The traditional solution is:

\[
m = 1/2
\]
\[
n = 1/2
\]
\[
p = 1
\]
\[
a = 1/6
\]
\[
d = 1/6
\]
\[
b = 1/3
\]
\[
c = 1/3
\]
which gives the following formula for the fourth order Runge-Kutta method:

\[
\begin{align*}
y_{i+1} &= y_i + \frac{1}{6} \left[ k_1 + 2k_2 + 2k_3 + k_4 \right] h \\
k_1 &= f(x_i, y_i) \\
k_2 &= f \left( x_i + \frac{1}{2}h, y_i + \frac{1}{2} k_1 h \right) \\
k_3 &= f \left( x_i + \frac{1}{2}h, y_i + \frac{1}{2} k_2 h \right) \\
k_4 &= f \left( x_i + h, y_i + k_3 h \right)
\end{align*}
\]
CE 563 COMPUTATIONAL METHODS

SECOND ORDER ODE’S

**Problem:**
Given the second order ordinary differential equation,

\[
\frac{d^2 y}{dx^2} = f \left(x, y, \frac{dy}{dx}\right)
\]

determine \(y(x)\) using a Runge-Kutta method.

**Solution:**
We begin by writing the equation as two, first-order equations as follows:

\[
\frac{dy'}{dx} = f \{x, y, y'\}
\]

\[
\frac{dy}{dx} = F \{x, y, y'\}
\]

Next, we apply our Runge-Kutta formulas to each of the two equations as follows:
Euler’s Method (First-order R-K):

<table>
<thead>
<tr>
<th>( \frac{dy'}{dx} = f(x, y, y') )</th>
<th>( \frac{dy}{dx} = y' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_{i+1} = y_i + f_{1i} h )</td>
<td>( y_{i+1} = y_1 + F_{1i} h )</td>
</tr>
<tr>
<td>( f_1 = f(x_i, y_i, y'_i) )</td>
<td>( F_1 = y'_i )</td>
</tr>
<tr>
<td>( y'_{i+1} = y'<em>i + f</em>{1i} h )</td>
<td>( y_{i+1} = y_i + y'_i h )</td>
</tr>
<tr>
<td>( f_1 = f(x_i, y_i, y'_i) )</td>
<td></td>
</tr>
</tbody>
</table>
Modified Euler's Method (Second-order R-K):

\[
\frac{dy'}{dx} = f(x, y, y') \quad \frac{dy}{dx} = y'
\]

\begin{align*}
y'_{i+1} &= y_i + \frac{1}{2} (f_1 + f_2) h \\
\quad &\quad \quad \quad \quad \quad \quad \quad y_{i+1} = y_i + \frac{1}{2} (F_1 + F_2) h
\end{align*}

\begin{align*}
f_1 &= f(x_i, y_i, y'_i) \\
F_1 &= y'_i
\end{align*}

\begin{align*}
f_2 &= f(x_i + h, y_i + F_1 h, y'_i + f_1 h) \\
F_2 &= y'_i + f_1 h
\end{align*}

\begin{align*}
y'_{i+1} &= y'_i + \frac{1}{2} \{ f_1 + f_2 \} h \\
y_{i+1} &= y_i + y'_i h + \frac{1}{2} \{ f_1 \} h^2
\end{align*}

\begin{align*}
f_1 &= f(x_i, y_i, y'_i)
\end{align*}

\begin{align*}
f_2 &= f(x_i + h, y_i + y'_i h, y'_i + f_1 h)
\end{align*}
Fourth-order Runge-Kutta Method:

\[
\begin{align*}
\frac{dy}{dx} &= f(x, y, y') \\
\frac{dy}{dx} &= y'
\end{align*}
\]

<table>
<thead>
<tr>
<th>( y'_{i+1} = y_i + \frac{h}{6} (f_1 + 2f_2 + 2f_3 + f_4) h )</th>
<th>( y_{i+1} = y_1 + \frac{h}{6} (F_1 + 2F_2 + 2F_3 + F_4) h )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_1 = f(x_i, y_i, y'_i) )</td>
<td>( F_1 = y'_i )</td>
</tr>
<tr>
<td>( f_2 = f \left( x_i + \frac{h}{2}, y_i + F_1 \frac{1}{2} h, y'_i + f_1 \frac{1}{2} h \right) )</td>
<td>( F_2 = y'_i + f_1 \frac{1}{2} h )</td>
</tr>
<tr>
<td>( f_3 = f \left( x_i + \frac{h}{2}, y_i + F_2 \frac{1}{2} h, y'_i + f_2 \frac{1}{2} h \right) )</td>
<td>( F_3 = y'_i + f_2 \frac{1}{2} h )</td>
</tr>
<tr>
<td>( f_4 = f \left( x_i + h, y_i + F_3 h, y'_i + f_3 h \right) )</td>
<td>( F_4 = y'_i + f_3 h )</td>
</tr>
<tr>
<td>( y'_{i+1} = y_i + \frac{h}{6} (f_1 + 2f_2 + 2f_3 + f_4) h )</td>
<td>( y_{i+1} = y_i + y'_i h + \frac{h}{6} { f_1 + f_2 + f_3 } h^2 )</td>
</tr>
<tr>
<td>( f_1 = f(x_i, y_i, y'_i) )</td>
<td>( f_2 = f \left( x_i + \frac{h}{2}, y_i + y'_i \frac{1}{2} h, y'_i + f_1 \frac{1}{2} h \right) )</td>
</tr>
<tr>
<td>( f_3 = f \left( x_i + \frac{h}{2}, y_i + y'_i \frac{1}{2} h + f_1 \frac{1}{4} h^2, y'_i + f_2 \frac{1}{2} h \right) )</td>
<td>( f_4 = f \left( x_i + h, y_i + y'_i h + f_2 \frac{1}{2} h, y'_i + f_3 h \right) )</td>
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