Example – Kinematic Wave

A rectangular channel of width $B = 200$ ft is 24000 feet long, has a bed slope of $S_o = 0.01$ and a Manning’s roughness factor $n = 0.035$. The inflow hydrograph to the channel is tabulated below. Implement a linear finite-difference solution of the kinematic wave equations to route the inflow hydrograph to the end of the channel. The initial conditions correspond to uniform flow along the channel at a rate of 2000 cfs. Use a $\Delta t$ of 3 min (180 sec) and a $\Delta x$ of 3000 ft.

<table>
<thead>
<tr>
<th>Inflow Time (min)</th>
<th>Inflow Rate (cfs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2000</td>
</tr>
<tr>
<td>12</td>
<td>2000</td>
</tr>
<tr>
<td>24</td>
<td>3000</td>
</tr>
<tr>
<td>36</td>
<td>4000</td>
</tr>
<tr>
<td>48</td>
<td>5000</td>
</tr>
<tr>
<td>60</td>
<td>5000</td>
</tr>
<tr>
<td>72</td>
<td>5000</td>
</tr>
<tr>
<td>84</td>
<td>4000</td>
</tr>
<tr>
<td>96</td>
<td>3000</td>
</tr>
<tr>
<td>108</td>
<td>2000</td>
</tr>
<tr>
<td>120</td>
<td>2000</td>
</tr>
</tbody>
</table>

As discussed in class, for a kinematic wave, the momentum equation reduces to,

$$S_o = S_f$$  \hspace{1cm} (1)

implying that the energy grade line is parallel to the channel bottom, and that the flow is steady and uniform. The above momentum equation can be shown to be equivalent to the following relationship between discharge, $Q$, and area of flow, $A$,

$$A = \alpha Q^b$$  \hspace{1cm} (2)

which, for example, can be satisfied by Manning equation,

$$Q = \frac{\sqrt{S_f}}{n} R^{2/3} A \text{ (if using metric units)}$$  \hspace{1cm} (3a)

$$Q = \frac{1.49 \sqrt{S_f}}{n} R^{2/3} A \text{ (if using English units)}$$  \hspace{1cm} (3b)

which can be rearranged as,
A = \left[ \frac{nP^{2/3}}{\sqrt{S_f}} \right]^{3/5} Q^{3/5} \text{ or } A = \left[ \frac{nP^{2/3}}{1.49 \sqrt{S_f}} \right]^{3/5} Q^{3/5}

Thus, as shown in class,

\[ \alpha = \left[ \frac{nP^{2/3}}{\sqrt{S_f}} \right]^{3/5} \text{ or } \alpha = \left[ \frac{nP^{2/3}}{1.49 \sqrt{S_f}} \right]^{3/5} \text{ and } \beta = 3/5 = 0.6 \tag{4} \]

Together with the continuity equation,

\[ \frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q \tag{5} \]

these equations represent the kinematic wave flow routing approach.

If an observer moves with the kinematic wave at a speed equal to the kinematic wave celerity, the observer would see the flow rate increase at a rate equal to the lateral inflow rate, \( q \), as shown below,

\[ \frac{dQ}{dx} = \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial t} \frac{dx}{dt} = \frac{\partial Q}{\partial x} + \frac{1}{c_k} \frac{\partial Q}{\partial t} = q \tag{6} \]

It can be seen that for conditions of no lateral inflow, that is, for \( q = 0 \), \( dQ/dx = 0 \). Thus, kinematic waves do not attenuate; they simply translate downstream without dissipation. Given that at any cross section \( Q \) and \( A \) are functionally related as \( A = \alpha Q^\beta \), the continuity equation can be rewritten as,

\[ \frac{\partial Q}{\partial x} + \frac{\partial A}{\partial Q} \frac{dQ}{dt} = q \tag{7} \]

and,

\[ \frac{\partial A}{\partial t} = \frac{dA}{dQ} \frac{\partial Q}{dt} = \alpha \beta Q^{\beta-1} \frac{\partial Q}{\partial t} \text{ or } \frac{\partial Q}{\partial x} + \frac{\alpha \beta Q^{\beta-1}}{\partial x} \frac{\partial Q}{\partial t} = q \tag{8} \]

For the linear solution, the equation is linearized by substituting an average of known solutions for the coefficient of the nonlinear term. This leads to the following solution,

\[ \frac{Q_{i+1}^{j+1} - Q_i^{j+1}}{\Delta x} + \alpha \beta \left( \frac{Q_i^{j+1} + Q_{i+1}^{j+1}}{2} \right)^{\beta-1} \left( \frac{Q_i^{j+1} - Q_i^{j+1}}{\Delta t} \right) = \left( \frac{q_i^{j+1} + q_i^{j+1}}{2} \right) \tag{9} \]

\[ Q_i^{j+1} = \left[ \frac{\Delta t}{\Delta x} Q_i^{j+1} + \alpha \beta \left( \frac{Q_i^{j+1} + Q_{i+1}^{j+1}}{2} \right)^{\beta-1} Q_i^{j+1} + \Delta t \left( \frac{q_i^{j+1} + q_i^{j+1}}{2} \right) \right] \tag{10} \]
In this linear solution, the subscript refers to the space coordinate and the superscript refers to the time coordinate. The solution advances on a time line from upstream to downstream.

The table below is the application of Equation 10 to route the hydrograph given above.

Solving equation 3 for the flow depth, it can be shown that the flow depth, $y$, corresponding to $Q = 5000$ cfs is 2.92 ft. That is, the solution of

$$5000 = \frac{1.49\sqrt{0.01}}{0.035} \left[ \frac{200y}{200 + 2y} \right] 200y$$

is $y = 2.92$ ft. Therefore, $y \ll B$, and we have a wide rectangular channel. Thus, equation 4 yields,

$$\alpha = \left[ \frac{n P^{2/3}}{1.49 \sqrt{S_f}} \right]^{3/5} = \left[ \frac{0.035 \cdot 200^{2/3}}{1.49 \sqrt{0.01}} \right]^{3/5} = 3.49 \quad \text{and} \quad \beta = \frac{3}{5} = 0.6$$

Using these values of $\alpha$ and $\beta$ in equation 10 yields the solution tabulated below and shown in the figure.
The shaded cells above indicate the known values (in blue) for every application of equation 10 in order to compute the discharge at the end of the time step and the downstream end of the subreach (in green).

For example, for \( j = 5 \), and \( i = 1 \), equation 10 is:

\[
Q_2^6 = \frac{\Delta t}{\Delta x} \left( \frac{Q_1^i + \alpha \beta (Q_1^2 + Q_1^6)^{\beta - 1} Q_2^2 + \Delta t (Q_2^6 + Q_2^5)}{2} \right) - \frac{\Delta t}{\Delta x} + \alpha \beta \left( \frac{Q_2^2 + Q_1^5}{2} \right)^{\beta - 1}
\]

\( \alpha = 3.49, \beta = 0.6, \Delta t = 180 \text{ s}, \) and \( \Delta x = 3000 \text{ ft} \). In addition, \( Q_1^6 = 2250 Q_2^2 = 2000 Q_2^6 = 0 \) \( Q_2^5 = 0 \); these values yield \( Q_2^5 = 2095.10 \). The integration proceeds by increasing \( i \) until the end of the reach, and then resetting \( i \) to 1 and increasing \( j \).