4.10.1 Frequency Response of a System

\[ F_{\text{ext}}(t) = F_o \sin(\omega t) \]  
(4.76)

\[ x(t) = X_o \sin(\omega t + \phi) \]  
(4.77)

Analytical Procedure to Determine the Frequency Response of a System

1. **Find the Laplace transform** of the system differential equation assuming initial conditions are zero: \( x(0) = \frac{dx}{dt}(0) = 0 \).

\[
\frac{d^2 x(t)}{dt^2} + 2\zeta \omega_n \frac{dx(t)}{dt} + \omega_n^2 x(t) = \frac{\omega_n^2}{k} F_{\text{ext}}(t) 
\]  
(4.78)

\[
(s^2 + 2\zeta \omega_n s + \omega_n^2)X(s) = \frac{\omega_n^2}{k} F_{\text{ext}}(s) 
\]  
(4.79)

2. **Find the transfer function** of the system, which is the ratio of the output and input Laplace transforms.

\[
G(s) = \frac{X(s)}{F_{\text{ext}}(s)} = \frac{\frac{\omega_n^2}{k}}{(s^2 + 2\zeta \omega_n s + \omega_n^2)} 
\]  
(4.80)

3. To simulate a harmonic input, replace \( s \) with \( j\omega \) in the transfer function. This yields the frequency response behavior of the system.

\[
G(j\omega) = \frac{1/k}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + j\left(2\zeta \frac{\omega}{\omega_n}\right)} 
\]  
(4.81)

4. **Find the desired amplitude ratio between the output and input** by determining the magnitude of the complex transfer function:

\[
\text{mag}[G(j\omega)] = |G(j\omega)| 
\]  
(4.82)

\[
\frac{X_o}{F_o/k} = \frac{1}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2 \left(\frac{\omega}{\omega_n}\right)^2}^{1/2} 
\]  
(4.83)

5. **Find the phase angle** \( \phi \) between the output and input by determining the argument of the complex transfer function:

\[
\phi = \text{arg}[G(j\omega)] = \angle G(j\omega) 
\]  
(4.84)

\[
\phi = 0 - \tan^{-1}\left\{\frac{2\zeta \omega}{\omega_n}\right\} = -\tan^{-1}\left\{\frac{2\zeta}{\omega_n - \omega}\right\} 
\]  
(4.85)