1) Prove the “max column sum” formula for the induced 1-norm:

\[ \|A\|_{1} = \max_{j} \left( \sum_{i} |a_{ij}| \right) \]

2) Suppose I have a known matrix \(M\) and a perturbation \(\Delta\). I wish to consider the question of what is the smallest perturbation \(\Delta\) which makes \(I - M\Delta\) singular:

i) Show that if \(\Delta\) satisfies:

\[ \sigma(\Delta) < \frac{1}{\sigma(M)} \]

then \(I - M\Delta\) cannot be singular.

ii) Construct a \(\Delta\) with

\[ \sigma(\Delta) = \frac{1}{\sigma(M)} \]

for which \(I - M\Delta\) is singular.

3) Verify that the vector 2-norm

\[ \|w\|_{2} = \sqrt{\sum_{i} |w_{i}|^2} \]

is indeed a norm (i.e., satisfies all four axioms).

4) Prove the following:

i) The vector 2-norm is unitary-invariant, i.e., for any unitary matrix \(U\) and compatible vector \(v\):

\[ \|Uv\|_{2} = \|v\|_{2} \]

ii) Hence show that the induced 2-norm of a matrix equals its maximum singular value:

\[ \|A\|_{2} = \sigma(A) \]
5) Prove that the maximum singular value satisfies the sub-multiplicative property:

$$\sigma(AB) \leq \sigma(A)\sigma(B)$$

6) Prove that for any vector norm:

$$||u|| - ||v|| \leq ||u - v||$$

7) Define the sequence of functions $f_k(x)$, belonging to $C[0,1]$, as:

$$f_k(x) = 0 \quad 0 \leq x \leq \frac{1}{k}$$
$$f_k(x) = 2(k^{\frac{3}{2}}x - k^{\frac{1}{2}}) \quad \frac{1}{k} \leq x \leq \frac{3}{2k}$$
$$f_k(x) = 2(-k^{\frac{3}{2}}x + 2k^{\frac{1}{2}}) \quad \frac{3}{2k} \leq x \leq \frac{2}{k}$$
$$f_k(x) = 0 \quad \frac{2}{k} \leq x \leq 1$$

and answer the following:

i) Sketch a typical function $f_k(x)$.

ii) Prove the following:

$$\|f_k\|_1 = \frac{1}{2}k^{-\frac{1}{2}} \rightarrow 0 \quad \text{as } k \rightarrow \infty$$
$$\|f_k\|_2 = \frac{1}{\sqrt{3}} \quad \text{for all } k$$
$$\|f_k\|_\infty = k^{\frac{3}{2}} \rightarrow \infty \quad \text{as } k \rightarrow \infty$$

iii) Note that with these signal norms I have defined a sequence $\|f_k\|$ which is going to zero in one norm, while blowing up in another. Is it possible to construct such an example using vector norms on $C^n$ (explain)?

8) For a SISO LTI system with transfer function $G(s)$, show that the induced $\infty$-norm of the system is given by the 1-norm of the impulse response $g(t)$:

$$\|G\|_\infty = \|g\|_1 = \int_0^\infty |g(t)|dt$$