Lecture 47
Switch Mode Converter Transfer Functions: 
\( T_{vd}(s) \) and \( T_{vg}(s) \)

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( this section is optional)
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2. \( n^{th} \) order case
3. Cuk example
4. Forth Order Filter
   Problem 8.19

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Lecture 47
Switch Mode Converter Transfer Functions: $T_{vd}(s)$ and $T_{vg}(s)$

A. Guesstimating Roots of Complex Polynomials : $P(s)$

1. Quick Insight

$$a_2 s^2 + a_1 s + 1 = (1 + a_1 s)(1 + \frac{a_2}{a_1} s)$$

$$= 1 + s[a_1 + \frac{a_2}{a_1}] + a_2 s^2$$

OK approximation only if $\frac{a_2}{a_1} \ll a_1$

Why bother?
Left Side Roots are hard to guess
Right Side Roots are easy to guess

$$s_1 = -\frac{1}{a_1} \quad \text{low f root coeff. if } a_1 \text{ is big}$$
$$s_2 = -\frac{a_1}{a_2} \quad \text{high f root if } a_1 \text{ is big}$$

This is the Widely separated root approximation:

$X-------------X$

$s_1 = 1/a_1 \quad \text{Means} \quad s_2 = a_1/a_2$

This is valid if $a_2/a_1 \ll a_1$

Can we generalize this to higher order polynomials?
Can we avoid root finder programs? It’s really academic as root finders are widely available on calculators.

2. $n^{th}$ order case
a. ideal case for approximation

$$R(s) = 1 + a_1 s + a_2 s^2 + a_n s^n \quad \text{All } a's \text{ are real}$$
Form is arbitrary but USEFUL
Beat/Cajole/Force into standard form
\[ R(s) = (1 + t_1 s)(1+ t_2 s) \ldots (1+t_n s) \rightarrow \text{easy to see by expanding out all the factors. t’s or time constants are either real or complex} \]

\[ a_1 = t_1 + t_2 + \ldots + t_n \]
\[ a_2 = \tau_1 (\tau_2 + \tau_3 + \ldots \tau_n) + \tau_2 (\tau_3 + \tau_4 + \ldots \tau_n) \]
\[ a_3 = \tau_1 \tau_2 (\tau_3 + \tau_4 + \ldots \tau_n) + \tau_2 \tau_3 (\tau_4 + \tau_5 + \ldots \tau_n) \]
\[ \ldots \]
\[ a_n = t_1 t_2 t_3 \ldots t_n \rightarrow \text{easy to see by expanding out factors. We ask under what conditions can solutions be easily approximated?} \]

We can use above for approximating roots if the roots are real and well seperated. Then only the first term of each equation is dominant.
\[ |t_1| > |t_2| > |t_3| > \ldots > |t_n| \]

This is a **big if** and the goal is to be clever in choosing the t’s in descending order when one does the factoring by hand of polynomials. Few cases are ideal.

If successful,
\[ a_1 = t_1 \rightarrow t_1 = a_1 \]
\[ a_2 = t_1 t_2 \rightarrow t_2 = a_2/a_1 \quad \} \text{Works for simple second order insight case} \]
\[ a_3 = t_1 t_2 t_3 \rightarrow t_3 = a_3/a_2 \]
\[ \ldots \]
\[ a_n = t_1 t_2 \ldots t_n \rightarrow t_n = a_n/a_{n-1} \]

In summary this assumes no inequalities are violated.

If \[ |a_1| > \left| \frac{a_2}{a_1} \right| > \left| \frac{a_3}{a_2} \right| \ldots \left| \frac{a_n}{a_{n-1}} \right| \]

the \[ P(s) = 1 + a_1 s + a_2 s^2 + \ldots + a_n s^n \]
P(s) must be in this standard form then the polynomial can be approximately factored by:

\[ P(s) = (1 + a_1 s) \left(1 + \frac{a_2}{a_1} s\right) \left(1 + \frac{a_3}{a_1} s\right) \]

\[ \downarrow \quad \downarrow \quad \downarrow \]

lowest f  second f  third highest f

pole       pole       pole

b. What if one inequality is not met

ideal: \[ P(s) = (1 + a_1 s)(1 + \frac{a_2}{a_1} s) \ldots (1 + \frac{a_n}{a_{n-1}} s) \]

\[ |a_1| \gg |\frac{a_2}{a_1}| \gg |\frac{a_3}{a_2}| \ldots \gg |\frac{a_n}{a_{n-1}}| \]

Non-ideal case where one inequality is not satisfied we leave the corresponding roots in quadratic form:

\[ \frac{a_k}{a_{k-1}} \gg \frac{a_{k+1}}{a_k} \]

\[ P(s) = (1 + a_1 s)(1 + \frac{a_2}{a_1} s) \ldots \left[1 + \frac{a_k}{a_{k-1}} s + \frac{a_{k+1}}{a_{k-1}} s^2\right] \]

\[ \bullet \ldots (1 + \frac{a_n}{a_{n-1}}) \]

The quadratic equation below

\[ \left[1 + \frac{a_k}{a_{k-1}} s + \frac{a_{k+1}}{a_{k-1}} s^2\right] \]

\[ \downarrow \quad \downarrow \]

we solve as a usual quadratic } Valid only for this pair of roots only \} \[ a_k > a_{k-1} \times a_{k+1} \]

When several inequalities are violated then we must use cubic or higher order roots and often the roots are close in value.
c. Cubic Polynomial: In general for a cubic polynomial this may occur two ways\[1 + a_1 s + a_2 s^2 + a_3 s^3\]

1. First and Second Close Roots Case

First guess \[(1 + a_1 s)(1 + \frac{a_2}{a_1} s)(1 + \frac{a_3}{a_2} s)\]

---X--------X-----------------------------X---------

s_1 s_2 s_3

close pair isolated root

\[(1 + a_1 s)(1 + \frac{a_2}{a_1} s)\] \[(1 + \frac{a_3}{a_2} s)\]

solution still valid

not valid approximation

Use: \[(1 + a_1 s + a_2 s^2)\]

\[s_3 = -\frac{a_2}{a_3}\]

to solve for unknown \n
\[s_1\] and \[s_2\] just

solve quadratic

quadratic validity

\[(1 + a_1 s)(1 + \frac{a_2}{a_1} s)\]

\[a_k > a_{k-1} a_{k+1}\]

\[1 + (a_1 + \frac{a_2}{a_1}) s + a_2 s^2\]

\[k = 1\]

\[a_1 > \frac{a_2}{a_1} - a_1^2 > a_0 a_2\]

\[a_1^2 > a_2\]

Summary:

\[1 + a_1 s + a_2 s^2\]

\[1 + \frac{a_3}{a_2} s\]

\[\downarrow\]

\[s_1\] and \[s_2\]

\[\downarrow\]

\[s_3\]

2. Second and Third Roots Close Case

---X-----------------------------X---------X--------

s_1 s_2 s_3
Original equation $1 + a_1 s + a_2 s^2 + a_3 s^3$

(1 + $a_1 s$) \quad (1 + \frac{a_2}{a_1} s)(1 + \frac{a_3}{a_2} s)

Isolated root \quad \text{Invalid First guess}
valid approximation \quad \text{Close roots}
\text{Not valid}

(1 + \frac{a_2}{a_1} s)(1 + \frac{a_3}{a_2} s)

Use: $1 + (\frac{a_2}{a_1}) s + (\frac{a_3}{a_1}) s^2$

Use this quadratic to solve for $s_2$

and $s_3$ only if

\[
\frac{a_3}{a_2} \ll \frac{a_2}{a_1}
\]

\[a_2^2 > a_1 a_3\]

Summary:

(1 + $a_1 s$) \quad 1 + \frac{a_2}{a_1} s + \frac{a_3}{a_1} s^2

\[\downarrow\]

\[s_1 = - \frac{1}{a_1}\]
yields $s_2$ and $s_3$

3. Cubic Polynomial Case for CuK

CuK converter with $D$, $D'$ left out for simplicity

\[G(s) = \frac{G_0}{1 + s(\frac{L_1 + L_2}{R}) + s^2 L_1 C + s^3 \frac{L_1 L_2 C}{R}}\]

\[\downarrow\quad \downarrow\quad \downarrow\]

\[a_1 \quad a_2 \quad a_3\]

How to factor $D(s)$ depends on circuit values. We want (but not always valid)

\[D(s) = (1 + a_1 s)(1 + \frac{a_2}{a_1} s)(1 + \frac{a_3}{a_2} s)\]
\[ T(s) = \left[ 1 + s\left(\frac{L_1 + L_2}{R}\right) \right] \left[ 1 + s\frac{RCL_1}{L_1L_2} \right] \left[ 1 + s\frac{L_2}{R} \right] \]

a. \( a_1 > \frac{a_2}{a_1} > \frac{a_3}{a_2} \) ideal case of well separated roots

If \( L_1 \) is chosen \( >> L_2 \) the approximate factorization is OK because:

\[ \frac{L_1}{R} > RC > \frac{L_2}{R} \]

for the ideal case of well separated roots

\[ \left[ 1 + s\frac{L_1}{R} \right] \left[ 1 + s RC \right] \left[ 1 + s\frac{L_2}{R} \right] \to \text{easy to get roots from circuit} \]

elements. Consider \( D(s) = (1 + a_1s) (1 + \frac{a_2}{a_1}s) (1 + \frac{a_3}{a_2}s) \)

Whereas if no approximations are made:

\[ \left[ 1 + s\left(\frac{L_1 + L_2}{R}\right) \right] \left[ 1 + s\frac{RCL_1}{L_1 + L_2} \right] \left[ 1 + s\frac{L_2}{R} \right] \]

What if you find the second inequality is violated \( a_1 \gg \frac{a_2}{a_1} \gg \frac{a_3}{a_2} \) due to component choices!

\[ \frac{L_1 + L_2}{R} \gg RC \frac{L_1}{L_1 + L_2} \text{ but not } \frac{L_2}{R} \]

This will require a different approach.

b. Second and Third Terms Roots Close

Then for easy root solving we break \( T(s) \) into two terms: one with a single real root and the second with two possibly complex roots.

Still valid Use this Quadratic

\[ \left[ 1 + s\left(\frac{L_1 + L_2}{R}\right) \right] \left[ 1 + sRC\frac{L_1}{L_1L_2} + s^2C^*L_1||L_2 \right] \]
Again this is valid only if both $L_1 >> L_2$ and $L_1/R > RC$

\[
\left[ 1 + s \frac{L_1}{R} \right] \left[ 1 + sRC + s^2CL_1 \right]
\]

↓

If also we find low Q for the quadratic then we get two well separated roots from the quadratic term as well

c. First and Second Terms Have Close Roots
This is a case when the first inequality is violated.

\[
\frac{L_1 + L_2}{R} \gg \frac{RCL_1}{L_1 + L_2} \gg \frac{L_2}{R}
\]

The inequalities are the same as $L_1 > L_2$ and $RC > L_2/R$. It is no longer required that $L_1/R >> RC$

Return to Quadratic \((1 + s \frac{L_2}{R})\) form for first and second roots $s_1$ and $s_2$. We still have a valid approx. for $s_3 = -a_2/a_3 = L_2/R$ and the quadratic is simply

\[
(1 + s(\frac{L_1 + L_2}{R}) + s^2L_1C) = (1 + sL_1/R + s^2L_1C)
\]

↓

Solve this quadratic for $s_1$ and $s_2$

4. Fourth Order Polynomial from a two section L-C Filter
Erickson Pbm. 8.19.

a. Two Section L-C Filter
A two-section L-C filter has the following transfer function:

\[
G(s) = \frac{1}{1 + s\left(\frac{L_1 + L_2}{R}\right) + s^2(L_1(C_1 + C_2)) + s^3\left(\frac{L_1L_2C_1}{R}\right) + s_4(I_1L_2C_1C_2)}
\]

The element values are:

- $R$ 50 mW
- $C_1$ 680 mF
- $C_2$ 4.7 mF
\[ G(s) = \frac{\frac{L_1}{L_1 + L_2} + s^2[4(C_1 + C_2) + L_2 C_2] + \frac{L_1 L_2 C_1}{R} + s^4(L_1 L_2 C_1 C_2)}{1 + s \left( \frac{L_1 + L_2}{R} \right) + s^2[4(C_1 + C_2) + L_2 C_2] + \frac{L_1 L_2 C_1}{R} + s^4(L_1 L_2 C_1 C_2)} \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\[ a_1 \quad a_2 \quad a_3 \quad a_4 \]

\[ R = 50 \text{ mW} \quad L_1 = 500 \text{ mH} \]
\[ C_1 = 680 \text{ mF} \quad L_2 = 50 \text{ mH} \]
\[ C_2 = 4.7 \text{ mF} \]

(a) Factor \( G(s) \) into approx. real and quadratic poles.
Compute \( a_1, a_2, a_3, a_4 \) and \( t_1, t_2, t_3, t_4 \)

\[ a_1 = 0.011 \quad t_1 = a_1 = 0.011 \]

\[ a_2 = 342.6 \times 10^{-9} \quad t_2 = \frac{a_2}{a_1} = 31.15 \times 10^{-6} \text{ Non ideal sequence} \]

\[ a_3 = 340 \times 10^{-12} \quad t_3 = \frac{a_3}{a_2} = 991.4 \times 10^{-6} \text{ we must use a quadratic to solve } s_2, s_3 \]

\[ a_4 = 80 \times 10^{-18} \quad t_4 = \frac{a_4}{a_3} = 235.3 \times 10^{-9} \]

Check: \( a_1 >> \frac{a_2}{a_1} \) but \( \frac{a_2}{a_1} \) is not >> \( \frac{a_3}{a_2} \)

\[ a_1 >> \frac{a_2}{a_1} >> \frac{a_1 a_4}{a_2^2} \] >> \( \frac{a_4}{a_3} \) Valid to break out one quadratic

\[ (=) \quad 0.011 \quad 31.15 \times 10^{-6} \quad 2.81 \times 10^{-6} \quad 235.3 \times 10^{-9} \]

\[ \Rightarrow G(s) = \frac{1}{(1 + a_1 s)(1 + \frac{a_2}{a_1} s + \frac{a_3}{a_1} s^2)(1 + \frac{a_4}{a_3} s)} \]
$\approx \frac{10}{(1 + \frac{s}{w_0})(1 + \frac{s}{Qw_1} + \frac{s^2}{w_1^2}) + (1 + \frac{s}{w_2})}$

Compare coefficients, we get

\[
\begin{align*}
 w_0 &= \frac{1}{0.011} \quad \Rightarrow \quad f_o = 90.9 \text{ rad/s} \\
 w_1 &= \sqrt{a_1} \quad \Rightarrow \quad f_1 = 5687.96 \text{ rad/s} \\
 Qw_1 &= \frac{a_1}{a_2} \quad \Rightarrow \quad Q = 5.65 \text{ db} \\
 w_2 &= \frac{a_3}{a_4} \quad \Rightarrow \quad f_2 = 4255319 \text{ rad/s}
\end{align*}
\]

Justification:

Actual exp. \[ G(s) = \frac{1}{a_0 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4} \]

Approx. exp. \[ G(s) = \frac{1}{1 + (\frac{a_2}{a_1} + a_1 + \frac{a_4}{a_3}) s + (a_3 + a_2 + \frac{a_2 a_4}{a_1 a_3}) s^2 + (a_3 + \frac{a_4}{a_3} + \frac{a_4 a_2}{a_4 s^2})} \]

Numerical values:

<table>
<thead>
<tr>
<th>Pole Finder</th>
<th>Actual</th>
<th>Factoring Approx.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_0$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$A_1$</td>
<td>$1.1 \times 10^{-2}$</td>
<td>$1.103 \times 10^{-2}$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$3.43 \times 10^{-7}$</td>
<td>$3.76 \times 10^{-7}$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$3.4 \times 10^{-10}$</td>
<td>$3.4008 \times 10^{-10}$</td>
</tr>
<tr>
<td>$A_4$</td>
<td>$7.99 \times 10^{-17}$</td>
<td>$7.99 \times 10^{-17}$</td>
</tr>
</tbody>
</table>
We wish to reduce $Q$ from 5.65 db to 2 db without changing break frequencies. Is this possible?

(c) Now $Q = 2$

From 1, 2 $\Rightarrow \frac{a_2}{a_1} = \frac{1}{Qw_1}$, $\frac{a_3}{a_1} = \frac{1}{w_1^2} - w_1 = \sqrt{\frac{a_1}{a_3}}$

$$\Rightarrow Q \sqrt{\frac{a_1}{a_3}} = \frac{a_1}{a_2} = \sqrt{\frac{a_1a_3}{a_2}}$$

but $C_1 >> C_2 \Rightarrow$ 

$L_1 >> L_2 \Rightarrow$ approx. $a_1 \approx \frac{L_1}{R}$

$a_2 \approx L_1C_1$

$a_3 \approx \frac{L_1L_2C_1}{R}$

$a_4 \approx L_1L_2C_1C_2$
Then \[ Q = \frac{\sqrt{L_1 L_1 L_2 C_1}}{R} \frac{R}{L_1 C_1} = \frac{1}{R} \sqrt{\frac{L_2}{C_1}} \]

\[ w_0 = \frac{1}{a_1} = \frac{R}{L_1} \]

\[ w_1 = \sqrt{\frac{a_1}{a_3}} = \frac{1}{\sqrt{L_2 C_1}} \]

\[ w_2 = \frac{a_3}{a_4} = \frac{1}{RC_2} \]

So \( Q \) depends only on \( R, L_2, C_1 \). Let’s change \( L_2 \) and \( C_1 \).

Can’t change \( R \) as it changes \( w_0 \).

\( R = 50 \text{ mW}, \quad Q = 2 \)

\[ \Rightarrow L_2 = 0.01 \text{ C}_1 \quad \text{and} \quad C_1 = \frac{10}{w_1} = \frac{10}{5688} = 1.76 \text{ mF} \]

\[ L_2 = 0.01 \text{ C}_1 = 17.581 \text{ mH} \]

Check:

\[ w_1 = \sqrt{\frac{a_1}{a_3}} = \frac{\sqrt{L_1 + L_2}}{\sqrt{L_1 L_2 C_1}} = 5687.96 \text{ rad/s} \]

\[ \Rightarrow f = 905.3 \text{ Hz} \Rightarrow \text{same} \]

**B. Buck-Boost \( T_{vd}(s) \) and \( T_{vg}(s) \) Analysis**

This is the crucial material for this lecture as part A can be replaced by using a root solver on a basic calculator.

1. Equivalent ac Small Signal model

The Buck-Boost small signal circuit model was developed earlier.
The buck-boost small signal equations follow.

Small-signal ac equations of the buck-boost converter, derived in section 7.2:

\[
L \frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) + D\hat{v}(t) + \left[V_g - V\right] \hat{d}(t)
\]

\[
C \frac{d\hat{v}(t)}{dt} = -D\hat{i}(t) - \frac{\hat{v}(t)}{R} + i(t)
\]

\[
\hat{i}(t) = D\hat{i}(t) + i(t)
\]

We aim to develop these equations to yield two separate transfer functions as described below \(G_{v_g}\) and \(G_{v_d}\):

The converter contains two inputs, \(\hat{d}(s)\) and \(\hat{v}_g(s)\), and one output, \(\hat{v}(s)\).

Hence, the ac output voltage variations can be expressed as the superposition of terms arising from the two inputs:

\[
\hat{v}(s) = G_{v_d}(s) \hat{d}(s) + G_{v_g}(s) \hat{v}_g(s)
\]

The control-to-output and line-to-output transfer functions can be defined as

\[
G_{v_d}(s) = \frac{\hat{v}(s)}{\hat{d}(s)} \bigg|_{\hat{v}_g(s) = 0} \quad \text{and} \quad G_{v_g}(s) = \frac{\hat{v}(s)}{\hat{v}_g(s)} \bigg|_{\hat{d}(s) = 0}
\]

This is our goal to extract these two functions for the buck-boost from the equivalent circuit. Setting one input to zero will result in great simplifications to the loop equations of the full circuit.

2. Loop Equations:

\[\hat{i}_g(t) = D\hat{i}(t) + \hat{i}(t) \quad \text{(for the first circuit loop on left above)}\]

\[L \frac{d\hat{i}}{dt} = Dv_g(t) + D'v(t) + (v_g - v) \hat{d} \quad \text{(middle circuit loop)}\]
\[ C \frac{dv}{dt} = -D\hat{i} - \frac{\hat{v}(t)}{R} + I\hat{d}(t) \quad \text{(output loop)} \]

Next do the Laplace transform for the small signal output voltage.

\[ \hat{v}(s) \equiv G_{vd}(s)\hat{d}(s) + G_{vg}(s)\hat{v}_g(s) \]

\[ \downarrow \quad \downarrow \]

\[ \begin{align*}
\hat{v}(s) \\
\hat{d}(s)
\end{align*} \begin{align*}
\hat{v}_g(s) = 0 \\
\hat{v}_g(s) = 0
\end{align*} \]

Simple but tedious algebra leads to:

\[ sC\hat{v}(s) = -\frac{D'}{sL} \left( D\hat{v}_g(s) + D'\hat{v}(s) + \left( V_g - V \right)\hat{d}(s) \right) - \frac{\hat{v}(s)}{R} + I\hat{d}(s) \]

\[ \hat{v}(s) = \frac{-DD'}{D'' + s \frac{L}{R} + s^2 LC} \hat{v}_g(s) - \frac{V_g - V - sLI}{D'' + s \frac{L}{R} + s^2 LC} \hat{d}(s) \]

Write in normalized form:

\[ \hat{v}(s) = \left( \frac{-D'}{D'} \right) \frac{1}{1 + s \frac{L}{D'' R} + s^2 \frac{LC}{D''}} \hat{v}_g(s) - \left( \frac{V_g - V}{D''} \right) \frac{1 - s \frac{LI}{V_g - V}}{1 + s \frac{L}{D'' R} + s^2 \frac{LC}{D''}} \hat{d}(s) \]

Or in the form, that we will use later, after putting DC analysis in.

\[ \hat{v}(s) = -\frac{V_g - V - sLI}{D'' + s \frac{L}{R} + s^2 LC} \hat{d}(s) - \frac{DD' \hat{v}_g(s)}{D'' + s \frac{L}{R} + s^2 LC} \]

\[ \downarrow \quad \downarrow \]

\[ \begin{align*}
\text{yields } T_{vd}(s) \\
\text{yields } T_{vg}(s)
\end{align*} \]

\[ T_{vd}(s) = \frac{\hat{v}(s)}{\hat{d}(s)} \quad T_{vg}(s) = \frac{\hat{v}(s)}{\hat{v}_g(s)} \]
3. Line to Output Transfer Function

Hence, the line-to-output transfer function is

\[ G_v(s) = \left. \frac{\tilde{v}(s)}{\tilde{v}_e(s)} \right|_{\alpha = 0} = \left( -\frac{D}{D'} \right) \frac{1}{1 + s \frac{L}{D'^2 R} + s^2 \frac{LC}{D'^2}} \]

which is of the following standard form:

\[ G_v(s) = G_{v_0} \frac{1}{1 + \frac{s}{Q_w} + \left( \frac{s}{\omega_0} \right)^2} \]

In standard form, such that coefficients of \( s^0 \) are unity, \( \Rightarrow \) The line (mains) to output transfer function is:

\[ T_vg(s) = \left( -\frac{D}{D'} \right) \frac{1}{1 + s \frac{L}{D'^2 R} + s^2 \frac{LC}{D'^2}} \]

Pole location in the dynamic transfer function depend on the choice of DC operating point, \( D \).

\[ L_{\text{eff}}(D) \equiv \frac{L}{(D')^2} = f(D, L, C, R). \]

In standard quadratic pole pair form:

\[ T_vg(s) = \frac{T_{go} \text{ (constant low f gain)}}{1 + \frac{s}{Q_w} + \left( \frac{s}{\omega_0} \right)^2} \]

with Q in db units

Identify like terms

\[ T_{go} = -\frac{D}{D'} \quad \text{Buck-boost Gain @ dc} \]

\[ \omega_0 = \frac{1}{\sqrt{L_{\text{eff}}C}} = \frac{D'}{\sqrt{LC}} \quad \text{The dynamic transfer function rollover frequency depends on the choice of DC operating point f(D')} \]

\[ \frac{1}{Q_w} = \frac{L_{\text{eff}}}{R} \]

\[ Q = D'R \sqrt{\frac{C}{L}} \quad \text{Note also that the second order Q also depends on} \]
the DC operating point chosen for the converter Q = f(D')

4. **Control to Output Transfer Function**: 

\[ T_{vd}(s) = \frac{\hat{v}(s)}{\hat{d}(s)} \bigg| \hat{v}_i(s) = 0 \]

\[ T_{vd}(s) = f(D', V_g, V, I, L, C, R) \]

All DC voltages/currents as well as D and component values of L and C effect the dynamic response.

In standard form such that \( s^0 \) coefficients are unity:

\[ T_{vd}(s) = \begin{bmatrix} -\frac{V_g - V}{D^2} \end{bmatrix} \frac{1 - s}{V_g - V} \frac{LI}{1 + s\frac{L_{\text{eff}}}{R} + s^2 L_{\text{eff}}C} \]

or in the standard form below,

\[ G_{vd}(s) = G_{ah} \left( 1 - \frac{s}{\omega_c} \right) \left( 1 - s \frac{L}{V_c - V} \right) \left( 1 + s \frac{L}{D^2 R} + s^2 \frac{L_{\text{eff}}C}{D^2} \right) \]

Standard form:

\[ G_{vd}(s) = G_{ah} \left( 1 + \frac{s}{Q \omega_0} + \left( \frac{s}{\omega_0} \right)^2 \right) \]

Poles of \( T_{vd}(s) \) are the same as poles of \( T_{vg}(s) \), but the zero is unique. **It is a right half plane zero.** We effect \( w_z \) location by choice of all DC values for \( V_g, V, I \) etc. \( w_z(\text{RHP ZERO}) = D'R/DL \) we will show below. Compare to standard quadratic form:

\[ \frac{T_{do}(\text{constant low f gain})[1 - s/ w_z]}{1 + \frac{s}{Q \omega_0} + \left( \frac{s}{\omega_0} \right)^2} = T_{VD}(s) \]

It’s easy to design the DC gain for Buck-Boost given \( V(\text{out}) \) and
the duty cycle.

\[ T_{dc}(dc) = \frac{V_g - V}{D^2} = -\frac{V_g}{D^3} = \frac{V}{DD'} \]

Recall for Buck-Boost DC conditions: \( \frac{V}{V_g} = -\frac{D}{D'} \) and \( I = -\frac{V}{R} \cdot \frac{1}{D'} \)

Be sure to note that \( I(DC) \) is in inner loop of the buck-boost circuit model on page 12. Then we find from the DC conditions the zero location:

\[ w_z = \frac{V_g - V}{LI} = \frac{D'}{D} \cdot \frac{R}{L} \quad \text{right-half plane} \]

\[ G_m = -\frac{V_g - V}{D^3} = -\frac{V_g}{D^3} = \frac{V}{DD'} \]

\[ \omega_c = \frac{V_g - V}{LI} = \frac{D'}{D} \cdot \frac{R}{L} \quad \text{(RHP)} \]

\[ \omega_n = \frac{D'}{\sqrt{LC}} \]

\[ Q = D'R \sqrt{\frac{C}{L}} \]

— Simplified using the dc relations: \( V = -\frac{D}{D'} V_g \)

\( I = -\frac{V}{D'R} \)

Next we try to get more specific and quantitative.

5. Specific Buck-Boost Design

\[
\begin{align*}
D &= 0.6 \quad R &= 10 \\
V_g &= 30V \quad L &= 160mH \\
& \quad C &= 160mF \\
\end{align*}
\]

\( f_{sw} = \)

a. We know DC operating conditions

\[ V = -\frac{0.6}{0.4} \cdot 30 = -45V \]
\[ I = \frac{-45}{10} \cdot 10 \cdot 0.4 = 11.25 \text{A} \quad \text{in middle loop of model, on page 12.} \]

We also find \( I_{\text{out}} = -4.5 \text{A} \) in the outer loop of the page 12 circuit.

Recall that \( I(\text{middle}) = I_{\text{out}} + I_{\text{in}} ? \) for CCM conditions.

**b. AC values in std. transfer functions** (using db for plots) for these DC conditions.

First a \( T_{\text{VG}}(s) \) having only poles is considered.

\[ G_{\text{go}} = \frac{D}{D'} = 1.5 \text{ or } 3.5 \text{ in db units} \]

\[ f_0 = \frac{w_0}{2\pi} = \frac{D'}{2\pi \sqrt{LC}} = 400 \text{ Hz} \]

\[ Q = D' R \sqrt{\frac{C}{L}} = 4 \text{ in normal units or } 12 \text{ in db units} \]

\[ T_{\text{vg}}(s) = -\frac{D}{D'} = 1.5 = 3.5\text{db} \]

\[ 1 + \frac{s}{(12)(2\pi f_0)} + \left( \frac{s}{2\pi f_0} \right)^2 \]

\[ \downarrow \quad \downarrow \quad Q = 400 \text{ Hz} \quad \text{low resonance} \]

\[ Q = D' R \sqrt{\frac{C}{L}} = 4 = 12\text{db} \]

Now for a \( T_{\text{VD}}(s) \) having a right hand plane zero we have a new set of values.

\[ G_{\text{do}} = \frac{V}{D(D')^2} = 469 \text{ or } 53.4 \text{ db} \]

\[ f_z = \frac{D'R}{2\pi DL} = 6.6 \text{ kHz} \]

\[ T_{\text{vd}}(s) = G_{\text{do}} \left( 1 - \frac{s}{w_z} \right) \]

\[ 1 + \frac{s}{(12)(2\pi f_0)} + \left( \frac{s}{2\pi f_0} \right)^2 \]

\[ \downarrow \quad \downarrow \]
Q in db 400 Hz

In summary:

Suppose we are given the following numerical values:

\[ D = 0.6, \quad R = 10\Omega, \quad V_o = 30V, \quad L = 160\mu H, \quad C = 160\mu F \]

Then the salient features have the following numerical values:

\[ |G_{so}| = \frac{D}{D} = 1.5 \Rightarrow 3.5\text{dB} \]
\[ |G_{so}| = \left| \frac{|V|}{D D^2} \right| = 469V \Rightarrow 53.4\text{dBV} \]
\[ f_0 = \frac{\omega_0}{2\pi} = \frac{D}{2\pi\sqrt{LC}} = 400\text{Hz} \]
\[ Q = D^2 R \sqrt{\frac{C}{L}} = 4 \Rightarrow 12\text{dB} \]
\[ f_c = \frac{\omega_0}{2\pi} = \frac{D R}{2\pi DL} = 6.6\text{kHz} \]

If L goes to small values (DCM) what happens to the zero location? In \( G_{vd}(f) \) Phase Plots we will find:

Two poles cover 180° over a frequency range, where now Q is in real units and not in db

\[ f_{\text{lower}} = 10^{-\frac{1}{2Q}} \quad f_0 = 10^{-\frac{1}{8}}(400) = 300\text{Hz} \]
\[ f_{\text{upper}} = 10^{\frac{1}{2Q}} \quad f_0 = 10^{\frac{1}{8}}400 = 533\text{Hz} \]

In \( G_{vd}(f) \) Gain Plots we will find: We have a double pole at 400 Hz showing a resonant “Q peak”.

The RHP zero will add amplitude above \( f_z = 6.6 \text{kHz} \) but will cause additional phase lag threatening stability if we use feedback.
Below are plotted $|G_{vd}|$ and $\angle G_{vd}$

Note carefully the unity gain line at “0” db.

RH plane zero starts to contribute phase angle at 660 Hz and LIMITS phase margin at unity gain. Additional phase contribution starts at $f_z/10$ @ 660 Hz.

The $G_{vg}$ amplitude and phase is much simpler as shown below.

Here the f margin at unity gain is also small but not as bad as $T_{vd}$. 

C. Right Half Plane Zero Issues and Indicators of This

\[ \langle i_D \rangle_{T_s} = \frac{d}{dt} \langle i_L \rangle_{T_s} \]

The above two switch mode circuits have right-half planes zeros.

1. Transient Response Indicators of a RHP zero

   a. Typical of 2nd order poles

   \[ V_G \]
   \[ t \]

   \[ V_{o(\text{final})} \]
   \[ V_{o(\text{initial})} \]

   versus

   b. Typical of (RH Plane/zero)

   \[ d_1 \]
   \[ d_2 \]

   \[ V_o \]
   \[ V_{o(\text{final})} \]
   \[ V_{o(\text{initial})} \]

This RH plane zero occurs for several PWM dc-dc converters:
   - boost
   - buck-boost
   - flyback

It does not occur for buck converter. WHY?
2. Origin of RH Plane Zero

Differing time delays for D’ which changes \( V_o \) immediately & \( i_m \) which changes \( V_o \) only slowly due to switching over several switch cycles, causing initial \( V_o \) to fall for all converters with output driven by \( i_m \). We have therefore the case that: (\( i_m \)) stays the same over short \( \Delta t \) time periods but D and D’ change fast. D changes instantaneously by the control circuit knob, this instantaneously lower D’ and thus \( V_o \) goes down initially and only later increases when the effect of bigger D makes it to the output.

* \( i_{Lm} \) cannot change instantaneously
* time interval \( d' = 1-d \) can change instantaneously which is the duration \( i_m \) is delivered to the output. This causes a phase lag and momentary drop in \( V_o \) followed by subsequent increase in \( V_o \) sudden change in \( d \) that builds up \( i_m \) slowly.

\( i_L \) varies slowly \( i_D \) duration changes instantaneously but \( i_{Lm} \) does not \( \Rightarrow V_o \downarrow \) then \( \uparrow \)
3. T(s) of a single right–half plane zero insight

\[
\frac{V_o}{V_d} \sim (1 - \frac{s}{w_z})
\]

A RHP zero acts as two parallel paths to \( V_o \)

\[
\begin{align*}
\text{in}(s) & \quad \rightarrow \quad 1 \\
\text{s/wz} & \quad \rightarrow \quad \text{out}(s)
\end{align*}
\]

Overall transient response:

4. Phase Margin Limitations of Any Closed Loop with RHP Zero Present

\[
A_{CL} = \frac{A_{oL}}{1 + A \beta}
\]

Want \((A\beta) = 1\)

Then \(< A\beta \) far from 180°

How far? 76° away (for conservative designer), 60° for average designer and 10° away for poor designer!

Flyback converter bode plots:

Oddity is gain (low f) is non-linear w.r.t. \( V_d \)

f range with -40db/decade drop depends on low f gain. Not enough \( \phi \) margin come close to instability at unity gain \( \phi \rightarrow 180° \)
Note the unity gain location at “0” db. Flyback operating DCM does not have right half plane zero! Without LH plane zero above the phase margin would be nil. Moreover, RH plane zero returns phase to $180^\circ$. 