Thevenin theorem

Any linear circuit can be replaced by an equivalent voltage source with a resistor in series.
To find the $V_{th}$ and $R_{th}$

a) $V_{th}$: is the voltage at open circuit
b) $R_{th}$: Input equivalent resistance at the terminals with the independent sources turned off

Remember: to turn off a source \( V \): short circuit \( I \): open circuit

**Example**

Find $I$ through $R_L$ if 

\[
\begin{align*}
R_L &= 6 \text{ k}\Omega \\
R_L &= 16 \text{ k}\Omega \\
R_L &= 36 \text{ k}\Omega
\end{align*}
\]

a) Find $R_{th}$: We turn off the sources
b) Find $V_{th}$

\[
\begin{align*}
\{ & i_2 = -2A \\
& -32 + 4i_1 + 12(i_1 - i_2) = 0
\end{align*}
\]

\[
\begin{align*}
& -32 + 16i_1 + 24 = 0 \rightarrow i_1 = 0.5A \\
& V_{th} = (i_1 - i_2) \times 12 \Omega = 2.5A \times 12 \Omega = 30V
\end{align*}
\]

We can use nodal analysis.

Upper node

\[
\frac{32 - V_{th}}{4} - \frac{V_{th}}{12} + 2 = 0
\]

\[
\frac{32}{4} - \frac{V_{th}}{4} - \frac{V_{th}}{12} + 2 = 0 \rightarrow V_{th} = 30V.
\]
Thus the circuit is equivalent to

\[ I_L = \frac{V_{th}}{R_{th} + R_L} \]

Now we have a variable load

\[ I_L = \frac{30 \, V}{4 + R_L} \]

\( R_L = 6 \quad I_L = 3 \, A \)
\( R_L = 16 \quad I_L = 1.5 \, A \)
\( R_L = 36 \quad I_L = 0.75 \, A \)

**Norton's Theorem**

\[ R_N = R_{th} \quad I_N = I_{SC} \]

Related due to Thevenin's Theorem, \( I_N = \frac{V_{th}}{R_{th}} \).

**Example**
\( R_N : \)
\[
\frac{6}{(3+3)} = 3 \, \Omega
\]

\( I_N \)
\[
-15 + 3i_1 + 3i_2 = 0 \quad \text{Supernmesh}
\]
\[
i_1 + i_1 - i_2 = 0 \quad \text{Node}
\]

Solving: \( i_1 = 0.5 \, \text{A} \quad i_2 = 4.5 \, \text{A} \)

\( I_N = 4.5 \, \text{A} \). Thus the circuit is equivalent to

\[
4.5 \, \text{A} \uparrow \quad \frac{3 \, \Omega}{\downarrow}
\]
Relation between Thevenin and Norton's:

For a linear circuit:

* Find the open circuit voltage
  \[ \rightarrow V_{oc} = V_{th} \]

* Find the short circuit current
  \[ \rightarrow I_{sc} = I_N \]

* Find the equivalent resistance (input resistance) \( R_{in} \)
  \[ \rightarrow R_{in} = R_{th} = R_N = \frac{V_{oc}}{I_{sc}} \]

Example: Find the Norton's equivalent circuit looking from a, b.

1) \( R_N \): turn off the source and find the Ref

\[
R_N = \frac{(8+4+8)}{5} \]

\[= \frac{20}{5} = 4 \]
2) Find $I_N$, calculate the short circuit current.

This circuit has 2 meshes (notice that the 5Ω resistor is short circuited, thus can be neglected).

In the 2 meshes the equations are

\[
\begin{align*}
    i_1 &= 2\,\text{A} \\
    -12 + 4(i_2 - i_1) + 8i_2 + 8i_2 &= 0
\end{align*}
\]

Solving the system $i_1 = 2\,\text{A}$, $i_2 = 1\,\text{A}$.

The short circuit current $I_N = i_2 = 1\,\text{A}$.

Thus the circuit is equivalent to

Let us solve the same circuit finding the Thévenin's equivalent.

The equations for the 2 meshes are

\[
\begin{align*}
    i_3 &= 2\,\text{A} \\
    25i_4 - 4i_3 - 12 &= 0
\end{align*}
\]
Solving we obtain $i_3 = 2A$

$i_4 = 0.8A$

Thus, $V_{th} = i_4 \times 5 \Omega = 0.8A \times 5 \Omega = 4V$

The circuit is equivalent to

![Circuit Diagram]

Notice that $I_N = \frac{V_{th}}{R_{th}} = \frac{4V}{4\Omega} = 1A$

The Norton and Thévenin’s equivalent circuits are related by the source transformation relationship

**Thévenin and Norton with Dependent Sources**

When we have dependent sources, we cannot “turn off” because the value of the source depends on a magnitude (voltage) somewhere else in the circuit which cannot be arbitrarily set to zero. Thus in this case to find the $R_{th}$ or $R_N$ we need to follow a special procedure which is:

1. Connect in the circuit an arbitrary current (or voltage) source
2. Solve for the voltage (or current)
3. The value of $R_N = R_{th} = V/I$
Example

\[ \begin{align*}
6 & \quad \text{10A} \\
\downarrow & -2V_x \\
\uparrow & \quad V_x \\
\downarrow & 2V_x \\
\uparrow & \quad V_x \\
\downarrow & I_0 \\
\downarrow & 1V
\end{align*} \]

(a) We set the independent sources to zero and connect a 1V voltage source in the output.

\[ \begin{align*}
6i_1 + 2V_x + 2i_1 - 2i_2 &= 0 \\
2i_2 - 2i_1 + 1 &= 0 \\
V_x &= 1
\end{align*} \]

Solving

\[ \begin{align*}
i_1 &= -0.5 \\
i_2 &= -1
\end{align*} \]

\[ I_0 = -i_2 = 1A, \quad R_N = R_{th} = \frac{V_x}{I_0} = \frac{1V}{1A} = 1\Omega \]

To find \( I_N \) we short circuit the output.

\[ \begin{align*}
6 & \quad \text{10A} \\
\downarrow & -2V_x \\
\uparrow & \quad V_x \\
\downarrow & 2V_x \\
\uparrow & \quad V_x \\
\downarrow & I_N
\end{align*} \]
When we short circuit the output, we are setting $V_x = 0$.

Thus the circuit is:

$G$ → $9$ $10A$ ↓ $I_N = 10A$

Finally, the circuit is equivalent to:

$\uparrow$ $10$ $\frac{1}{1\Omega}$
Maximum Power Transfer.

When we design a circuit, usually we would like to set the conditions to obtain a maximum power transfer \( P \) between our circuit and our load. This is said that the circuit and the load are "adapted". For example, this is important in communications, or in a audio system.

We can consider any circuit as a Thévenin equivalent.

\[
\begin{align*}
V_{\text{th}} & \quad \pm \quad \text{M} \quad R_{\text{th}} \quad \text{RL}
\end{align*}
\]

Power: 
\[
P = V \cdot I = I^2 \cdot R = \left( \frac{V_{\text{th}}}{R_{\text{th}} + R_L} \right)^2 \cdot R_L
\]

If we plot this function we obtain

\[ P \]

\[ R_{\text{th}} \quad R_L \]
To prove that we take the last equation, differentiate and equate to zero to find the maximum value.

\[ P = V_{th}^2 \frac{R_L}{(R_{th} + R_L)^2} \]

\[ \frac{dP}{dR_L} = V_{th}^2 \left[ \frac{(R_{th} + R_L)^2 - 2(R_{th} + R_L)R_L}{(R_{th} + R_L)^4} \right] \]

\[ = V_{th}^2 \left[ \frac{(R_{th} + R_L) - 2R_L}{(R_{th} + R_L)^3} \right] = 0 \]

\[ \Rightarrow R_{th} + R_L - 2R_L = 0 \Rightarrow R_L = R_{th} \]

The maximum power transfer occurs for \( R_L = R_{th} \).

In this case (for max power transfer)

\[ P_{\text{max}} = \frac{V_{th}^2}{4R_{th}} \]

Otherwise

\[ P = V_{th}^2 \frac{R_L}{(R_{th} + R_L)^2} \]
Source Modeling

Ideal Source
\[ V \rightarrow \text{cte Voltage} \]
\[ I \rightarrow \text{cte current} \]
\[ \Rightarrow \text{independent of } R_L \]

Real Sources

\[ V \] has a small series resistance \( R_s \)
\[ I \] has a large parallel resistance \( R_p \)

These are called "inherent resistances."

For good sources
\[ R_s \rightarrow 0 \]
\[ R_p \rightarrow \infty \]

Let us see what happens when a load is connected to a source

\[ V_L = V_s \cdot \frac{R_L}{R_L + R_s} \]

\[ V_L \] vs. \[ R_L \] plot
Conclusion: When $R_S \ll R_L$, the load voltage $V_L$ is close to $V_S$.

When disconnected ($R_L = \infty$) → $V_{OC} = V_S$

The load produces a drop in the output voltage of the source.

Current Source

\[ i_L = i_S \cdot \frac{R_P}{R_P + R_L} \]

We want $R_P >> R_L$.

How can we measure $R_S$ or $R_P$?

1) \[ \text{Source} \quad V_{OC} = V_S \]

2) \[ \frac{1}{2} R_L \rightarrow \text{change } R_L \text{ till } V_L = \frac{V_{OC}}{2} \]
   \[ \Rightarrow R_L = R_S \]
Example

The following result is obtained from measurements taken between the test terminals of a resistive network.

1) Measurement: Terminal voltage = 120 V - Current 0
2) Measurement: Terminal voltage = 0 V - Current = 4 A

What is the Thévenin equivalent of the network?

Solution: Measurement 1 says that the "open circuit" voltage is 120 V (because I = 0). Thus \( V_{th} = 120 \text{V} \).

Measurement 2 says! The "short circuit" current is 4 A (because \( V = 0 \)). Thus \( I_N = 4 \text{A} \).

We know \( R_{th} = R_N = \frac{V_{th}}{I_N} = \frac{120 \text{V}}{4 \text{A}} = 30 \Omega \).

The equivalent circuits are

\[ \begin{align*}
&\text{30} \Omega \\
&120 \text{V} \\
\end{align*} \text{ or } \begin{align*}
&4 \text{A} \\
&3 \text{30} \Omega \\
\end{align*} \]

Example: The Norton equivalent at terminals A-B of a linear network is to be determined by measurement. When 10 \( \Omega \) is connected to A-B the measured voltage is \( V_{AB} = 12 \text{V} \). When 30 \( \Omega \) is connected the measured voltage is 24 V. What is the Norton equivalent circuit?

We know

\[ \begin{align*}
&V = 12 \text{V} \\
&\frac{3}{10} \Omega \\
\end{align*} \text{ AND } \begin{align*}
&V = 24 \text{V} \\
&\frac{3}{30} \Omega \\
\end{align*} \]
The linear circuit can be replaced by its Norton's equivalent.

Thus if the load is $R_L$ we will have

$$V_0 = R_L \cdot I_L = R_L \cdot I_N \left( \frac{R_N}{R_N + R_L} \right)$$

We have 2 data points:

$$\begin{align*}
V_1 &= 12 \text{ V}, \quad I_N \left[ \frac{R_N}{R_N + 10R} \right] \\
V_2 &= 24 \text{ V}, \quad I_N \left[ \frac{R_N}{R_N + 30R} \right]
\end{align*}$$

$$\begin{align*}
I_N &= \frac{12}{10} \cdot \frac{R_N + 10}{R_N} \\
24 &= 30 \cdot \frac{12}{10} \cdot \frac{R_N + 10}{R_N} \cdot \frac{R_N}{R_N + 30}
\end{align*}$$

$$\begin{align*}
24 \left( R_N + 30 \right) &= 36 \left( R_N + 10 \right) \quad \Rightarrow \quad R_N = 30 R \\
I_N &= \frac{12}{10} \cdot \frac{(30+10)}{30} = 1.6 \text{ A}
\end{align*}$$

Thus the equivalent circuit is
Example: An 8 V source is to be used with 2 standard resistors to design a voltage divider with an output of 5 V. When connected to a 100 μA load, while keeping the consumed power as low as possible, we wish to minimize the error between the actual output and the required 5 V.

In the node we have this equation \[ \frac{8-V_o}{R_1} = 100 \mu A + \frac{V_o}{R_2} \]

Now we replace \( V_o = 5 \text{ V} \) and the equation is:

\[ \frac{3}{R_1} = 100 \mu A + \frac{5}{R_2} \quad \Rightarrow \quad R_1 = \frac{3R_2}{5 + 100 \mu A \cdot R_2} \]

This is an equation that gives the relation between \( R_1 \) and \( R_2 \) to have 5 V in the output. However, now in a real case, we should replace \( R_1 \) and \( R_2 \) by standard resistor values (see the list in the web page).

Because this is a design problem, it does not have a single solution. One possible selection is:

\( R_1 = 10 \text{ k} \Omega \quad R_2 = 27 \text{ k} \Omega \) (if we plug 27 kΩ in the above equation, we obtain \( R_1 = 10519.5 \Omega \))

With these standard values, we can calculate back the actual output voltage.
\[
\frac{8-V_0}{R_1} = 100 \mu A + \frac{V_0}{R_2} \implies \frac{8-V_0}{R_1} = \frac{100 \mu A \cdot R_2 + V_0}{R_2}
\]

\[
(8-V_0) \cdot R_2 = (100 \mu A \cdot R_2 + V_0) \cdot R_1
\]

\[
8R_2 - V_0 R_2 = 100 \mu A \cdot R_2 \cdot R_1 + V_0 R_1
\]

\[
V_0 (R_1 + R_2) = 8R_2 - 100 \mu A R_1 R_2
\]

\[
V_0 = \frac{8R_2 - 100 \mu A \cdot R_1 R_2}{R_1 R_2}
\]

If we replace \( R_1 = 10 \, k\Omega \quad R_2 = 27 \, k\Omega \) (tabulated actual values of standard Resistors) we get \( V_0 = 5.108 \, V \) that is 98% of the design value.
Operational Amplifiers

An operational amplifier or "opamp" is an integrated circuit that behaves as a voltage-controlled voltage source can be used to implement circuits capable of performing mathematical operations such as addition, subtraction, multiplication, division, differentiation and integration.

Scheme

The opamp is an active element and must be powered by a voltage supply.

By KCL

By simplicity, the power supplies are often ignored in the schematics, but the power supply currents must not be overlooked.
The equivalent circuit of an op amp is

\[ V_1 \]
\[ V_d \]
\[ V_2 \]
\[ V_0 \]
\[ R_0 \]
\[ A \cdot V_d \]

The differential voltage is \( V_d = V_2 - V_1 \).

The opamp senses the difference voltage between the 2 inputs, multiplies it by a gain factor \( A \), and causes the resulting voltage to appear at the output.

Thus

\[ V_0 = A \cdot V_d = A (V_2 - V_1) \]

\( A \) is the "open loop" gain, is the gain without any feedback between output and input.

**Typical Values**

<table>
<thead>
<tr>
<th></th>
<th>Typical</th>
<th>Ideal</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>Open loop gain</td>
<td>( 10^5 ) to ( 10^8 )</td>
</tr>
<tr>
<td>( R_i )</td>
<td>Input resistance</td>
<td>( 10^5 ) to ( 10^{13} ) ( \Omega )</td>
</tr>
<tr>
<td>( R_o )</td>
<td>Output resistance</td>
<td>( 10 ) to ( 100 ) ( \Omega )</td>
</tr>
<tr>
<td>( V_{cc} )</td>
<td>Supply voltage</td>
<td>5 to 24 V</td>
</tr>
</tbody>
</table>

When a feedback is connected between input and output, the ratio of the output voltage to the input voltage is called closed-loop gain. We will do this in an example.
A limitation in op amps is that the output voltage cannot exceed the power supply voltage.

\[ \begin{align*} 
& V_C, \quad \text{positive saturation} \\
& -V_C, \quad \text{linear region} \\
& V_C \leq V_O = A, V_D \leq V_C \\
& \text{negative saturation} 
\end{align*} \]

**Example** A 741 op amp has an open loop voltage gain \( A = 2 \times 10^5 \), input resistance \( R_i = 2 \, \text{M} \Omega \), output resistance \( R_o = 50 \, \Omega \). Find the closed loop gain \( V_o/V_s \) and determine the current \( i \) when \( V_s = 2 \, \text{V} \).

We replace the op amp by the equivalent circuit.
Now we can use nodal analysis to solve the circuit.

In the node 1, we have

$$\frac{V_S - V_1}{10 \times 10^3} = \frac{V_1}{2 \times 10^6} + \frac{V_1 - V_0}{20 \times 10^3}$$

Multiplying by $2,000 \times 10^3$

$$200 (V_S - V_1) = V_1 + 100 V_1 - 100 V_0$$

$$\Rightarrow 200 V_S = 301 V_1 - 100 V_0$$

**Or**

$$2 V_S = 3 V_1 - V_0 \Rightarrow V_1 = \frac{2 V_S + V_0}{3}$$

At the node 0,

$$\frac{V_1 - V_0}{20 \times 10^3} = \frac{V_0 - A V_d}{50}$$

But $V_d = -V_1$ and $A = 2 \times 10^5$ - Thus

$$V_1 - V_0 = \frac{20 \times 10^3}{50} (V_0 + 200000 V_1)$$

$$V_1 - V_0 = 400 (V_0 + 200000 V_1)$$

Replacing $V_1$ from (2)

$$0 \approx 26,667.067 V_0 + 53,333.333 V_S \Rightarrow \frac{V_0}{V_S} = -1.9999699$$

When $V_S = 2 V$, $V_0 = -3.9999398 V$
Using equation (2) we can obtain \( V_1 = 20.066667 \mu V \)

thus \( i = \frac{V_1 - V_0}{20 \cdot 10^3} = 0.19999 \text{ mA} \)

**The Ideal OP AMP**

For an ideal op amp we assume

1) Infinite open loop gain \( A \approx \infty \)

2) Infinite input resistance \( R_i \approx \infty \)

3) Zero output resistance \( R_o \approx 0 \)

The ideal op amp has two important characteristics.

1. Currents in both inputs are zero \( i_1 = 0 \), \( i_2 = 0 \)

2. The voltage across the input terminal is zero

\[ V_d = V_1 - V_2 = 0 \implies V_1 = V_2 \]

**Example**

Let us solve the former example assuming that the 741 op amp is ideal.

\[
\frac{V_5 - 0}{10 \cdot 10^3} = \frac{0 - V_0}{20 \cdot 10^3} \implies V_5 = -\frac{V_0}{2} \implies \frac{V_0}{V_5} = -2
\]
If \( V_s = 2V \) \( V_o = -4V \)

The current \( i \) is

\[
i = \frac{0+4V}{20 \times 10^3} = 0.2mA.
\]

**Inverting Amplifier**

\( i_1 = i_2 \) because the current in the input is zero.

Thus using Ohm's law.

\[
i_1 = i_2 = \frac{V_i - 0}{R_1} = \frac{0 - V_o}{R_f}
\]

because the \( + \) input is grounded. \( \odot \) input is ground

Then

\[
\frac{V_i}{R_1} = - \frac{V_o}{R_f} \Rightarrow V_o = - \frac{R_f}{R_1} V_i
\]

**Non inverting Amplifier**
In this circuit again \( i_1 = i_2 \) (because \( i \) in the input is zero).

\[
i_1 = i_2 \implies \frac{0 - V_i}{R_i} = \frac{V_i - V_o}{R_f}
\]

\[
\implies -\frac{V_i}{R_f} = \frac{V_i - V_o}{R_f} \implies V_o = V_i \left(1 + \frac{R_f}{R_1}\right)
\]

**Example** Calculate \( V_o \) in the following circuit

\[
V_i = 3V. \quad \frac{8k\Omega}{4k\Omega + 8k\Omega} = 2V
\]

Thus, the voltage of the input is 2V. We calculate now the current \( i_o \)

\[
i_o = \frac{V_o - V_i}{5k\Omega} = \frac{V_i - 0}{2k\Omega} \implies \frac{V_o - 2V}{5k\Omega} = \frac{2V}{2k\Omega}
\]

\[
\implies V_o - 2V = 5V \implies V_o = 7V
\]
Summing Amplifier

It combines several inputs and produces an output that is a weighted sum of the inputs.

\[
\begin{align*}
V_1 & \rightarrow R_1 \rightarrow V_i \\
V_2 & \rightarrow R_2 \\
V_3 & \rightarrow R_3 \\
\end{align*}
\]

The current \( i \) after the resistors may flow is equal to the current through \( R_f \), because there is no current in the \( \Theta \) input. Also, the voltage at the \( \Theta \) input is ground (equal to \( \Theta \) input).

Using KCL,
\[
i = i_1 + i_2 + i_3
\]

\[
\Rightarrow \frac{0 - V_0}{R_f} = \frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3}
\]

\[
\Rightarrow V_0 = - \left( \frac{R_f}{R_1} V_1 + \frac{R_f}{R_2} V_2 + \frac{R_f}{R_3} V_3 \right)
\]

Example: Find \( V_0 \) and \( i_0 \) in the following circuit.

\[
\begin{align*}
1.5V & \rightarrow 20k\Omega \\
2V & \rightarrow 10k\Omega \\
1.2V & \rightarrow 6k\Omega \\
\end{align*}
\]
Summing the currents in the 3 branches in the input

\[
\frac{1.5}{20k} + \frac{2}{10k} + \frac{1.2}{6k} = -\frac{V_0}{8k}
\]

0.075 mA + 0.2 mA + 0.2 mA = \(-\frac{V_0}{8k}\).

\[V_0 = -0.475 \, mA \times \frac{0.00008}{R} = -3.8 \, V\]

To calculate \(i_0\) (the current in the output) we use KCL in the node 1:

\[i_0 + i_1 + i_2 = 0\]

\[i_0 = \frac{0 - (-3.8V)}{4k\Omega} + \frac{0 - (-3.8V)}{8k\Omega} = 0\]

\[i_0 + 0.95 \, mA + 0.475 \, mA = 0 \quad \Rightarrow \quad i_0 = -1.425 \, mA\]
Difference Amplifier

- A signal connected to the non-inverting input produces a signal at the output which is proportional to the input.
- A signal connected to the inverting input produces an output proportional to the input and with inverted polarity.

The difference amplifier combines these two characteristics:

\[ \frac{V_1 - V_a}{R_1} = \frac{V_a - V_0}{R_2} \quad \rightarrow \quad V_0 = \left( \frac{R_2}{R_1} + 1 \right) V_a - \frac{R_2}{R_1} V_1 \]

In the node ②
\[ \frac{V_2 - V_b}{R_3} = \frac{V_b - 0}{R_4} \quad \rightarrow \quad V_b = \frac{R_4}{R_3 + R_4} \cdot V_2 \]

But we know \( V_b = V_a \) - Thus
\[ V_0 = \left( \frac{R_2}{R_1} + 1 \right) \cdot \frac{R_4}{R_3 + R_4} \cdot V_2 - \frac{R_2}{R_1} \cdot V_1 \]

or
\[ V_0 = \frac{R_2}{R_1} \cdot \frac{1 + \frac{R_1}{R_2}}{1 + \frac{R_3}{R_4}} \cdot V_2 - \frac{R_2}{R_1} V_1 \]
If this is a difference amplifier we will want that $V_0 = 0$ when $V_1 = V_2$. This is true when

$$\frac{R_1}{R_2} = \frac{R_3}{R_4}$$

In this case $V_0 = \frac{R_2}{R_1} (V_2 - V_1)$

If $R_1 = R_2 \implies V_0 = (V_2 - V_1)$

**Voltage Follower**

This circuit helps to insulate the input source from variations in the load. For example, let us consider a circuit and a load $R_L$. The circuit can be represented by its Thévenin equivalent.

![Thévenin Circuit](image)

The output voltage $V_0$ is in this case

$$V_0 = V_{th} \cdot \frac{R_L}{R_L + R_{th}}$$

which obviously depends on $R_L$ and $R_{th}$. Now let insert in between a voltage follower implemented with an opamp.
Voltage follower or "buffer"

In this case $V_0 = V_n = V_p = V_{th}$, independently of the values of $R_{th}$ and $R_L$.

"The output of the voltage follower "follows" the input signal while remaining immune to changes in $R_2"$

Example: Find $V_0$ in terms of $V_1, V_2$ and $V_3$

\[ \frac{V_1}{0.5} + \frac{V_2}{1} + \frac{V_3}{2} = -\frac{V_x}{3} \]

\[ \frac{V_x}{5} = -\frac{V_0}{10} \Rightarrow V_x = -\frac{V_0}{2} \]

Thus

\[ 2V_1 + V_2 + 0.5V_3 = \frac{V_0}{6} \Rightarrow V_0 = 12V_1 + 6V_2 + 3V_3 \]
Instrumentation Amplifier

This is a set of 3 op amps that allows the detection and amplification of very small signals superimposed to two much larger (and otherwise identical) signals.

\[ i = \frac{V_2 - V_1}{R_2} = \frac{V_{o2} - V_{o1}}{R_1 + R_2 + R_3} \rightarrow V_{o2} - V_{o1} = (V_2 - V_1) \cdot \frac{R_1 + R_2 + R_3}{R_2} \]

\[ \Rightarrow V_{o2} - V_{o1} = G_1 (V_2 - V_1) \]

The third opamp works as a difference amplifier. We observe that the resistors in the upper and lower loop are the same. \( \frac{R_5}{R_4} \) in the upper loop

\( \frac{R_5}{R_4} \) in the lower loop
Under these circumstances, the output is

\[ V_0 = \frac{R_u}{R_5} \cdot (V_{o2} - V_{o1}) \]

If we call \( G_2 \)

\[ V_0 = G_2 \cdot (V_{o2} - V_{o1}) = G_2 \cdot G_1 \cdot (V_2 - V_1) \]

Thus

\[ V_0 = \frac{R_u}{R_5} \cdot \left( \frac{R_1 + R_2 + R_3}{R_2} \right) \cdot (V_2 - V_1) \]

To simplify the circuit, and improve precision, all resistors with the exception of \( R_2 \) are chosen to be identical.

In this case

\[ V_0 = \left( 1 + \frac{2R}{R_2} \right) \cdot (V_2 - V_1) \]

The instrumentation amplifier circuit comes in integrated packages - \( R_2 \) controls the gain. This circuit can amplify signals in the microvolt range up to volts.

The instrumentation amplifier is a high sensitivity, high gain, deviation sensor.
Digital to Analog converter

\[ V_0 = -\frac{R_f}{R_1} V_1 + \frac{R_f}{R_2} V_2 + \frac{R_f}{R_3} V_3 + \frac{R_f}{R_4} V_4 \]

This connection is a summing amplifier. The output \( V_0 \) is:

The inputs \( V_1, \ldots, V_4 \) can assume only two voltage values, 0 or 1.

**Example**

In the circuit above let assume:

\[ R_f = 10 \, \text{K} \Omega, \quad R_1 = 10 \, \text{K} \Omega, \quad R_2 = 20 \, \text{K} \Omega, \quad R_3 = 40 \, \text{K} \Omega, \quad R_4 = 80 \, \text{K} \Omega \]

The inputs are binary signals \([0000], [0010], \ldots\)

The equation of the summing amplifier with these values is:

\[ V_0 = V_1 + 0.5 V_2 + 0.25 V_3 + 0.125 V_4 \]

Thus an input \([0, 0, 0, 0]\) gives \( V_0 = 0 \)

An input \([0, 0, 0, 1]\) gives \( V_0 = -0.125 \)

In this case the resolution is the least significant bit = 0.125 V.

The maximum signal will be \([1, 1, 1, 1]\) ->

\[ V_0 = 1 + 0.5 + 0.25 + 0.125 \rightarrow V_0 = -1.875 \text{ V} \]
Example Electrocardiogram amplifier.

Unfortunately the leads capture noise signal from the room. And note that the EKG leads are antennas and the expressive (noise) signal usually is many times larger than the EKG signal.

We can consider that the signal (actual signal) is the EKG signal plus a random noise.

\[ V_1 = v_i(t) + v_n(t) \quad V_2(t) = v_2(t) + v_n(t) \]

We can connect these 2 signals \((V_1, \text{and} \ V_2)\) to a difference amplifier.
With this circuit, the output is

\[ V_{out} = \frac{R_2}{R_1} \left[ (V_{i1}(t) + V_{i2}(t)) - (V_{i2}(t) - V_{i2}(t)) \right] \]

or

\[ V_{out} = \frac{R_2}{R_1} (V_{i1}(t) - V_{i2}(t)) \]

This circuit is very effective to filter out common noise signals in the input leads. The most typical common noise is the 60 Hz noise from the line.

**SUMMARY**

Non-inverting

\[ V_0 = \frac{R_1 + R_2}{R_2} \cdot V_s \]

Inverting

\[ V_0 = -\frac{R_2}{R_1} \cdot V_s \]

Summer

\[ V_0 = -(\frac{R_1}{R_1} + \frac{R_2}{R_2} + \frac{R_3}{R_3}) \cdot V_s \]

Subtracting

\[ V_0 = \frac{R_2}{R_1} (V_2 - V_1) \]

for \( \frac{R_1}{R_2} = \frac{R_3}{R_4} \)

Followed

\[ V_0 = V_s \] (independently)
Capacitors

Capacitors are passive elements designed to store energy in its electric field. Capacitors are used extensively in electronics, communications, computers, etc.
A capacitor consists of two conducting plates, separated by an insulator.

When the voltage source is unconnected, it deposit charge $+$ in the positive electrode and charge $-$ in the negative electrode.

The amount of charge stored, represented by $Q$ is proportional to the voltage $V$.

$$Q = \varepsilon \cdot V$$

$$\frac{[Q]}{[V]} = F \quad \text{(Farad)}$$

The capacitance depends on the shape of the capacitor and on the properties of the insulator that fills the capacitor. For a plane parallel plate capacitor, it is

$$C = \frac{\varepsilon \cdot A}{d}$$

$\varepsilon$ is the "permittivity" of the insulator (or dielectric).

Symbol for capacitor

$$\frac{1}{=}$$

The capacitance depends on the shape of the capacitor and on the properties of the insulator that fills the capacitor.
To find the current-voltage relationship we take the derivative relative to time in the former expression

\[ Q = C \cdot V \rightarrow \frac{dQ}{dt} = C \cdot \frac{dV}{dt} \rightarrow i = C \cdot \frac{dV}{dt} \]

From here we can compute

\[ V = \frac{1}{C} \cdot \int_{-\infty}^{t} i \cdot dt \]

Thus

\[ V = \frac{1}{C} \cdot \int_{t_0}^{t} i \cdot dt + V(t_0) \]

The instantaneous power delivered to the capacitor is

\[ p = V \cdot i = C \cdot V \cdot \frac{dV}{dt} \]

The energy stored in the capacitor is

\[ W = \int_{-\infty}^{t} p \cdot dt = C \int_{-\infty}^{t} V \cdot \frac{dV}{dt} \cdot dt = C \int \frac{V^2}{2} \bigg|_{V(\infty)}^{V(t)} \]

\[ = \frac{1}{2} \cdot C \cdot V^2 \bigg|_{V(-\infty)}^{V(t)} \]

If we assume that \( V(-\infty) = 0 \) (no charge)

\[ W = \frac{1}{2} \cdot C \cdot V^2 \quad \text{or} \quad W = \frac{Q^2}{2C} \]
Notice that

1) If the voltage across the capacitor (DC) then
   \[ i = C \frac{dV}{dt} = 0 \Rightarrow \text{the cap is an open circuit in DC} \]

2) The voltage in the cap must be continuous. (Otherwise \( i \to \infty \))
   \( \Rightarrow \) the capacitor opposes any abrupt change of voltage across its terminals.

3) The ideal capacitor does not dissipate energy. It stores energy.

4) A real (non-ideal) capacitor can be modeled with some "leakage." This is current that flows in the insulator due to an "imperfect" material. The model equivalent is

\[ R \text{ leakage resistance (very high)} \]

Series and Parallel

\[ i \quad \frac{1}{C_1} \quad \frac{1}{C_2} \quad \frac{1}{C_3} \quad \cdots \quad \frac{1}{C_n} \quad \frac{1}{\text{Vin}} \]

All caps have the same voltage across (as in parallel).

\[ i = i_1 + i_2 + i_3 \ldots + i_n = C_1 \frac{dV}{dt} + C_2 \frac{dV}{dt} + \ldots + C_n \frac{dV}{dt} \]

\[ = ( \sum_{k=1}^{n} C_k ) \cdot \frac{dV}{dt} = C_{\text{eq}} \cdot \frac{dV}{dt} \]

\[ C_{\text{eq}} = C_1 + C_2 + C_3 \ldots + C_n = \sum_{k=1}^{n} C_k \]
Connection in Series.

\[ V = V_1 + V_2 + V_3 + \ldots + V_n \]

Replacing \( V_k = \frac{1}{C_k} \int_{t_0}^{t} i(t) \, dt + V_k(t_0) \)

\[ V = \frac{1}{C_1} \int_{t_0}^{t} i(t) \, dt + V_1(t_0) + \frac{1}{C_2} \int_{t_0}^{t} i(t) \, dt + V_2(t_0) + \]
\[ + \ldots + \frac{1}{C_n} \int_{t_0}^{t} i(t) \, dt + V_n(t_0) \]

\[ \Rightarrow \quad V = \left( \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots + \frac{1}{C_n} \right) \cdot \int_{t_0}^{t} i(t) \, dt + V_1(t_0) + V_2(t_0) + \]
\[ + \ldots + V_n(t_0) \]

\[ V = \frac{1}{C_{eq}} \cdot \int_{t_0}^{t} i(t) \, dt + V(t_0) \]

Where
\[ \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots + \frac{1}{C_n} = \sum_{k=1}^{n} \frac{1}{C_k} \]
Example

Find the equivalent capacitance of the network.

\[ \frac{1}{20 \mu F} = \frac{1}{20 \mu F} \]

Inductors. Symbol \[ \text{---} \]

In an inductor

\[ V = L \cdot \frac{di}{dt} \]

Units of inductance \[ \text{H: Henry} \]

The value of the inductance depends on the shape and size of the inductor.

For example, for a solenoid \[ L = \frac{N^2 \mu A}{L} \]

Thus the current–voltage characteristic is

\[ V = L \cdot \frac{di}{dt} \quad \rightarrow \quad i = \frac{1}{L} \int_{t_0}^{t} V(t) \, dt + i(0) \]
Now we can calculate the instantaneous power.

\[ p = V \cdot i = L \cdot \frac{di}{dt} \cdot i \]

and the energy

\[ W = \int_{-\infty}^{t} p \cdot dt = \int_{-\infty}^{t} L \cdot \frac{di}{dt} \cdot i \cdot dt = \]

\[ = L \cdot \int_{-\infty}^{t} i \cdot di = \frac{1}{2} L \cdot i^2(t) - \frac{1}{2} L \cdot i^2(\infty) \]

If we assume that the current at \( t = -\infty = 0 \)

\[ W = \frac{1}{2} L \cdot i^2 \]

**Remark**

② The voltage across the inductor is zero when the current is constant

\[ V = L \cdot \frac{di}{dt} \]

→ \( L \) is a short circuit in DC.

② \( i \) has to be continuous (otherwise \( V \rightarrow \infty \))

Real inductor
Series and Parallel

\[ V = V_1 + V_2 + V_3 + \cdots + V_N = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + \cdots + L_N \frac{di}{dt} = \left( L_1 + L_2 + L_3 + \cdots + L_N \right) \frac{di}{dt} \]

\[ \text{Leq} = \sum_i L_i \]

\[ i = i_1 + i_2 + i_3 + \cdots + i_N \quad \text{We replace} \quad i_k = \frac{1}{L_k} \int_{t_0}^{t} V(t, t) \, dt + i(t_0) \]

\[ i = \frac{1}{L_1} \int_{t_0}^{t} V(t, t) \, dt + i_1(t_0) + \frac{1}{L_2} \int_{t_0}^{t} V(t, t) \, dt + i_2(t_2) + \cdots + \frac{1}{L_N} \int_{t_0}^{t} V(t, t) \, dt + i_N(t_0) \]

\[ = \left( \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N} \right) \int_{t_0}^{t} V(t, t) \, dt + i_1(t_0) + i_2(t_2) + i_3(t_3) + \cdots + i_N(t_0) \]

\[ \frac{1}{\text{Leq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N} \]

\[ \int_{t_0}^{t} V(t) \, dt + i(t_0) \]
Summary

\[ \begin{align*}
V - i & \quad V = R \cdot i \\
\dot{i} - \dot{V} & \quad \dot{i} = \frac{1}{C} \int i \, dt + V_o \\
p \text{ or } w & \quad \dot{w} = \frac{1}{2} L \cdot \dot{i}^2 \\
\text{Series} & \quad R_{eq} = \sum R_i, \quad \frac{1}{C_{eq}} = \sum \frac{1}{C_i} \\
\text{Parallel} & \quad \frac{1}{R_{eq}} = \sum \frac{1}{R_i}, \quad C_{eq} = \sum C_i, \quad \frac{1}{L_{eq}} = \sum \frac{1}{L_i} \\
\text{At DC} & \quad \text{same}, \quad \text{open } C, \quad \text{short } C.
\end{align*} \]

Example

Calculate \( L_{eq} \)

\begin{align*}
\text{Diagram:} & \quad 20 \, \text{mH} \quad 100 \, \text{mH} \quad 40 \, \text{mH} \\
& \quad 50 \, \text{mH} \quad 40 \, \text{mH} \quad 30 \, \text{mH} \\
& \quad 20 \, \text{mH}
\end{align*}

\[ L_{eq} = \left[ \left( \frac{40 + 20}{30} \right) \parallel 100 \right] \parallel \left( \frac{40}{20} \right) \parallel 50 \]

\[ = \left[ \left( \frac{60}{30} \right) \parallel 100 \right] \parallel \left( \frac{40}{20} \right) \parallel 50 \]

\[ = \left( \left( \frac{20 + 100}{40} \right) \parallel 20 \right) \parallel \left( \frac{30}{20} \right) \parallel 50 = \frac{50}{50} = 2 \, \text{mH} \]
Example

Find \( i, V_c, i_L \).

For DC

\[ i = i_L = \frac{12V}{6\Omega} = 2A \quad V_c = 5\Omega \times 2A = 10V \]

What are \( W_c \) and \( W_L \)?

\[ W_c = \frac{1}{2} \cdot C \cdot V^2 = \frac{1}{2} \cdot 1F \cdot (10V)^2 = 50J \]

\[ W_L = \frac{1}{2} \cdot L \cdot i^2 = \frac{1}{2} \cdot 2H \cdot 4A^2 = 4J \]
**OpAmps applications**

**Differentiator**

\[ i_C = i_R \]
\[ i_R = -\frac{V_o}{R} \]
\[ i_C = C \cdot \frac{dV_i}{dt} \]

Thus, \[ V_o = -RC \cdot \frac{dV_i}{dt} \]

**Integrator**

\[ i_R = i_C \]
\[ i_R = \frac{V_i}{R} \]
\[ i_C = -C \frac{dV_o}{dt} \]

Thus, \[ \frac{V_i}{R} = -C \frac{dV_o}{dt} \Rightarrow dV_o = -\frac{1}{RC} V_i \, dt \]

\[ V_o(t) - V_o(0) = \frac{-1}{RC} \int_0^t V_i(t) \, dt \]
Example

\[ V_0 = -\frac{1}{Re} \int_0^t V_i(t) \, dt = -\frac{1}{25 \times 10^3 \times 10 \times 10^{-6}} \int_0^t 10 \times 10^{-3} \, dt \]

\[ V_0 = -40 \times 1 \text{ mV}. \]

Example

This is a differentiator

\[ RC = 0.2 \times 10^{-6} \text{ F.} \times 5 \times 10 \times n \]

\[ = 10^{-3} \text{ s} \]

The input voltage is

\[ V_i = \begin{cases} 
2000t & 0 < t < 2 \text{ ms} \\
8 - 2000t & 2 < t < 4 \text{ ms} 
\end{cases} \]

\[ V_0 = \begin{cases} 
-2V & 0 < t < 2 \text{ ms} \\
2V & 2 < t < 4 \text{ ms} 
\end{cases} \]
Example

Calculate $V_c, i_L, W_c, W_L$.

\[ i_L = 4A \times \frac{3}{3+1} = 3A \]

Thus

\[ V_c = i \times 1 \Omega = 3V \]

\[ W_c = \frac{1}{2} C V_c^2 = \frac{1}{2} \times 2F \times 9V^2 = 9J \]

\[ W_L = \frac{1}{2} L i^2 = \frac{1}{2} \times 0.25H \times 9A^2 = 1.125J \]
First Order Circuits

1) Source-free RC circuit. We consider an initially charged capacitor

\[ V(0) = V_0 \]
\[ W(0) = \frac{1}{2} \cdot V_0^2 \cdot C \]

Using KCL,

\[ i_C + i_R = 0 \]

We know \( i_C = C \cdot \frac{dV}{dt} \)

\[ i_R = \frac{V}{R} \]

Thus,

\[ C \cdot \frac{dV}{dt} + \frac{V}{R} = 0 \]

1st order diff. eq.

\[ \Rightarrow \frac{dV}{dt} = -\frac{1}{RC} \cdot V \]

\[ \Rightarrow \ln V = -\frac{1}{RC} \cdot t + \ln A \]

\[ \Rightarrow V(t) = A e^{-t/RC} \]

From the initial conditions:

\[ V(t=0) = V_0 = A \]

Finally

\[ V(t) = V_0 \cdot e^{-t/RC} \]

"natural" response

\[ \tau = RC \text{ is the "time constant" } \]

\[ V(t) = V_0 \cdot e^{-t/\tau} \]
If we plot \( V(t) \)

Key to solve the circuit:
1. Find the initial voltage in \( C \)
2. Find \( \tau \)

Example

To calculate \( \tau \), we need \( R \). Looking to the circuit, \( R \) is

\[
R = \left( \frac{6}{12} \right) + 8 = 4 + 8 = 12 \, \Omega
\]

Thus \( \tau = RC = \frac{4}{3} \, F \cdot 12 \, \Omega = 4.5 \)

Finally \( V_C = 30 \, e^{-t/4} \, V \)

To calculate \( V_x \), we realize that this is a voltage divider
\[ V_x = V_c \cdot \frac{4}{4+8} = \frac{V_c}{3} \]

Thus \[ V_x = 10 \cdot e^{-t/4} \, V \]

Finally the current \[ i_0 \] is

\[ i_0 = \frac{V_x - V_c}{8L} = \frac{(10 - 30)e^{-t/\tau}}{8L} = -2.5 \, e^{-t/\tau} \, A \]

**Source-free RL circuit**

\[ L \begin{array}{c}
\cdots \\
\vdots \\
\end{array} \]

\[ L, R \]

\[ i(0) = I_0 \]

\[ v(0) = \frac{1}{2} L \cdot I_0^2 \]

Using KVL, \[ V_L + V_R = 0 \rightarrow L \cdot \frac{di}{dt} + Ri = 0 \]

Solving the 1st order diff eq

\[ i(t) = I_0 \, e^{-t/\tau} \quad \text{where} \quad \tau = \frac{L}{R} \]

With \[ i(t) \] we can calculate \[ v(t) \]

\[ v_R(t) = i \cdot R = I_0 \cdot R \, e^{-t/\tau} \]

and the power

\[ p = V_i \cdot i = V_R \cdot i = I_0^2 \cdot R \cdot e^{-2t/\tau} \]
The energy absorbed in the R
\[ W_R = \int_0^t I_0^2 R e^{-\frac{2t}{\tau}} \, dt \]
\[ = -\frac{1}{2} \tau I_0^2 R e^{-\frac{2t}{\tau}} \bigg|_0^t \] replacing \( \tau = \frac{1}{R} \)
\[ = \frac{1}{2} L I_0^2 \left( 1 - e^{-\frac{2t}{\tau}} \right) \]

When \( t \to \infty \) \( W_R \to \frac{1}{2} L I_0^2 \checkmark \) 0k

Key to solve RL circuits
1) Find the initial current
2) Find \( \tau \)

**Example**

5A \( \uparrow \)
\[ \frac{12 \Omega}{3} \] \[ \frac{8 \Omega}{5} \] \( 2 \mu \)

the switch was closed and at \( t = 0 \) we open it.

1) Find \( I_{0} \)

5 \( \uparrow \)
\[ \frac{12 \Omega}{3} \]
\[ \frac{5 \Omega}{8} \] \( 2 \mu \)

\( \downarrow \)
\[ i_1 = 5A \times \frac{8R}{12R+8R} = 2A \rightarrow I_0 = 2A \]

2) Find \( \tau \)

When we open the switch the circuit becomes:

\[
\begin{array}{c}
12 \\
2H \\
\end{array}
\begin{array}{c}
5R \\
\hline
8R \\
\end{array}
\begin{array}{c}
2H \\
\end{array}
\]

\[
\begin{array}{c}
2H \\
\end{array}
\begin{array}{c}
5R \\
\hline
12R \\
\end{array}
\begin{array}{c}
8R \\
\end{array}
\]

Thus \( R = \frac{5}{12+8} = \frac{5}{20} = 0.25 \Omega \)

\( \tau = \frac{L}{R} = 2H / 0.25 \Omega = 0.5 \text{ s} \).

Finally \( I(t) = I_0 e^{-t/\tau} = 2 \cdot e^{-t/0.5} \text{ A} \)
Singularity functions: are functions that either are discontinuous or have discontinuous derivatives.

1) Step function

\[ u(t) = \begin{cases} 
0 & t < 0 \\
1 & t > 0
\end{cases} \]

If the change occurs at \( t = t_0 \)

\[ u(t-t_0) = \begin{cases} 
0 & t < t_0 \\
1 & t > t_0
\end{cases} \]

We can use a step function to represent an abrupt change in the voltage or current, like the change that occurs in a digital circuit.

For example

\[ v(t) = \begin{cases} 
0 & t < t_0 \\
V_0 & t > t_0
\end{cases} \quad \rightarrow \quad v(t) = V_0 \ u(t-t_0) \]

Thus we can represent switches with step functions.

\[ \begin{align*}
\begin{array}{c}
+ \\
V_0 \ u(t) \\
- \\
\end{array}
\end{align*} = \begin{cases} 
V_0 & t = 0 \\
\end{cases} \]

\[ \begin{align*}
\begin{array}{c}
\text{\textbullet} \\
\lambda_0 \ u(t) \\
\end{array}
\end{align*} = \begin{cases} 
\lambda_0 & t = 0 \\
\end{cases} \]
The derivative of the step function is the unit impulse function, or "delta" function.

The delta function is zero everywhere except at \( t=0 \) when it is undefined.

\[
\delta(t) = \begin{cases} 
0 & t<0 \\
\text{undef} & t=0 \\
0 & t>0 
\end{cases}
\]

It can be regarded as a very short duration pulse of unit area - the total area under the function is 1. This can be written as

\[
\int_{-\infty}^{\infty} \delta(t) \, dt = 1
\]

When the impulse has a strength other than unit, the area of the impulse equals to its strength. For example, a series of short pulses with strengths 5, 10 and -4 at times -2, 0 and +3 is expressed as:

\[
5 \delta(t+2) + 10 \delta(t) - 4 \delta(t-3)
\]
Important property of the $S(t)$ function

Let us calculate

$$\int_{a}^{b} f(t) \cdot S(t-t_0) \, dt$$

Where $a < t_0 < b$. Since $S(t-t_0)$ is zero always except at $t = t_0$, we can write

$$\int_{a}^{b} f(t) \cdot S(t-t_0) \, dt = \int_{a}^{b} f(t_0) \cdot S(t-t_0) \, dt$$

$$= f(t_0) \int_{a}^{b} S(t-t_0) \cdot dt = f(t_0)$$

The integral of a function with the $S(t)$ is the function evaluated at the argument of the $S(t)$.

The special case when $t_0 = 0$

$$\int_{a}^{b} f(t) \cdot S(t) = f(0)$$

This property is called the "sampling" property.
Unit Ramp

The integral of the step is the unit ramp. It is zero for negative values of \( t \) and has a slope 1 for \( t > 0 \).

\[
r(t) = \begin{cases} 
0 & t < 0 \\
1 & t > 0 
\end{cases}
\]

We can write also the advanced or delayed ramps.

\[
r(t + t_0) = \begin{cases} 
0 & t \leq t_0 \\
t + t_0 & t > t_0 
\end{cases} 
\]

\[
r(t - t_0) = \begin{cases} 
0 & t < t_0 \\
t - t_0 & t > t_0 
\end{cases} 
\]

Remember

\[
\text{\( s(t) = \frac{d}{dt} u(t) \)}
\]

\[
\text{\( u(t) = \int_{-\infty}^{t} s(t) \, dt \)}
\]

\[
\text{\( v(t) = \frac{d}{dt} r(t) \)}
\]

\[
\text{\( r(t) = \int_{-\infty}^{t} u(t) \, dt \)}
\]
These three functions allow us to write different temporal signals.

**Example** Express a gate function (\( V(t) \))

\[
V(t) = \begin{cases} 
0 & t < 2 \\
10 & 2 < t < 5 \\
0 & t > 5 
\end{cases}
\]

We can write this square signal as a gate function as

\[
V(t) = 10 \ \mu(t-2) - 10 \ \mu(t-5)
\]

\[
= 10 \ [ \mu(t-2) - \mu(t-5) ]
\]

If we want to calculate the derivative of this signal,

\[
\frac{dV(t)}{dt} = 10 \ [ \delta(t-2) - \delta(t-5) ]
\]
Example

Express the current in the figure in terms of step functions and calculate the integral.

\[ i(t) = 10 \mu(t) - 20 \mu(t-2) + 10 \mu(t-4) \]

To calculate the integral

\[ \int i(t) \, dt = 10 \left[ r(t) - 2 r(t-2) + r(t-4) \right] \]
**Example**

Express a sawtooth function.

\[ v(t) = \begin{align*}
5 \cdot r(t) - 5 \cdot r(t-2) - 10 \cdot u(t-2)
\end{align*} \]

It is possible to solve this problem in different ways.

1. \[ v(t) = \begin{align*}
5 \cdot r(t) - 5 \cdot r(t-2) - 10 \cdot u(t-2)
\end{align*} \]

   a) \[ \begin{align*}
   &+ \\
   &+ \\
   &=
\end{align*} \]

   b) \[ \begin{align*}
   &+ \\
   &+ \\
   &=
\end{align*} \]
2) The sawtooth can be expressed as the product of a ramp and a gate function:

\[ v(t) = 5, t \left[ u(t) - u(t-2) \right] \]

\[ = 5t \ u(t) - 5 \ t \ u(t-2) \]

\[ = 5 \ r(t) - 5 \left( t-2+2 \right) \ u(t-2) \]

\[ = 5 \ r(t) - 5 \ (t-2) \ u(t-2) - 10 \ u(t-2) \]

\[ = 5 \ r(t) - 5 \ r(t-2) - 10 \ u(t-2) \]

3) We can also think of this function as the product of a ramp times a step function:

\[ v(t) = 5 \ r(t) \cdot u(-t+2) \]

or \[ 5 \ r(t) \cdot [1 - u(t-2)] \]
Step Response of an RC Circuit.

We will study the response of a RC circuit to a sudden change in the voltage or current applied to it.

![Circuit Diagram]

We assume that the capacitor is initially charged at a voltage $V_0$.

$\nu(0^-) = \nu(0^+) = V_0$

Because the capacitor cannot change the voltage instantaneously.

Now we apply KCL in the node (1)

$$C \cdot \frac{d\nu}{dt} + \frac{\nu - V_0 \cdot u(t)}{R} = 0$$

Or

$$\frac{d\nu}{dt} + \frac{\nu}{RC} = \frac{V_0 \cdot u(t)}{RC}$$

Where $\nu$ is the voltage across $C$

For $t > 0 \quad u(t) = 1$ Thus we have

$$\frac{d\nu}{dt} + \frac{\nu}{RC} = \frac{V_0}{RC} \quad \rightarrow \quad \frac{d\nu}{dt} = -\frac{\nu - V_0}{RC}$$
OR \[ \frac{dW}{V-V_s} = -\frac{dt}{RC} \]

Integrating both sides
\[ \ln (V-V_s) \Big|_{V_0}^{V(t)} = -\frac{t}{RC} \bigg|_{0}^{t} \]

\[ \rightarrow \ln (V(t)-V_s) - \ln (V_0-V_s) = -\frac{t}{RC} \]

\[ \rightarrow \ln \left( \frac{V-V_s}{V_0-V_s} \right) = -\frac{t}{RC} \rightarrow \text{ convince } \]

\[ \rightarrow \frac{V-V_s}{V_0-V_s} = e^{-t/\tau} \rightarrow V-V_s = (V_0-V_s) e^{-t/\tau} \]

\[ \rightarrow V(t) = V_s + (V_0-V_s) e^{-t/\tau} \quad \text{for } t>0 \]

Thus the solution is

\[ V(t) = \begin{cases} V_0 & t < 0 \\ V_s + (V_0-V_s) e^{-t/\tau} & t > 0 \end{cases} \]

Complete response of a RC circuit

\[ V(t) \quad t(t) \]
If \( V_0 = 0 \) \( V(t) = \begin{cases} 0 & t < 0 \\ V_s \left(1 - e^{-t/\tau}\right) & t \geq 0 \end{cases} \)

Which can be written as \( V(t) = V_s \left(1 - e^{-t/\tau}\right), \mu(t) \)

The current is obtained using \( i(t) = C \frac{dV}{dt} \)

\[ i(t) = C \frac{d}{dt} \left(V_s \left(1 - e^{-t/\tau}\right)\right) = \frac{C}{\tau} V_s e^{-t/\tau} \]

or \( i(t) = \frac{V_s}{R} e^{-t/\tau} \mu(t) \)

How to analyze 1st order RC circuits

We can see that the response of the RC circuit to a step voltage is composed of 2 parts:

Complete Response = Natural Response + Forced Response

or \( V = V_n + V_f \)

Where \( V_n = V_0 e^{-t/\tau} \) (natural "source less" response)

\[ V_f = V_s \left(1 - e^{-t/\tau}\right) \]

We can also consider the complete response as a steady state (long term) response superposed with a transient (time dependent) response

\( V = V_t + V_{ss} \)
Where \[ V_t = (V_o - V_s) \, e^{-t/\tau} \]
\[ V_{ss} = V_s \]

OR \[ V(t) = V(\infty) + [V(0) - V(\infty)] \, e^{-t/\tau} \]

Where \( V(\infty) \) is the steady state voltage, \( V_0 \) is the initial voltage. Thus, to calculate the step response of a RC circuit we will need:

1) The initial voltage in the capacitor \( V(0) \)
2) The final \& voltage in the capacitor \( V(\infty) \)
3) The time constant \( \tau \)

**Example**

1) \( V(0) = 10 \, V \) (is the initial voltage in the \( C \), you should consider the initial circuit)

\[ \begin{array}{c}
\quad 10 \\
\quad \downarrow \\
\quad \quad \downarrow \downarrow \\
\quad \quad \quad 1/3 \, F \\
\quad \quad \downarrow \\
\quad \quad 50 \\
\quad \downarrow \\
\quad 2 \, R \\
\quad \uparrow \\
\quad 6 \, R \\
\quad \uparrow \\
\end{array} \]

2) \( V(\infty) \) - The final voltage is obtained solving the following circuit

\[ -10 + 2i + 6i - 50 = 0 \]

\[ 8i = 60 \rightarrow i = 7.5 \, A \]
then \( V(\infty) = 10V - 7.5 \times 2 \cdot \frac{\pi}{2} = -5V \). 

3) Finally \( \tau = RC \)

\( R \) is the equivalent resistance from the capacitor terminals.

\[ R = \frac{2\pi}{3} \approx 3.2 \ \Omega \]

Thus \( \tau = RC = \frac{3}{2} \pi \times \frac{1}{3} \text{ F} = \frac{1}{2} \text{ s} \)

The final temporal response of \( V \) is:

\[ V(t) = -5 + \left[ 10 - (-5) \right] e^{-t/1/2} \]

\[ \Rightarrow -5 + 15 e^{-2t} \]
Step response of a RL circuit.

Let us consider the following circuit:

\[ V_S \quad \begin{array}{c} \text{+} \\ \hline \end{array} \quad R \quad \begin{array}{c} \text{+} \\ \hline \end{array} \quad L \quad \begin{array}{c} \text{+} \\ \hline \end{array} \quad V_S \quad \begin{array}{c} \text{+} \\ \hline \end{array} \quad \mu(t) \quad R \quad \begin{array}{c} \text{+} \\ \hline \end{array} \quad L \quad \begin{array}{c} \text{+} \\ \hline \end{array} \]

The current in the circuit will have a natural response \( i_n \) and a forced response \( i_f \)

\[ i = i_n + i_f \quad \text{where} \quad i_n = A \cdot e^{-t/\tau} \quad (\tau = \frac{1}{R}) \]

The natural response dies out after \( 5\tau \), and the only remaining term is \( i_f \) which corresponds to the current in the circuit after the natural response disappears.

\[ i_f = \frac{V_S}{R} \]

Thus

\[ i = A \cdot e^{-t/\tau} + \frac{V_S}{R} \]

"A" is a constant to be determined from the initial conditions. Let us assume that the initial current is \( I_0 \)

\[ i(0^+) = i(0^-) = I_0 \]

Thus

\[ I_0 = A + \frac{V_S}{R} \quad \rightarrow \quad A = I_0 - \frac{V_S}{R} \]
Replacing in the formula \( \Box \)

\[ i(t) = \frac{V_s}{R} + \left( I_0 - \frac{V_s}{R} \right) e^{-t/\tau} \]

That can be written as

\[ i(t) = i(\infty) + \left[ i(0) - i(\infty) \right] e^{-t/\tau} \]

To calculate the step response of a RL circuit we need

1) The initial current in the inductor \( i(0) \)
2) The final current in the inductor \( i(\infty) \)
3) The time constant \( \tau \)

Once we know the current \( i(t) \), the voltage \( v(t) \) is obtained

\[ v(t) = L \frac{di}{dt} = V_s \cdot \frac{L}{\tau R} e^{-t/\tau} \quad (\tau = L/R) \]

\[ v(t) = V_s \cdot e^{-t/\tau} u(t) \]

---

**Example**

[Diagram of RL circuit with initial conditions: \( I = 3 \) A, \( V = 10 \) V, \( L = 1.5 \) H, and \( R = 5 \) Ohm at \( t = 0 \).]
We need to calculate \( i(0), i(\infty) \), and \( \tau \).

\( i(\infty) \): corresponds to the current in the following circuit.

\[
\begin{aligned}
\text{The inductor in DC is a short circuit \( \rightarrow \) current value:} \\
i'(\infty) &= 3A \times \frac{10}{15} = 2A
\end{aligned}
\]

\( i(0) \): corresponds to the current in the following circuit

\[
\begin{aligned}
\text{The inductor in DC is a short circuit \( \rightarrow \) current value:} \\
i(0) &= 3A
\end{aligned}
\]

\[
\tau = \frac{L}{R}
\]

\( R \) is the resistor connected to the inductor, the series of 5\( \Omega \) and 10\( \Omega \), \( R = 5 + 10 = 15\Omega \)

\( \tau = 1.5H / 15\Omega = 0.1s \)

Thus

\[
\begin{aligned}
i(t) &= 2 + [3 - 2] e^{-t/0.1} \\
i'(t) &= 2 + e^{-10t}
\end{aligned}
\]
Exercise 1: In the circuit below, what is the value of $R$ such that $W_c = W_L$? (For steady state)

\[ W_c = W_L \rightarrow \frac{1}{2}, \; C V^2 = \frac{1}{2} \; I^2 \]

\[
\frac{1}{2} \cdot 10^{-6} \cdot \left( \frac{10}{20+R} \right)^2 = \frac{1}{2} \cdot 10^{-2} \left( \frac{10}{20+R} \right)^2 \rightarrow R = 10 \Omega
\]

Exercise 2: The circuit is in steady state before the switch opens. Find the current $i(t)$ for $t > 0$

Before the switch opens, the voltage in the capacitor is 2V.

After the switch is open, the circuit we have is:

We solve for the voltage in the capacitor.
1) $V(0) = 2V$.

2) $V(\infty) = 4V$ (voltage divider).

3) $RC = 30k + (60k/160k) = 60k \Omega$.

Thus the voltage in the capacitor is

$$V(t) = 4 + [2-4] e^{-t/0.12} = 4 - 2 e^{-t/0.12}$$

We need to find now the voltage of node A ($V_A$)

We can write the node equation for A

$$\frac{8-V_A}{60k} + \frac{V(t)-V_A}{30k} - \frac{V_A}{60k} = 0$$

Solving for $V_A$

$$V_A(t) = 4 - e^{-t/0.12}$$

Finally $i(t)$ is obtained with Ohm's law

$$i(t) = \frac{4 - e^{-t/0.12}}{60k \Omega} = (66.7 - 16.7 e^{-t/120}) \mu A.$$
Exercise. Find the current in the inductor after the switch opens.

\[ i(0) = \frac{6 \text{V}}{15 \Omega} = \frac{2}{5} \]

\( a \) \( i(0) \). We consider the following circuit.

\[ i(\infty) = \frac{12 \text{V}}{20 \Omega} = \frac{3}{5} \]

\( b \) \( i(\infty) \). We consider the following circuit.

\( c \) \( \tau = \frac{L}{R} = \frac{15+5}{20 \Omega} = 1/5 \\text{s} \).

Thus

\[ i(t) = \frac{3}{5} + \left[ \frac{2}{5} - \frac{3}{5} \right] e^{-st} = 0.6 - 0.2 e^{-5t} \]

For \( t < 0 \) \( i(t) = \frac{2}{5} = 0.4 \text{ A} \).
Exercise: Find the current in the inductor after the switch closes.

\[ i(t) = 40 \text{A} \]

\[ v(t) = 10 \text{V} \]

\[ R = 10 \text{ohms} \]

\[ L = 20 \text{H} \]

\[ V = 12 \text{V} \]

\[ V_1 = 4 \text{V} \]

\[ i(0) = \frac{4 \text{V}}{40 \text{ohms}} = 0.1 \text{A} \]

\[ i(\infty) = \frac{12 \text{V}}{40 \text{ohms}} = 0.3 \text{A} \]

\[ R = 40 \text{ohms} \]

\[ L/R = \frac{1}{2} \] ohms

Finally, \[ i(t) = 0.3 + [(0.1-0.3) e^{-2t}] \]

\[ \rightarrow 0.3 - 0.2 e^{-2t} \]
Thévenin and Norton

Find the Thévenin equivalent circuit of

We recognize that $i_x = 0$ (there is no return path).

The $V_{th}$ voltage is the voltage across the 25 kΩ resistor
$V_{th} = V_{ab} = (-20i) \times 25 = -500i$  

Now we replace $i$ by $i = \frac{5-3V_0}{2kΩ}$
thus $i = \frac{5-3V_{th}}{2000}$

Combining the two equations: 

$V_{th} = \frac{-500(5-3V_{th})}{2000}$ 

We can also find $I_N$ calculating the S.C. current

$v_{sc} = -20i$
$V_{ab}$ is measured with a voltmeter with an internal resistance $R_i = 100\, k\Omega$. What is the voltmeter reading?

The equivalent circuit is

$$R_{th} = 15\, k + (60k/12k) = 15k + 10k = 25\, k\Omega$$

$$V_{th} = \frac{36 - V_o}{12k} + 18\, mA - \frac{V_o}{60k} = 0$$

$$-3\, mA - \frac{V_o}{12k} + 18\, mA - \frac{V_o}{60k} = 0$$

$$15\, mA = \frac{V_o}{10k} \quad \rightarrow \quad V_o = 150\, V = V_{th}$$

Thus the circuit is equivalent to

$$150V \quad \frac{1}{25\, k\Omega}$$
Now we use a voltmeter with an internal resistance of 100 kΩ.

\[ V_{\text{measured}} = 150 \text{V} \times \frac{100 \text{kΩ}}{100 \text{kΩ} + 25 \text{kΩ}} = 120 \text{V} \]

The reading will be what is measured in the 100 kΩ resistor.

Notice that the larger the internal resistance, the more accurate is the measurement. For example, if the voltmeter has an internal resistance of 10 MΩ, then the measured voltage will be

\[ V_{\text{meas}} = 150 \text{V} \times \frac{10 \text{MΩ}}{10 \text{MΩ} + 25 \text{kΩ}} = 149.62 \text{V} \] (closer to the real value)
Second Order circuits

Second order circuits are described by 2nd order diff. eqns. They comprise 2 storage (energy storage) devices, this is caps or inductors. Examples of 2nd order circuits are as follows.

\[
\begin{align*}
  & V_s \quad R \quad L \\
  & V_s \quad R \quad C \\
  & V_s \quad R \quad L \\
  & V_s \quad R \quad C
\end{align*}
\]

1. Finding the initial and final values of \( V_c \) and \( I_L \)

We have presented before examples of circuits and guidelines how to calculate the voltage across the capacitor and the current through the inductor by taking into account that in a DC circuit the caps behave as open circuits and the inductors as short circuits.

Now we must also learn how to find the initial and final values of the derivatives of these quantities.

Thus we will see how to find

\[
\begin{align*}
  & V(0) \\
  & i(0) \\
  & \frac{\Delta V(0)}{\Delta t} \\
  & \frac{\Delta i(0)}{\Delta t} \\
  & i(\infty) \\
  & V(\infty)
\end{align*}
\]
When it is understood that \( V \) is across the capacitor and \( i \) is through the inductor.

Remember that:

1) In a capacitor the voltage is always continuous

\[
\rightarrow V(0^+) = V(0^-)
\]

2) The current in an inductor is always continuous.

\[
\rightarrow i(0^+) = i(0^-)
\]

Where \( 0^+ \) and \( 0^- \) are the times just after and before the switch.

Let us use an example to clarify ideas: (Prob. 8.1)

\[
\begin{align*}
\frac{2\Omega}{10\Omega} & \frac{1/20\text{F}}{0.4\text{H}} \quad \text{24V} \\
\end{align*}
\]

\[
\begin{align*}
\text{Calculate} \\
\text{a) } i(0^+) \\
\text{b) } V(0^+) \\
\text{c) } \frac{di(0^+)}{dt} \\
\text{d) } \frac{dv(0^+)}{dt} \\
\text{e) } i(\infty) \\
\text{f) } V(\infty)
\end{align*}
\]

Initial values of \( i \) and \( V \).

The initial circuit is:

\[
\begin{align*}
\frac{2\Omega}{10\Omega} & \frac{1/20\text{F}}{0.4\text{H}} \quad \text{24V} \\
\end{align*}
\]

\[
\begin{align*}
i(0^+) &= \frac{24\text{V}}{12\Omega} = 2 \text{ A} \\
V(0^+) &= 24\text{V} \times \frac{2\Omega}{12\Omega} = 4\text{V}
\end{align*}
\]
Final values at $t = \infty$.

The final circuit is:

\[\begin{array}{c}
\text{2Ω} \\
\text{1/20 F} \\
\text{24V} \\
\end{array}\]

\[\begin{array}{c}
\text{0.14 H}
\end{array}\]

\[i(\infty) = \frac{24V}{2\Omega} = 12\ A\]

\[V(\infty) = 24\ V\]

Now calculate $\frac{di(0)}{dt}$ and $\frac{dV(0^+)}{dt}$.

On the capacitor $i(0^+) = i(0^-) = 0 = c \cdot \frac{dv}{dt} \Rightarrow \frac{dV(0^+)}{dt} = 0$

For $\frac{di(0^+)}{dt}$ we use KVL in the circuit:

\[-24V + V_L + 2.5 \times 2A \times i(0^+) = 0\]

\[-24V + V_L + 2A = 0 \Rightarrow V_L(0^+) = 20\ V.\]

Thus

\[L \cdot \frac{di(0^+)}{dt} = V_L(0^+) \Rightarrow \frac{di(0^+)}{dt} = \frac{20V}{0.14\ H} = 50\ V/s\]
Source Free RLC Series circuit

\[ R \rightarrow I_0 \]
\[ L \quad + \]
\[ C \quad - \]
\[ V_0 \]

We assume that the circuit is being excited by energy stored in \( L \) and \( C \).

Thus we assume an initial current \( I_0 \) and voltage \( V_0 \).

\[ V_0 = \varphi(0) = \frac{1}{C} \int_{\infty}^{0} i(t) \, dt \quad \quad I_0 = i(0) \]

Now we apply KVL in the circuit

\[ R \cdot i + L \cdot \frac{di}{dt} + \frac{1}{C} \int_{\infty}^{0} i(t) \, dt = 0 \]

or

\[ \frac{d^2 i}{dt^2} + \frac{R}{L} \cdot \frac{di}{dt} + \frac{i}{LC} = 0 \]

Second order diff.

To solve we need 2 initial conditions.

1) \( i(0) = I_0 \quad \checkmark \)

2) \[ R \cdot i(0) + L \cdot \frac{d i(0)}{dt} + \frac{1}{C} \int_{\infty}^{0} i(t) \, dt = 0 \]

\[ \Rightarrow R I_0 + L \cdot \frac{d i(0)}{dt} + V_0 = 0 \]

or

\[ \frac{d i(0)}{dt} = - \frac{1}{L} \left( R I_0 + V_0 \right) \quad \checkmark \]
From our analysis of 1st order circuits we assume a solution
\[ x(t) = A \cdot e^{st} \]

Substituting
\[ A s^2 e^{st} + \frac{AR}{L} s e^{st} + \frac{A}{LC} e^{st} = 0 \]

or
\[ A e^{st} (s^2 + \frac{R}{L} s + \frac{1}{LC}) = 0 \]

\[ \Rightarrow s^2 + \frac{R}{L} s + \frac{1}{LC} = 0 \]

This eq. define 2 values for \( s \).

\[ S_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \]

\[ S_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \]

or
\[ S_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \]
\[ S_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \]

\( \alpha = \frac{R}{2L} \)
\( \omega_0 = \sqrt{\frac{1}{LC}} \)

\( S_1, S_2 \) are the "natural frequencies"

\( \alpha \) : damping factor \hspace{1cm} \omega_0 : resonant freq.
The total solution for \( i(t) \) is a linear combination of these 2 solutions.

\[ i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \]

There are 3 cases or 3 different types of solutions:

\( \lambda > \omega_0 \rightarrow \text{overdamped circuit} \)

\( \lambda = \omega_0 \rightarrow \text{critically damped} \)

\( \lambda < \omega_0 \rightarrow \text{underdamped} \)

1) Overdamped. If \( \lambda > \omega_0 \) both roots are negative and real. Thus,

\[ i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \]

Current approaches to zero for \( t \rightarrow \text{increasing} \).

2) Critically damped \( \lambda = \omega_0 \rightarrow \frac{R}{2L} = \sqrt{\frac{1}{LC}} \)

\[ \Rightarrow \quad C = \frac{4L}{R^2} \quad \text{and} \quad s_1 = s_2 = -\frac{R}{2L} \]
In this case, the solution is
\[ i(t) = A_1 e^{-\alpha t} + A_2 e^{-\alpha t} = A_3 e^{-\alpha t} \]

This is wrong! Because 2 initial conditions cannot be satisfied by one constant. There is an error!

Let us go back to the initial equation
\[ \frac{d^2 i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{1}{LC} = 0 \]

When \( \lambda = \omega_0 = \frac{R}{2L} = \sqrt{1/LC} \)
\[ \frac{di}{dt} + 2\lambda \frac{di}{dt} + \lambda^2 i = 0 \]

On
\[ \frac{d}{dt} \left( \frac{di}{dt} + \alpha i \right) + \lambda \left( \frac{di}{dt} + \alpha i \right) = 0 \]

Can be written
\[ \frac{d}{dt} f + \lambda f = 0 \implies f = A_1 e^{-\alpha t} \]

Thus
\[ \frac{di}{dt} + \alpha i = A_1 e^{-\alpha t} \]
\[ \implies e^{\alpha t} \frac{di}{dt} + e^{\alpha t} \alpha i = A_1 \]
\[
\frac{d}{dt} (e^{\alpha t} i) = A_1
\]

Integrating

\[
e^{\alpha t} i = A_1 t + A_2
\]

or

\[
i'(t) = (A_1 t + A_2) e^{-\alpha t}
\]

3) Undamped circuit, \( \alpha < \omega_0 \)

In this case, the roots of the equation \( S_1 \) and \( S_2 \) are imaginary,

\[
S_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d
\]

\[
S_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d
\]

where \( j = \sqrt{-1} \) \( \omega_d = \sqrt{\omega_0^2 - \alpha^2} \) damping frequency.

In this case

\[
i(t) = A_1 e^{-(\alpha - j\omega_d) t} + A_2 e^{-(\alpha + j\omega_d) t}
\]

\[
= e^{-\alpha t} (A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t})
\]
Using Euler's identity

\[ n(t) = e^{-\alpha t} \left( B_1 \cos \omega_d t + B_2 \sin \omega_d t \right) \]

This is a oscillatory current with exponential decay. The decay has a time constant \(1/\alpha\). The oscillatory current has a period \(T = 2\pi / \omega_d\).
Source Free parallel RLC circuit

\[ R \begin{array}{c}
\ \uparrow \\
- \end{array} \begin{array}{c}
\ \downarrow \\
\ 1/2L \\
\ = \ \frac{1}{C} \\
\ - \end{array} \begin{array}{c}
\ \uparrow \\
+ \end{array} u(t) \]

We assume an initial current \( I_0 \) and a voltage \( V_0 \).

\[ i(0) = I_0 = \frac{1}{L} \int V(t) \, dt \]

\[ V(0) = V_0. \]

Using KCL in the upper node:

\[ \frac{V}{R} + \frac{1}{L} \int V(t) \, dt + C \frac{dV}{dt} = 0 \]

\[ 0 \rightarrow \ \frac{d^2 V}{dt^2} + \frac{dV}{dt} \cdot \frac{1}{RC} + \frac{1}{LC} \ V = 0 \]

Following the same reasoning used in series RLC circuits:

\[ s^2 + \frac{1}{RC} \ s + \frac{1}{LC} = 0 \]

Thus:

\[ s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \]

And defining

\[ \omega_0 = \sqrt{\frac{1}{LC}} \]

\[ s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \]

1. Overdamped: \( \alpha > \omega_0 \rightarrow s_{1,2} \text{ negatives and real} \)

\[ V(t) = A_1 e^{s_{1t}} + A_2 e^{s_{2t}} \]
2) Critically damped, \( \alpha = \omega_0 \), \( L = 4R^2C \)
Both roots are real and equal.
\[
V(t) = (A_1 + A_2 t) e^{-\alpha t}
\]

3) Underdamped, \( \alpha < \omega_0 \)
Roots are complex \( S_{1,2} = \alpha \pm j \omega_d \)
where \( \omega_d = \sqrt{\omega_0^2 - \alpha^2} \)
\[
V(t) = e^{-\alpha t} \left( A_1 \cos \omega_d t + A_2 \sin \omega_d t \right)
\]
\( A_1 \) and \( A_2 \) in each case are obtained from the initial conditions. We need
\[ V(0) \quad \text{and} \quad \frac{dV(0)}{dt} \]
\[ V(0) = V_0 \] is a demand.
To find \( \frac{dV(0)}{dt} \) we write the equation for \( t=0 \)
\[
\frac{V_0}{R} + I_0 + C \cdot \frac{dV(0)}{dt} = 0 \rightarrow \frac{dV(0)}{dt} = -\frac{(V_0 + RI_0)}{RC}
\]

**Example**

\[
\begin{array}{c}
\text{Example} \\
\end{array}
\]

\[
\begin{array}{c}
\text{R} \quad \text{L} \quad \text{C} \\
\hline
\end{array}
\]

\[
\begin{array}{c}
V(0) = 5V \\
I(0) = 0 \\
L = 1H \\
C = 10 \text{ mF} \\
R = 1.923, 5, 6.25
\end{array}
\]
Case 1 \( R = 1.923 \Omega \)

\[ \lambda = \frac{1}{2RC} = \frac{1}{2 \times 1.923 \times 10^{-3}} = 26. \]

\[ \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 10^{-3}}} = 10 \]

Thus \( \lambda > \omega_0 \) \( \rightarrow \) overdamped circuit

The roots of the equation are \( \lambda_1, \lambda_2 = -\lambda \pm \sqrt{\lambda^2 - \omega_0^2} = -50 \)

Thus the solution is \( v(t) = A_1 e^{-\lambda t} + A_2 e^{-50t} \)

The constants \( A_1 \) and \( A_2 \) are obtained from the initial conditions.

\[ v(0) = 5V = A_1 + A_2 \quad (1) \]

\[ \frac{dv(0)}{dt} = -\frac{v(0) + R i(0)}{RC} = -\frac{5 + 0}{1.923 \times 10^{-3}} = -260 \]

Differentiating \( v(t) : \frac{dv(t)}{dt} = -2A_1 e^{-\lambda t} - 50A_2 e^{-50t} \)

At \( t = 0 \)

\[ -2A_1 - 50A_2 = -260 \quad (2) \]

From (1) and (2) equate the constants \( \begin{cases} A_1 = -0.2083 \\ A_2 = 5.208 \end{cases} \)

Case 2 \( R = 5 \Omega \)

\[ \lambda = \frac{1}{2RC} = \frac{1}{2.5 \times 10^{-3}} = 10 \]

\[ \omega_0 = 10 \]

Thus \( \lambda = \omega_0 \) \( \rightarrow \) critically damped.
For a critically damped circuit \( S_1 = S_2 = -10 \)
\[ v(t) = (A_1 + A_2 t) e^{-10t} \]

Using the initial conditions,
\[ v(0) = 5V = A_1, \quad \checkmark \]

\[ \frac{dv(0)}{dt} = - \frac{v(0) + Ri(0)}{RC} = - \frac{5 + 0}{R5, 10 \times 10^{-3}} = -100 \]

Differentiating \( v(t) \)
\[ \frac{dv(t)}{dt} = (-10A_1 - 10A_2 t + A_2) e^{-10t} \]

at \( t = 0 \)
\[ -10A_1 + A_2 = -100 \rightarrow A_2 = -50 \]

Case 3 \( R = 6.25 \Omega \)
\[ \lambda = \frac{1}{2RC} = \frac{1}{2 \times 6.25 \times 10^{-3}} = 8 \quad \omega_0 = 10 \]

Thus \( \lambda < \omega_0 \rightarrow \text{underdamped} \).

The roots are
\[ \alpha_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega_0^2} = -8 \pm j6 \]

Thus
\[ \omega_d = \sqrt{\omega_0^2 - \lambda^2} = 6 \]

Thus
\[ v(t) = (A_1 \omega_d 6t + A_2 \sin 6t) e^{-8t} \]

And we find \( A_1 \) and \( A_2 \) as
\[ v(0) = 5V = A_1, \quad \checkmark \]
\[
\frac{dV(0)}{dt} = - \frac{V(0) + Ri(0)}{RC} = - \frac{5 + 0}{6.25 \times 10^{-3}} = -80
\]

Differentiating the expression of \(V(t)\)

\[
\frac{dV}{dt} = (-8A_1 \omega_0 b + 6t - 8A_2 \sin \omega t - 6A_1 \sin \omega t + 6A_2 \omega b t) e^{-8t}
\]

and evaluating at \(t = 0\)

\[
-8A_1 + 6A_2 = -80
\]

Notice that increasing \(R\), the level of damping decreases.