Charge and Current

\[ \text{electrons } e^- \quad \text{protons } p^+ \] \[ 1.602 \times 10^{-19} \text{ C} \]

A simple (simpler) circuit: source element (for example: battery) wires.

The current \( I \) (or \( i \)) is defined as the flow of \( + \) charge (it is opposite to the flow of electrons).

Current: the time rate of change of electric charge
\[ i = \frac{dq}{dt} \quad q : \text{charge} \]
\[ i : \text{current} \]

Units of current: A (ampere)
\[ A = \frac{\text{Coulomb}}{\text{second}} \text{ (the unit of electric charge)} \]
— When the current is constant in time we call "direct current" or dc
— When the current changes in time like a sin or a cos — alternating current or ac

Example of alternating current

\[ i = i_0 \sin (\omega t) \]  
\[ \text{time} \]
\[ \text{magnitude} \]  
\[ \text{frequency} \]

If we measure the current as a function of time we will see something like

In the analogy with a water (pipe) circuit:
current \leftrightarrow \text{gallons per second}
change \leftrightarrow \text{gallons of water}
Voltage: The current is produced by the movement of charges. To move a charge, we need to apply some work or (in other words) to transfer some energy. This work is performed by an external source or "electromotive force" represented typically by a battery. The emf is also known as a voltage or potential difference.

Definition: \[ V_{ab} = \frac{dW}{dq} \]

The voltage between two points a and b is the amount of energy to move a charge through an element.

\[ V = \lim_{\Delta q \to 0} \frac{\Delta W}{\Delta q} = \frac{J}{C} = \frac{N \cdot m \cdot \text{meter}}{C} \]

One Volt is the required energy to move a charge of one Coulomb a distance of 1 meter.
Power and Energy

Power is the time rate of expending or absorbing energy

\[ p = \frac{dW}{dt} = \frac{dW}{dq} \cdot \frac{dq}{dt} = V \cdot i \]

Units: power units are \textbf{Watts} : W

\[ W = J/s \]

Sign convention: Power can be supplied or absorbed by an element. Passive sign convention: When the current enters through the positive terminal of an element, \( p = +V \cdot i \).
If the current enters through the negative terminal \( p = -V \cdot i \).

Important: Power is a time-varying quantity. It is called "instantaneous power" and equals to the product of the voltage across the element times the current through the element.

Important: The algebraic sum of the powers absorbed or supplied in any electric circuit is zero.
Sources: Ideal sources maintain its nominal value always. There are two types of sources: voltage sources and current sources.

Voltage source symbol: \[\begin{array}{c}
\downarrow \\
\uparrow \\
\hline
\end{array} \] or \[\begin{array}{c}
\uparrow \\
\downarrow \\
\end{array} \]

Current source symbol: \[\begin{array}{c}
\circ \\
\uparrow \\
\end{array} \]

Sources can be independent or dependent: Independent sources have a fixed value. Dependent sources have a value that depends on the magnitude of other elements in the circuit. The symbols are:

Independent sources: \[\begin{array}{c}
\uparrow \\
\downarrow \\
\hline
\end{array} \] or \[\begin{array}{c}
\circ \\
\uparrow \\
\end{array} \]

Dependent sources: \[\begin{array}{c}
\circ \\
\downarrow \\
\end{array} \] or \[\begin{array}{c}
\circ \\
\uparrow \\
\end{array} \]

Energy:

\[ p = \frac{dW}{dt} \rightarrow W = \int p \cdot dt \]

\[ W = \int V \cdot i \cdot dt \]
Examples

\[ I = 5 \cdot \cos(60\pi t) \]
\[ V = 15 \cdot \cos(60\pi t) \]

What is the power at \( t = 3 \text{ ms} \)?

Current enters through the \( \oplus \) terminal \(\rightarrow\) \[ p = + V \cdot i \]

The sign of the power is \(\oplus\) thus the element is absorbing power

\[ p = + V \cdot i = 5 \cdot \omega (60\pi t) \cdot 15 \cdot \omega (60\pi t) \]
\[ = + 75 \cdot \omega^2 (60\pi t) \]

at \( t = 3 \text{ ms} \)

\[ p = 75 \cdot \omega^2 (60\pi \cdot 3 \times 10^{-3}) \]

Example: How much energy a 100 Watt bulb consume in 2 hours?

\[ p = \frac{dW}{dt} \rightarrow W = \int p \cdot dt \]

because the power is constant in time \(\rightarrow\)

\[ W = p \cdot Dt = 100 \cdot W \cdot 2 \cdot 3600 \text{ s} \]

energy \(= 720,000 \text{ J} \)
Example: To move a charge $q$ from $a$ to $b$ requires $-30$ J. Find $V_{ab}$

$q = 2.0 \text{ C}$

by definition: $V_{ab} = \frac{dW}{dq} = \frac{-30 \text{ J}}{2 \text{ C}} = -15 \text{ V}$

Example: A stove draws 15 A at 120 V. How long takes to consume 30,000 J?

$P = \text{V} \times \text{i} = 15 \text{ A} \times 120 \text{ V} = 1800 \text{ W}$

Because the power is constant, in time we can write

$P = \frac{\Delta W}{\Delta t} \rightarrow \Delta t = \frac{\Delta W}{P} = \frac{30 \times 10^3 \text{ J}}{1800 \text{ J/s}} = 16.6 \text{ s}$

Example: The charge entering an element is

$g (\text{mc})^2$

80

\[ 2 \quad 8 \quad 12 \quad t (\text{ms}) \]

Calculate I
\[ 0 \leq t \leq 1 \text{ms} \quad I = \frac{dq}{dt} = \frac{80}{2 \times 10^{-3} s} = \frac{80 \times 10^{-3} \text{C}}{2 \times 10^{-3} \text{s}} = 40 \text{ A} \]

\[ 1 < t < 8 \text{ms} \quad I = \frac{dq}{dt} = 0 \]

\[ 8 \text{ms} < t \leq 12 \text{ms} \quad I = \frac{dq}{dt} = \frac{0 - 80 \times 10^{-3} \text{C}}{(12 - 8) \times 10^{-3} \text{s}} = -20 \text{ A} \]
Basic laws

Ohm's law

Materials have the characteristic behavior of resisting the flow of charge. This characteristic is described by the "resistivity"

\[ \text{Resistivity } \rho \quad (\rho = \text{"rho"}) \]

The resistance depends on the resistivity of the material and also on the physical (size) characteristic.

\[ R = \rho \cdot \frac{L}{A} \]

We can build a Resistor with a material of resistivity \( \rho \). The symbol for a Resistor is

\[
\begin{array}{c}
\frac{1}{2} \\
\text{fixed value resistor}
\end{array}
\quad \text{or} \quad 
\begin{array}{c}
\frac{1}{3} \\
\text{variable resistor}
\end{array}
\]

Ohm's law: The voltage across a resistor is directly proportional to the current flowing through it.

\[ V = R \cdot i \]

Voltage \( \rightarrow \) resistance

Current \( \rightarrow \) resistor
Limit of Resistance: \( R = \frac{V}{i} \rightarrow [R] = \frac{[V]}{[i]} \)

The units of \( R \) are \( \text{Volt/Ampere} = \text{Ohm} : \Omega \)

\[ 1 \Omega = 1V/1A \]

Two limit conditions:
- Open circuit: \( i = 0 \)
- Short circuit: \( V = 0 \)

In an open circuit: \( R \rightarrow \infty \) and \( i \rightarrow 0 \)
In a short circuit: \( R \rightarrow 0 \) and \( V \rightarrow 0 \)

Another way to measure how well an element conducts electric current is the reciprocal of the resistance, known as "conductance".

\[
\text{Conductance} \quad G = \frac{1}{R} = \frac{i}{V}
\]

Units of conductance: Siemens: \( S = \frac{1}{\Omega} \)

or "mho": \( \Omega^{-1} \)

The power dissipated by a resistor can be expressed in terms of \( R \)

\[
p = V \cdot i = i^2 R = \frac{V^2}{R}
\]
Important remarks.

* The power dissipated in a resistor is a \textit{non-linear} function of \( i \) or \( V \).
* Since \( R > 0 \) \( \rightarrow \) the power dissipated in a resistor is \textit{always} \textit{positive}.

\underline{Example} Find the power dissipated in the resistor.

\[ P = V \cdot i = \frac{V^2}{R} = \frac{(30)^2}{5} = 180 \text{ W}. \]

\[ P = V \cdot i = \frac{V^2}{R} = (2 \times 10^{-3} \text{ A})^2 \times 10 \times 10^3 \Omega = 40 \text{ mW}. \]

In the circuits find (a) the current (b) the voltage.

Using Ohm's Law.

\[ i = \frac{V}{R} = \frac{30 \text{ V}}{5 \Omega} = 6 \text{ A} \quad (1^{st} \text{ circuit}) \]

\[ V = R \cdot i = 10 \times 10^3 \Omega \times 2 \times 10^{-3} \text{ A} = 20 \text{ V} \quad (\text{for} \ 2^{nd} \text{ circuit}) \]
Branch: Single element (e.g., resistor)

Node: Point of connection between 2 or more branches.

Example: How many nodes are in the following circuit?

Answer: 4 nodes. Notice that this circuit can be drawn as

Where probably the nodes are more clearly distinguished.

Loop: Any closed path in the circuit

Independent Loop: A loop is an independent loop if it contains at least one branch which is not part of other independent loops.
How we can calculate how many independent loops a circuit has? The number of independent loops are related to the number of nodes and branches through:

\[ b = n + l - 1 \]

number of branches \quad \text{number of nodes} \quad \text{number of independent loops.}

**Example**

\[ b = 5 \]
\[ n = 3 \]
\[ l = 3 \]

\[ 5 = 3 + 3 - 1 \quad \checkmark \quad \text{OK} \]

**Kirchhoff's laws**

\[ \begin{align*}
\text{KCL} & \quad (\text{K. Current Law}) \\
\text{KVL} & \quad (\text{K. Voltage Law})
\end{align*} \]

**KCL:** The algebraic sum of currents entering a node \( \quad \text{(or a closed boundary)} \) is zero

\[ \sum i_n = 0 \]
Example

\[ I_1 - I_2 - I_3 + I_4 + I_5 - I_6 = 0 \]

or a boundary

\[ I_1 - I_2 - I_3 = 0 \]

K.V.L: the algebraic sum of voltages around a closed path (or loop) is zero

\[ \sum V_m = 0 \]

Example

\[-V_1 + V_2 + V_3 + V_4 + V_5 = 0\]

Notice the sign. If you enter the element through terminal 1, the term is positive. Otherwise, it is negative.
Example

Find \( V_1 \) and \( V_2 \). If we apply KVL:

\[-20V + V_1 + V_2 = 0\]

According to Ohm's law:

\( V_1 = 2\Omega \times i \)

\( V_2 = 3\Omega \times i \)

Thus, we replace and we obtain:

\[-20V + 2i + 3i = 0 \implies i = \frac{20V}{(2+3)\Omega} = 4A\]

Now we can calculate:

\( V_1 = 2\Omega \times 4A = 8V \)

\( V_2 = 3\Omega \times 4A = 12V \)

Notice that:

\[-20V + 8V + 12V = 0 \quad \checkmark\] (KVL)

Example

Find \( V_0 \) and \( V_x \)

Applying KVL:

\[-35V + V_x + 2V_x - V_0 = 0\]

Using Ohm's law:

\( V_x = 10\Omega \times i \)

\( V_0 = 5\Omega \times (-i) \)

Thus:

\[-35V + 10i + 2 \times 10i - 5(-i) = 0 \implies i = 1A\]

\( \implies V_x = 10V \quad V_0 = -5V \)
**Example**

This circuit has 2 nodes. In the upper node we can apply KCL:

\[ 6A - i_0 - \frac{i_0}{4} - \frac{V_0}{8} = 0 \]

We also notice that

\[ V_0 = 2\Omega \times i_0 \]

Thus

\[ 6A - i_0 - \frac{i_0}{4} - \frac{i_0}{4} = 0 \Rightarrow i_0 = 4A \]

And \( V_0 = 2\Omega \times 4A = 8V \).

**Series Resistors**

Resistors are connected back to back. Thus they share the same current.

Resistors are connected in series when they have the same current.

Thus

\[ i_1 = i_2 = i_3 \ldots = i \Rightarrow \frac{V_1}{R_1} = \frac{V_2}{R_2} = \ldots = \frac{V_n}{R_n} \]

and according to KVL:

\[ V = V_1 + V_2 + V_3 + \ldots \]
Thus \( V = R_1i + R_2i + R_3i + \ldots + R_Ni \)
\[ = i [R_1 + R_2 + R_3 + \ldots + R_N] = i R_{eq} \]

\( R_{eq} \) is the **equivalent resistance** of the **series** of resistors.

\[ R_{eq} = \sum_{i=1}^{n} R_i \]

If we have 2 resistors:

\[ \begin{align*}
V & = i R_1 + i R_2 \\
V & = \frac{V}{R_1 + R_2}
\end{align*} \]

Thus
\[ \begin{align*}
V_1 & = i R_1 = V \cdot \frac{R_1}{R_1 + R_2} \\
V_2 & = i R_2 = V \cdot \frac{R_2}{R_1 + R_2}
\end{align*} \]

\( \left\{ \text{Voltage "divider"} \right\} \)

**Parallel resistors — Current divider**

Two resistors are **connected in parallel** when they are connected to the same **two nodes**. In resistors in parallel, the **voltage across is the same**

**Example**

\[ \begin{align*}
V & = \frac{1}{R_1 + R_2} \cdot (V_1 + V_2) \\
V & = \frac{1}{R_1 + R_2} \cdot (V_1 + V_2)
\end{align*} \]
We can calculate the current through each resistor.

\[ i_1 = \frac{V}{R_1} \quad i_2 = \frac{V}{R_2} \]

and apply KCL.

\[ i = i_1 + i_2 = \frac{V}{R_1} + \frac{V}{R_2} = V \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \]

We can write this as

\[ i = \frac{V}{\frac{1}{R_1} + \frac{1}{R_2}} \]

Where \( \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \)

\( R_{eq} \) stands for “equivalent resistor” - The two resistors connected in parallel behave as one equivalent resistor with the value

\[ R_{eq} = \frac{R_1 \cdot R_2}{R_1 + R_2} \]

This is:

\[
\begin{figure}
\begin{center}
\includegraphics[width=0.5\textwidth]{parallel_resistors}
\end{center}
\end{figure}
\]

In general, if we have \( N \) resistors in parallel

\[ \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N} = \sum_{i=1}^{N} \frac{1}{R_i} \]
**Current divider**  We obtained \( V = i \cdot R_{eq} = i \cdot \frac{R_1 R_2}{R_1 + R_2} \)

If we remember that \( i_1 = \frac{V}{R_1}, \) \( i_2 = \frac{V}{R_2} \)

Combining both equations,

\[
i_1 = i \cdot \frac{R_2}{R_1 + R_2} \quad i_2 = i \cdot \frac{R_1}{R_1 + R_2}
\]

**Current divider.**

In a parallel, the current splits in the different branches following the current divider equation.

**Example**  Find the \( R_{eq} \) of the following network.

![Image of a parallel circuit](image)

Let us go in steps.

1. Replace the last 3 resistance by a SERIES —
   
   Series: \( 4 \Omega + 5 \Omega + 3 \Omega = 12 \Omega \)

   We get

   ![Updated circuit diagram](image)
(2) Now we can observe that the 4Ω and the 12Ω resistors are connected in parallel.

\[
4\Omega \parallel 12\Omega = \frac{4 \cdot 12}{4 + 12} = 3\Omega
\]

means "parallel"

Thus we can write:

\[
\text{SERIES: } 3\Omega + 3\Omega = 6\Omega
\]

Thus we have:

(3) In this circuit we observe that the 2 resistors (3Ω) are in series.

Now we see that the 2 6Ω resistors are in parallel.

\[
6\Omega \parallel 6\Omega = \frac{3 \cdot 6}{3 + 6} = 3\Omega
\]

We have:

(4) Finally, there are 3 resistors in series.

\[
\text{SERIES: } 2\Omega + 3\Omega + 1\Omega = 6\Omega
\]
How to calculate current and charge

If we know how the charge changes in time, the current is calculated taking the time derivative.

Example: \( q(t) = 2 \sin(30\pi t) \)

\[ i(t) = \frac{dq}{dt} = 60\pi \omega (30\pi t) \]

If we know how the current changes in time, we calculate the charge accumulated taking the integral. However, we also have to consider the charge that the component holds at \( t = 0 \). (This is \( q(0) \)).

Example: Calculate the charge flowing into a device if

\[ i(t) = \sin(t + \frac{\pi}{6}) \text{A} \] and \( q(0) = 0.1 \text{C} \).

\[ q(t) = \int_0^t i(t) \, dt + q(0) \]

\[ = \int_0^t \sin(t + \frac{\pi}{6}) \, dt + q(0) \]

\[ = -\cos(t + \frac{\pi}{6}) \bigg|_0^t + q(0) \]

\[ = -\cos(t + \frac{\pi}{6}) + \frac{\sqrt{3}}{2} + q(0) \]

Thus,

\[ q(t) = \left[ -\cos(t + \frac{\pi}{6}) + \frac{\sqrt{3}}{2} + 0.1 \right] \text{C} \]
Wye - Delta transformation.

It may occur that a set of 3 resistors are not connected in series nor in parallel. For example

If we want to obtain the equivalent resistance of this network, we can see that the rules for parallel and series equivalent are not longer useful.

This can be solved using the 3 terminal equivalent network. These are the Wye (Y) and T networks.

or

\[
\begin{align*}
\text{Wye} & \quad \text{T} \\
1 & \quad 3 \\
2 & \\
\end{align*}
\]

\[
\begin{align*}
\text{Wye} & \quad \text{T} \\
1 & \quad 3 \\
2 & \\
\end{align*}
\]
Conversion Rules

Consider three nodes a, b, c. If we have a Δ configuration \((Ra, Rb, Rc)\) it can be replaced by a Y configuration \((R_1, R_2, R_3)\) and vice versa.

The equations to transform from \(Δ \leftrightarrow Y\) are:

**From Δ to Y**

\[
R_1 = \frac{R_b R_c}{R_a + R_b + R_c}
\]

\[
R_2 = \frac{R_c R_a}{R_a + R_b + R_c}
\]

\[
R_3 = \frac{R_a R_b}{R_a + R_b + R_c}
\]

**From Y to Δ**

\[
Ra = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_1}
\]

\[
R_b = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_2}
\]

\[
R_c = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_3}
\]
Example

In the following circuit

\[ \begin{array}{c}
\text{120 V} \\
| \\
\downarrow \\
\text{12.5} \\
| \\
\downarrow \\
\text{15} \\
| \\
\downarrow \\
\text{20} \\
\end{array} \]

Find the current \( i \). To solve this problem one possibility is to replace the resistor network by a single equivalent resistor such that the circuit will be

\[ \begin{array}{c}
\text{120 V} \\
| \\
\downarrow \\
\text{120} \\
| \\
\downarrow \\
\text{Ref} \\
\end{array} \]

and thus, \( i = \frac{120}{\text{Ref}} \).

In this circuit we identify resistors which are in a \( \Delta \) configuration. For example, the 5 \( \Omega \), 10 \( \Omega \) and 20 \( \Omega \) resistors are in a \( \Delta \) configuration.

If we set 
\[ R_1 = 10 \ \Omega \quad R_2 = 20 \ \Omega \quad R_3 = 5 \ \Omega \]

we can calculate the \( \Delta \) configuration as

\[ R_a = 35 \ \Omega \quad R_b = 17.5 \ \Omega \quad R_c = 70 \ \Omega \]

the circuit is thus equivalent to

\[ \begin{array}{c}
\text{120 V} \\
| \\
\downarrow \\
\text{12.5} \\
| \\
\downarrow \\
\text{15} \\
| \\
\downarrow \\
\text{70} \\
\end{array} \]
In this network we identify that

\[ 70 \Omega \parallel 30 \Omega = 21 \Omega \]

\[ 12.5 \Omega \parallel 17.5 \Omega = 7.292 \Omega \]

\[ 15 \Omega \parallel 35 \Omega = 10.5 \Omega \]

The network is equivalent to

\[ 120 \text{V} \]

\[ \frac{7.292}{2.1} \]

\[ \frac{10.5}{2.1} \]

And here it is clear that \( R_{eq} = \frac{(7.292 + 10.5)}{2.1} \)

\[ = 9.632 \Omega \]

Finally the circuit is

\[ 120 \text{V} \]

\[ 9.632 \Omega \]

And the current is

\[ i = \frac{120 \text{V}}{9.632 \Omega} = 12.458 \text{A} \]

This is not the only way to solve this problem — let us go back to the original circuit, and now let us consider a \( \Delta \) configuration with the resistors \( 10 \Omega, 5 \Omega \) and \( 12.5 \Omega \). If we set

\[ R_c = 10 \Omega \quad R_a = 5 \Omega \quad R_b = 12.5 \Omega \]
We can calculate the equivalent resistance in the Y.

We have

\[
\begin{array}{c}
12.5 \\
5 \\
10
\end{array}
\quad =
\begin{array}{c}
4,545 \\
2,273 \\
1,818
\end{array}
\]

If we replace the Δ by the Y (in the same 3 nodes), the original circuit becomes:

\[
\begin{array}{c}
120 \\
4,545 \\
2,273 \\
315 \\
20
\end{array}
\]

Here it is clear that 2.273 is in series with 1.818 and 20.

These 2 series are in // and the set is in series with 4.545.

The resultant is in // with 20.

This is

\[
Ref = \left[ \left\{ \frac{(2.273 + 15)\parallel (1.818 + 20)}{20} \right\} + 4.545 \right] \parallel 30
\]
Solving $R_{eq} = 9.63 \Omega$, this is the same result.

**Example** Christian tree lighting possibilities: series or in parallel.

Solve the following case:

We have 10 bulbs with a power rating 40 W that can be connected in parallel and 10 bulbs with a power rating 40 W that can be connected in series. If the voltage in the plug is 110 V, calculate the current through each bulb in each case.

A) Parallel.

We have:

$$P = \frac{V^2}{R} = I^2 \cdot R = 40 \text{ W} \implies \frac{110^2}{R} = 40 \implies R = 362.5 \Omega$$

Thus $I^2 \cdot R = 40 \implies I = \frac{40}{362.5} = 0.132 \implies I = 0.364 \text{ A}$
If we connect 10 times
\[ 110 \quad \text{V} \]
\[ P = I^2 \cdot R = \frac{V^2}{R} = 40 \, \text{W} \]

All resistors (bulbs) are identical. If we apply KVL, the voltage drop in each bulb is \( 110 \, \text{V} \div 10 = 11 \, \text{V} \)

Thus -
\[ P = I^2 \cdot R = \frac{V^2}{R} = 40 \, \text{W} \]

\[ V = 11 \, \text{V} \]
\[ P = \frac{11^2}{R} = 40 \, \text{W} \Rightarrow R = 3.025 \, \Omega \]

Now
\[ I^2 R = 40 \, \text{W} \Rightarrow I^2 = \frac{40 \, \text{W}}{3.025 \, \Omega} = 13.223 \, \text{A}^2 \]

\[ \Rightarrow I = 3.64 \, \text{A} \]
Methods of Analysis

Nodal Analysis

1. Select a node of reference
2. Apply KCL
3. Solve the equations.

Example

In Node 1, we apply KCL.

\[ 1A = i_1 + i_2 = \frac{V_1 - 0}{2 \Omega} + \frac{V_1 - V_2}{6 \Omega} \]  

In Node 2, using KCL.

\[ -i_2 + i_3 + 4A = 0 \Rightarrow -\frac{V_1 - V_2}{6 \Omega} + \frac{V_2 - 0}{7 \Omega} + 4 = 0 \]

We got 2 equations (*) and (**) with 2 variables \((V_1, V_2)\)

Solving this set of linear equations

\[ \begin{align*}
\frac{V_1}{2} + \frac{V_1 - V_2}{6} &= 1 \\
-\frac{V_1 - V_2}{6} + \frac{V_2}{7} &= -4
\end{align*} \]

Solving \(V_1 = -2V\)

\(V_2 = -14V\)
To remember: in a resistor current flows from a higher potential to a lower potential.

\[ V_1 \rightarrow i \rightarrow V_2 \]

\[ i = \frac{V_1 - V_2}{R} \]

\( V_1 > V_2 \)

Example

In this circuit we can identify 3 nodes: 1, 2, 3.

In Node 1

\[ 10 A = i_1 + i_2 = \frac{V_1 - V_2}{3} + \frac{V_1 - V_3}{2} = 10 \]

In Node 2

\[ i_1 + 4i_x - i_x = 0 \Rightarrow i_1 + 3i_x = 0 \]

\[ \Rightarrow \frac{V_1 - V_2}{3} + 3 \cdot \frac{V_2 - 0}{4} \]

In Node 3

\[ i_2 - 4i_x - i_3 = 0 \Rightarrow \frac{V_1 - V_3}{2} - 4 \cdot \frac{V_2 - 0}{4} - \frac{V_3 - 0}{6} = 0 \]

There are 3 eqs. with 3 unknowns.
Nodal Analysis with Voltage Sources.

Consider the following circuit.

We have 3 nodes, 1, 2, 3

The 10V source is connected between Node 1 and ground

\[ V_1 = 10V \] - The source directly fixes the voltage of Node 1

To find the voltages of Nodes 2 and 3 we try to apply KCL. However, it is not possible to determine the current flowing through the 5V source. We solve the dilemma considering a "Super Node".

We must use a Super node when we have non-referenced voltage sources (or "floating") voltage sources.

The super node is defined as a surface that encloses floating voltage sources and any element connected in parallel with it.
In the example the supernode is just the 5V source.

\[
\begin{align*}
\frac{V_2}{i_1} &= \frac{V_3}{i_2} = \frac{5V}{i_4} \\
i_1 - i_2 + i_4 - i_3 &= 0
\end{align*}
\]

Now we can apply KCL to the supernode

\[
i_1 - i_2 + i_4 - i_3 = 0
\]

And replacing the currents using Ohm's law

\[
\begin{align*}
i_1 &= \frac{10V - V_2}{2\Omega} \\
i_2 &= \frac{V_2 - 0}{8\Omega} \\
i_3 &= \frac{V_3 - 0}{6\Omega} \\
i_4 &= \frac{10V - V_3}{4\Omega}
\end{align*}
\]

\(\ast\) as one equation with two unknowns \((V_2, V_3)\)

The extra equation we need to solve the circuit is obtained from the supernode (using KVL)

\[
V_2 = V_3 + 5V
\]

\(\times\) is the equation we need to solve the circuit.
Mesh Analysis

To use this method we must define auxiliary "mesh currents" that in turn allow to calculate the real branch currents.

Steps

1. Assign mesh currents to the n meshes.
2. Apply KVL in each mesh. Use Ohm's law to express the voltages in terms of the mesh currents.
3. Solve the equations.

Example

![Image of a circuit diagram]

We define 2 meshes and 2 mesh currents, $i_1$ and $i_2$. Applying KVL in the meshes.

Mesh 1

$-12V + i_1 2\ \Omega + i_1 12\ \Omega - i_2 12\ \Omega + i_1 4\ \Omega = 0$

$\Rightarrow \quad i_1 18\ \Omega - i_2 12\ \Omega = 12V \quad \bigstar$

Mesh 2

$+8V + i_2 3\ \Omega + i_2 12\ \Omega - i_1 12\ \Omega + i_2 9\ \Omega = 0$

$\Rightarrow \quad i_1 12\ \Omega - i_2 24\ \Omega = 8V \quad \bigstar\bigstar$
\( \square \) and \( \square \) are 2 equations with 2 unknowns, the mesh currents \( i_1 \) and \( i_2 \).

Solving the equations \( i_1 = \frac{2}{3} \) \( i_2 = 0 \).

Now we can calculate the branch currents.

The current in the 2\,\Omega\,\text{resistor is } i_1 \#

\( i_1 = \frac{9}{3} \) \( i_2 = 0 \)

\( i_1 = \frac{12}{3} \) \( i_2 \)

\( i_1 - i_2 \)

(This particular example from the book isn’t very useful because one mesh current is zero).
Mesh analysis with current sources.

We can have 2 cases:

a) The current source is only in one mesh. In this case, the current source defines one of the mesh currents.

Example

\[ \begin{align*}
\text{10V} & \quad 4Ω & \quad 3Ω & \quad 6Ω & \quad 5A \\
& \quad \downarrow \quad i_1 & \quad \downarrow \quad i_2 & \quad \downarrow \quad i_3
\end{align*} \]

We define 2 mesh currents, \( i_1 \) and \( i_2 \).

The current source 5A defines \( i_2 = -5A \).

For the other mesh we have the equation

\[-10V + i_1 \cdot 4 + i_1 \cdot 6 - 6Ω \cdot (-5A) = 0\]

which gives \( i_1 = -2A \).

b) The current source exists between 2 meshes. In this case we should define a "super mesh".

Example

\[ \begin{align*}
\text{2Ω} & \quad \rightarrow \quad \text{super mesh} \\
\text{5A} & \quad \rightarrow \quad \text{super mesh} \\
\text{6Ω} & \quad \rightarrow \quad \text{super mesh} \\
\text{4Ω} & \quad \rightarrow \quad \text{super mesh} \\
\text{2Ω} & \quad \rightarrow \quad \text{super mesh} \\
\text{5Ω} & \quad \rightarrow \quad \text{super mesh} \\
\text{10V} & \quad \rightarrow \quad \text{super mesh}
\end{align*} \]
In this circuit, the 5A source and the 3Io source is shared between 2 meshes. We cannot determine the mesh currents and a "super mesh" is necessary. The supermesh includes dependent and/or independent current sources and anything connected in series within it.

The supermesh is defined by the dotted line in the circuit.
Thus to solve the circuit we can write the following equations:

\[ 2i_4 + 8(i_4 - i_3) + 10 = 0 \quad \text{(in the 4th mesh)} \]

\[ 2i_1 + 4i_3 + 8(i_3 - i_4) + 6i_2 = 0 \quad \text{(in the super mesh)} \]

These are 2 equations with 4 unknowns. The remaining 2 extra equations can be obtained using K.C.L in the nodes \( P \) and \( Q \).

For the node \( P \): \( i_2 - i_1 - 5A = 0 \)

For the node \( Q \): \( i_3 + 3Io - i_2 = 0 \)

These 2 equations complete the set of 4 equations.

Solving for \( i_1 \rightarrow i_4 \) obtain:

\[ i_1 = -7.5A \]
\[ i_2 = -2.5A \]
\[ i_3 = 3.93A \]
\[ i_4 = 2.143A \]
Examples

Use Mesh Analysis to find $I_0$.

1) Define mesh currents: $i_1, i_2, i_3$
2) Apply KVL in the meshes.

\[
\begin{align*}
-24 + (i_1 - i_2) \cdot 10 + (i_1 - i_3) \cdot 12 &= 0 \\
(i_2 - i_1) \cdot 10 + i_2 \cdot 24 + (i_2 - i_3) \cdot 4 &= 0 \\
4I_0 + (i_3 - i_1) \cdot 12 + (i_3 - i_2) \cdot 4 &= 0
\end{align*}
\]

(Notice that we are neglecting units for simplicity).

Using KCL we can also write $I_0 = i_1 - i_2$

If we replace the set of equations are

\[
\begin{align*}
-24 + 10i_1 - 10i_2 + 12i_1 - 12i_3 &= 0 \\
10i_2 - 10i_1 + 24i_2 + 4i_2 - 4i_3 &= 0 \\
4i_1 - 4i_2 + 12i_3 - 12i_1 + 4i_3 - 4i_2 &= 0
\end{align*}
\]
On

\[
\begin{align*}
22i_1 - 10i_2 - 12i_3 &= 24 \\
-10i_1 + 38i_2 - 4i_3 &= 0 \\
-8i_1 - 8i_2 + 16i_3 &= 0
\end{align*}
\]

This can be written in matrix format

\[
\begin{bmatrix}
22 & -10 & -12 \\
-10 & 38 & -4 \\
-8 & -8 & 16
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2 \\
i_3
\end{bmatrix}
= 
\begin{bmatrix}
24 \\
o \\
o
\end{bmatrix}
\]

\[\mathbf{M} \cdot \mathbf{I} = \mathbf{V} \quad \Rightarrow \quad \mathbf{I} = \mathbf{inv}(\mathbf{M}) \cdot \mathbf{V}\]

Example

\[
\begin{bmatrix}
6 & \mathbf{N} & 10
\end{bmatrix}
\]

1) Assign mesh currents \( i_1, i_2 \)

2) We have a current source between 2 meshes, thus we should use a super mesh.

The equation for the super mesh is

\[-20 + 6i_1 + 10i_2 + 4i_2 = 0 \quad \ast\]
The other equation we need is obtained applying KCL in the node N

\[ i_1 + 6 - i_2 = 0 \]

The 2 equations \( \bigstar \) solve the circuit. In this simple case we can use substitution

\[ i_2 = i_1 + 6 \quad \text{replacing in the other equation} \]

\[-20 + 6i_1 + 14i_1 + 84 = 0 \implies i_1 = -3.2 \text{ A} \]

and \[ i_2 = 2.8 \text{ A} \]

Example

Find the node voltages.

1) We define the reference node - This is the ground node.
2) We observe that the circuit has 4 other nodes.
   Because the 10V source we immediately obtain \( V_4 = 10 \text{ V} \)
3) There is a dependent voltage source (value 4 i_0) that is floating. Thus we need to apply a supernode.
The circuit with the supernode looks like:

Applying KCL in the supernode

\[1 + 2V_0 - \frac{V_1}{4} - \frac{V_2}{1} - \frac{V_1 - V_3}{1} = 0\]

\[V_0\] is the voltage drop in the 1Ω resistor (in the upper branch). We can write

\[V_0 = V_1 - V_3\]

Thus, replacing in the supernode equation

\[1 + 2V_1 - 2V_3 - \frac{V_1}{4} - \frac{V_2}{1} - \frac{V_1}{1} + \frac{V_3}{1} = 0\]

\[\Rightarrow 4 + 3V_1 - 4V_2 - 4V_3 = 0\]

This is the equation for the supernode. We need 2 more equations to solve the circuit.

Applying KVL in the supernode
\[ V_1 + 4i_0 = V_2 \implies \text{Notice that we are neglecting the units!! This equation in the super node says that the node 2 has a voltage } V_2 \text{ which is } (4i_0) \text{ volts higher than the voltage of node } 1. \]

From this equation:

\[ V_1 + 4i_0 = V_2 \implies V_1 + 4 \cdot \frac{V_3}{4} = V_2 \implies V_2 = V_1 + V_3 \quad \bigotimes \]

Finally, the third equation is obtained using KCL in the node 3:

\[ -2V_0 + \frac{V_1 - V_3}{1} - \frac{V_3}{4} + \frac{10 - V_3}{2} = 0 \]

\[ \implies -4V_1 + V_3 = -20 \quad \bigotimes \]

Summarizing, the set of equations to solve the circuit:

\[
\begin{cases}
3V_1 - 4V_2 - 4V_3 = -4 \\
-4V_1 + 0V_2 + V_3 = -20 \\
V_1 - V_2 + V_3 = 0
\end{cases}
\]
Superposition

"The voltage across (or the current through) an element in a linear circuit is the algebraic sum of the voltages across (or the current through) that element due to each independent source acting alone."

Method

1) "Turn off" all independent sources except one.
2) Find the output in the simplified circuit.
3) Repeat 1) for all the other sources.
4) Add all contributions.

Example: Calculate I.

\begin{circuit}\begin{tikzpicture}[scale=0.8, every node/.style={scale=0.8}]  \node (power1) at (0,0) {$\pm$};  \node (source1) at (1,0) {$G$};  \node (load1) at (2,0) {$Z$};  \node (load2) at (3,0) {$W$};  \node (load3) at (4,0) {$8$};  \node (ground) at (5,0) {$\pm$}; \end{tikzpicture}\end{circuit}

1) Turn off the sources one at the time.

\begin{circuit}\begin{tikzpicture}[scale=0.8, every node/.style={scale=0.8}]  \node (power1) at (0,0) {$\pm$};  \node (source1) at (1,0) {$G$};  \node (load1) at (2,0) {$W$};  \node (load2) at (3,0) {$4A$};  \node (load3) at (4,0) {$8$}; \end{tikzpicture}\end{circuit}
This circuit is very simple to solve (it's a current divider). Notice that it can be drawn in this way:

\[ I_1 = 4A \times \frac{2}{14+2} = 0.5\, A \]

Here we use KVL:

\[ I_2 = \frac{16\, V}{16\, \Omega} = 1\, A \]

Here we also use KVL:

\[ I_3 = -\frac{12\, V}{16\, \Omega} = -0.75\, A \]

Thus the current \( I \) in the original circuit is

\[ I = 0.5\, A + 1\, A - 0.75\, A = 0.75\, A \]

\[ \text{Example} \]

Find \( V_x \)
We can solve the circuit very easily using KCL (nodal analysis).

The equation for the upper node is

\[ 2 + 0.1V_x - \frac{V_x}{20} - \frac{V_x}{4} = 0 \rightarrow V_x = 10V. \]

b) Similarly, the equation for this circuit is

\[ 0.1V_x - \frac{V_x}{4} + \frac{10-V_x}{20} = 0 \rightarrow V_x = 2.5 \]

Thus, \( V_x \) in the original circuit is \( V_x = 10 + 2.5 = 12.5V \).
Source Transformation

\[ V_S \quad \leftrightarrow \quad I_S \] 
\[ V_S = I_S \cdot R \] 
\[ I_S = \frac{V_S}{R} \]

Example

Find \( V_0 \)

Taking the series 4 and 2

Now we can replace the 12V source by a current source.
In this circuit we notice that the current sources are connected in parallel. Thus the circuit is equivalent to

![Circuit Diagram]

This is $4 \uparrow + 2 \downarrow$. This is $6\Omega \parallel 3\Omega$.

In the last circuit we can calculate the current through the $8\Omega$ resistor very easily using current divider.

$$i = 2A \times \frac{2}{2+8} = 0.4A$$

Thus, $V_o = 0.4A \times 8\Omega = 3.2V$

Another way to solve (absolutely equivalent) is to calculate $8\parallel 2 = 1.6\Omega$

The voltage across the equivalent parallel is the same as $V_o$

$$V_o = 2A \times 1.6\Omega = 3.2V$$
Example

Find \( i_x \)

First we replace the voltage source by a current source.

We notice that the current sources are connected in parallel. Thus:

\[
4 - \frac{2i_x}{5}
\]

Here we can use a current divider.

\[
i_x = \left( 4 - \frac{2i_x}{5} \right) \times \frac{5}{5+10} \quad \Rightarrow \quad 3i_x = 4 - \frac{2i_x}{5}
\]

\[
\Rightarrow i_x = 1.176 \text{ A}
\]