Energy dissipation theories and optimal channel characteristics of river networks

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Abstract. The effects of energy dissipation on channel properties of a river network are explored. On the basis of a local and global hypothesis of optimality in energy expenditure, we investigate the relationships between channel hydraulic geometry, flow velocity, channel bed slope, and streamflow conditions in optimal river networks. Expressions for the rate of energy dissipation per unit channel area P_a are derived as functions of cumulative drainage area and river network parameters. Optimal channel characteristics are developed that satisfy the hypothesis of local optimality, and provide constant P_a throughout the river network. We show that these optimal channel characteristics are remarkably similar to those of many natural river systems in their downstream hydraulic geometry exponents, channel bed slope scaling, spatial distribution of average flow velocity, boundary shear, resistance to flow, etc. Optimal combinations of channel downstream hydraulic geometry and basin topography were analyzed on data from Goodwin Creek. We found ranges of optimality for the combination of the downstream hydraulic geometry exponent for width of *Leopold and Maddock* [1953] ($0.32 < b < 0.74$), and the channel bed slope scaling exponent ($-0.65 < z < -0.29$), and argue that river networks develop average channel properties within these ranges in order to attain constant P_a throughout the network. We propose that the hypothesis of local optimality is a central principle that explains the average behavior and adjustment of channel characteristics in natural river systems.

1. Introduction

River networks are dynamic systems with complex branched structures that exhibit a high degree of regularity and spatial organization. This regularity has inspired researchers to study the aggregation patterns of river networks and to search for fundamental principles that relate their structure and channel characteristics to the hydrology and sedimentology of their watersheds. The thesis emerged that drainage networks naturally evolve into structures that are most efficient in draining their watersheds [*Leopold and Langbein*, 1962]. Efficiency, in this context, was associated with work that a river network performs in transporting water and sediment, or with the rate and distribution of energy expenditure in the runoff process. It was postulated that river systems develop toward a state in which an approximate equilibrium between channel form and the imposed water and sediment load is produced [*Leopold and Maddock*, 1953]. It was argued that this is also a state of minimum energy expenditure [*Leopold and Langbein*, 1962; *Langbein and Leopold*, 1964]. Concepts of minimum energy expenditure have been applied to theoretically derive the downstream hydraulic geometry exponents of *Leopold and Maddock* [1953] observed in natural river systems [e.g., *Langbein*, 1964; *Williams*, 1978]. The conjecture that channels adjust toward an equilibrium state in which stream power or energy dissipation is minimum has also been successfully applied to numerous other fluvial problems [e.g., *Chang*, 1979; *Yang and Song*, 1986].

In addition to describing channels in equilibrium, the theory of optimal energy expenditure has been used to describe the structure and shape of optimal river networks [*Howard*, 1990] and to define a set of principles that govern the evolution of river networks to the optimal state [*Rodríguez-Iturbe et al.*, 1992]. In their seminal work, *Rodríguez-Iturbe et al.* [1992] postulated three principles of optimal energy expenditure that define the optimal topological structure of a network, as well as its channel characteristics, and proceeded to derive from them other important properties of drainage networks. Their work led to the definition and modeling of optimal channel networks (OCNs) that exhibit remarkable similarities with river networks extracted from digital elevation models (DEMs) in their fractal aggregation structure and other empirical geomorphological properties [*Rodríguez-Iturbe et al.*, 1992; *Rinaldo et al.*, 1992; *Ijjász-Vásquez et al.*, 1993; *Rigon et al.*, 1993; *Sun et al.*, 1994]. Models of OCNs describe the formation of optimal topological structures by minimizing the total rate of energy expenditure. River networks closely resembling OCNs have also been obtained without optimizing energy expenditure, using physically based models of water and sediment transport in network and catchment evolution [*Wilgoose et al.*, 1989] and models of self-organized criticality with a stable landscape dependent on a critical erosion threshold [*Rinaldo et al.*, 1993; *Rigon et al.*, 1994]. The principles of *Rodrígez-Iturbe et al.* [1992] are restated here in the form of a local and a global hypothesis of optimality in drainage network evolution. The local hypothesis states that a river network adjusts its channel properties toward an optimal state in which the rate of energy dissipation per unit channel area is constant throughout the network. The global hypothesis states that a river network adjusts its topological structure toward a state in which the total rate of energy dissipation in the network is minimum. An important feature

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of natural river networks is their dynamic nature. River systems respond to spatially and temporally variable external driving forces (discharge and sediment load) by adjusting their cross-sectional form, bed configuration, channel pattern, and bed slope. In this sense, the local optimality hypothesis addresses the short-term adjustment of internal channel geometry and roughness in response to short-term fluctuations in discharge and sediment load. On the other hand, the global optimality hypothesis addresses the long-term adjustment of the topological structure of the river network in response to geologic driving forces, continuous erosion, and long-term changes in the runoff amount and sediment supply. In addition to external driving forces, there are other constraints (such as geology, vegetation, human interference, etc.) that can control river network adjustment. It has been argued that a natural river will develop characteristic forms (such as average downstream hydraulic geometry) that are relatively stable on an appropriate timescale between short-term adjustment and long-term evolutionary tendencies [e.g., *Leopold and Maddock*, 1953; *Knighton*, 1984, p. 162; *Petts and Foster*, 1985]. In any natural river system, fluctuations will occur about the characteristic average behavior in response to fluctuations in the controlling variables and the resistance and susceptibility of the river system to change.

In this paper we concentrate mostly on the hypothesis of local optimality. We argue that if this hypothesis applies to drainage networks, then it must be traceable in the spatially and temporally averaged distribution of their channel characteristics (in particular the downstream hydraulic geometry relations of *Leopold and Maddock* [1953]). Our main objective is to develop relations between channel properties, discharge, sediment load, and the rate of energy expenditure throughout a river network, and from them derive so called optimal channel characteristics of river networks where the energy dissipation rate per unit channel area is constant. Our results from Goodwin Creek, Mississippi, indicate that the condition of a constant energy dissipation rate per unit channel area in an optimal river network can be attained by a complex interaction of channel geometry, topography, and discharge driving conditions that leads to channel properties remarkably similar to those observed in many natural river systems.

2. Rate of Energy Dissipation in a River Network

Rivers are nonconservative systems. In the runoff process, potential energy of water on the hillslopes is transformed into kinetic energy of the flowing fluid-sediment mixture in the river network. In this process, energy is dissipated from the system. It is generally accepted that fluvial systems perform work (1) against friction at the boundary, (2) against viscous shear and turbulence (internal friction), (3) in transporting sediment load and flood debris, and (4) in eroding the channel bed [*Knighton*, 1984, p. 54]. However, it is difficult to quantify energy dissipation rates associated with each process separately. In most rivers the main source of energy dissipation is due to friction of the fluid-sediment mixture at the boundary and can be computed from the power equation [e.g., *Yang and Song*, 1986; *Julien*, 1995, p. 42]. This section discusses the development of the energy expenditure relation and its application to a river network.

2.1. Energy Dissipation by Fluid in Motion

The power, or rate of work done by fluid in motion is a scalar quantity obtained by the product of the vectors of force and velocity. The nonconservative, dissipative term in the power equation for irrotational flow is a volume integral representing the rate at which mechanical energy is dissipated from the system (transformed into heat). It is the term that defines the work that the fluid element needs to do to overcome friction, and will be called the rate of energy dissipation, P_{diss} [e.g., *Molnár*, 1996]. After neglecting linear deformation by tension against the bed, banks, or surface, the final form of the rate of energy dissipation P_{diss} due to friction in an incompressible fluid under irrotational flow conditions can be expressed as

$$P_{\text{diss}} = \int_V \left[\tau_{xz}\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) + \tau_{xy}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) + \tau_{yz}\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right) \right] dV \tag{1}$$

where τ_{ij} are tangential and normal stresses, v_i are flow velocity components in the respective directions, and V is the control volume. To obtain the spatial distribution of the rate of energy dissipation in a river network using (1), assumptions need to be made about the flow conditions, as well as about the structure of the river network.

2.2. Application to a River Network

Assuming one-dimensional flow in the x direction leads to $v_y = v_z = 0$. Neglecting the effect of secondary flows is appropriate in the context of this study, where we are interested more in the large-scale behavior of a river system than in describing detailed hydraulics in a particular river cross-section. Transversal shear stresses are induced by secondary flows; therefore $\tau_{yz} = 0$ (for irrotational flow $\tau_{xy} = \tau_{yx}$ and $\tau_{xz} = \tau_{zx}$). Equation (1) then reduces to

$$P_{\text{diss}} = \int_V \left(\tau_{zx} \frac{\partial v_x}{\partial z} + \tau_{yx} \frac{\partial v_x}{\partial y} \right) dV \tag{2}$$

Thus the rate of energy dissipation is associated with bed and bank shear stresses, and with gradients of the velocity field in the vertical and horizontal directions. The product of these two terms, integrated over a control volume, gives the energy dissipation rate under given conditions.

A river network can be visualized as consisting of a series of rectangular channel links of different lengths, widths, and flow depths. Equation (2) can then be applied to any one of these rectangular channel links. We assume that in each link the bed shear stress τ_{zx} and the bank shear stress τ_{yx} are equal to an average boundary shear stress acting on both bed and banks, and that the velocity gradient near that boundary is equal to the vertical velocity gradient. We also assume that most energy dissipation occurs in a boundary layer of depth l and that within it, shear stress remains constant and equal to the average boundary shear stress τ_0. Under these assumptions, (2) reduces to

$$P_i = LP_w\tau_0 \int_0^l \frac{\partial v_x}{\partial z}\, dz = LP_w\tau_0 v(l) \tag{3}$$

where P_i is the energy dissipation rate of a fluid-sediment mixture in motion in link i; L and P_w are the channel link length and wetted perimeter, respectively; and $v(l)$ is the flow velocity as a function of the boundary layer. The depth of this layer l can be defined by the flow depth at which the actual velocity $v(l)$ equals the depth-averaged velocity v. Then (3) can be approximated as

$$P_i = LP_w \tau_0 v \tag{4}$$

This approximation is essential to the practical application of the energy dissipation theories to river networks. A depth-averaged velocity is fairly easily defined and determined from discharge and cross-sectional area, whereas a complete velocity distribution at every point in a river network is not.

Under uniform flow, boundary shear stress can be determined from

$$\tau_0 = \gamma_m R_h S \tag{5}$$

where γ_m is the submerged specific weight of the fluid-sediment mixture, R_h is the hydraulic radius of the link, and S is the link channel bed slope. The energy dissipation rate P_i then becomes

$$P_i = \gamma_m LSQ \tag{6}$$

where Q is discharge. All variables are particular to the studied link i. The total energy dissipation rate P_t in a river network composed of n links, is then

$$P_t = \sum_{i=1}^{n} P_i \tag{7}$$

The effect of sediment transport on energy expenditure in (6) is in the form of added weight to clear water. The fluid-sediment mixture then has a submerged specific weight

$$\gamma_m = g[\rho_s C_V + \rho_w(1 - C_V)] = g\ \rho_w[1 + C_V(G - 1)] \tag{8}$$

where g is gravitational acceleration, ρ_w and ρ_s are the mass densities of water and sediment, C_V is the volumetric sediment concentration, and G is the specific gravity of sediment ($G = \rho_s/\rho_w$; the specific gravity of quartz particles submerged in water is approximately equal to 2.65). Commonly, sediment concentrations are too low to add significant weight to the fluid-sediment mixture. However, sediment transport also affects the velocity profile in a river section and therefore the depth-averaged velocity in (4) [e.g., *Parker and Coleman*, 1986].

3. Hypotheses of Optimal Energy Expenditure in River Networks

Rodríguez-Iturbe et al. [1992] show that under certain assumptions, a joint application of optimality principles that relate the topological structure of a river network and its individual elements with the rate of energy expenditure in the system as a whole and each of its elements can be sufficient to explain the internal organization and the tree-like structure of a drainage network. The three principles of *Rodríguez-Iturbe et al*. [1992] are restated in this study as the following local and global optimal energy expenditure hypotheses.

Local optimal energy expenditure hypothesis: A river network adjusts its channel characteristics toward a state in which the rate of energy dissipation per unit channel area P_a is constant throughout the river network.

Global optimal energy expenditure hypothesis: A river network adjusts its topological structure toward a state in which the total rate of energy dissipation P_t in the river network is minimum.

The local and global optimality hypotheses define the optimal condition of a river network. In our work we concentrate on the hypothesis of local optimality as it applies to adjustment of channel geometry.

In this section we generalize the results of *Rodríguez-Iturbe et al*. [1992] to river networks with variable velocity and roughness, and discuss some implications of this generalization on total energy expenditure.

3.1. Velocity and Roughness in an Optimal Network

Rodríguez-Iturbe et al. [1992] state that the rate at which energy is expended due to friction of the fluid with the boundary and channel maintenance in a channel reach may be written as

$$P_i = C_f \rho_w \frac{v^2}{R_h} QL + K\tau_0^n P_w L \tag{9}$$

The first term on the right-hand side in (9) is the rate of energy expenditure due to friction with the boundary, with C_f as a resistance coefficient representing a measure of total roughness of the channel system. This term is identical to (6), with C_f related to the Darcy-Weisbach friction factor ff under conditions of no sediment transport as $C_f = ff/8$. The second term in (9) represents the rate of energy expenditure involved in the removal and transport of sediment which would otherwise accumulate in the channel. It has the form of a bed load sediment transport equation, with constants K and n depending on the soil and fluid properties. *Rodríguez-Iturbe et al*. [1992] assume that total roughness throughout a river system is approximately constant and apply the local optimal energy expenditure hypothesis of a constant energy dissipation rate per unit channel area

$$P_a = \frac{ff}{8} \rho_w v^3 + K\tau_0^n = \text{const} \tag{10}$$

to conclude that in an optimal river network, average velocity will be constant. They therefore limit their optimal river networks to those where roughness and flow velocity are constant.

It is clear that flow velocity, channel roughness, and geometry are closely related. If one neglects the channel maintenance term and allows channel roughness and velocity to vary, it follows from (10) that the local optimality hypothesis is satisfied when

$$ff\ v^3 = \text{const} \tag{11}$$

This interpretation defines an expanded class of optimal river networks by including networks with variable flow velocity and roughness. It is crucial to acknowledge this interconnectedness of flow velocity (through channel geometry) and channel roughness throughout the network in the context of the local optimal energy expenditure hypothesis.

The global optimal energy expenditure hypothesis requires that in an optimal river network the total rate of energy dissipation P_t be minimum, which in turn implies that the rate of energy dissipation in every link P_i is minimum. Minimizing (9) by letting $d(P_i)/dH = 0$ led *Rodríguez-Iturbe et al*. [1992] to the following scaling relationships between optimal channel depth, width, and discharge throughout the network:

$$H \propto Q^{0.5} \qquad W \propto Q^{0.5} \tag{12}$$

The scaling exponent 0.5 for both the optimal channel depth and width is a consequence of assuming constant velocity and roughness in the river network.

The effect of a consistent trend in velocity on the scaling exponent in an optimal river network can be shown by manipulating (9) and neglecting the energy term associated with channel maintenance to obtain

$$P_i = \frac{QL}{H}\frac{1}{v}\rho_w \tfrac{1}{8}[ff\ v^3] + HL\rho_w \tfrac{1}{4}[ff\ v^3] \qquad (13)$$

According to (11), the term in brackets in (13) representing a function of channel roughness and velocity is constant under the local hypothesis of optimal energy expenditure. Let us assume that average flow velocities in the river network exhibit a downstream trend with discharge of the form

$$v \propto Q^m \qquad (14)$$

where Q is discharge of the same frequency everywhere in the network and m is a constant for the network [*Leopold and Maddock*, 1953]. Then minimizing P_i generalizes the scaling relationships between optimal channel depth, width, and discharge of *Rodríguez-Iturbe et al.* [1992] to

$$H \propto Q^{0.5-(m/2)} \qquad W \propto Q^{0.5-(m/2)} \qquad (15)$$

Thus, in a river network that satisfies the local optimal energy expenditure hypothesis of a constant energy dissipation rate per unit channel area, the scaling relationships between channel depth, width, and discharge can be different from 0.5 (less than 0.5 assuming an overall increase in velocities downstream). Observations of average flow velocities in natural river systems show m ranging mostly between 0 and 0.3 [*Park*, 1977]. Although we neglected the channel maintenance term in (9) in the above analysis, it can readily be shown that the developed scaling relationships in (15) also hold for the condition where the channel maintenance term would be constant throughout the network (which is not a bad assumption for stable natural rivers). It should also be pointed out that a rectangular channel cross section and a constant width-depth ratio throughout the river network has been assumed ($b = f$). However, the scaling relationships in (15) also hold for trapezoidal cross sections where the width-depth ratio can vary throughout the network as a function of the bank declination angle [see *Rodríguez-Iturbe and Rinaldo*, 1997, p. 256].

3.2. Models of Optimal Channel Networks

By adding the rate of energy expenditure from all links of the network, we obtain the total rate of energy dissipation in a river network P_t. Assuming a scaling relationship between channel bed slope and discharge of the form [*Langbein and Leopold*, 1964; *Carlston*, 1968]

$$S \propto Q^{\zeta} \qquad (16)$$

P_t becomes

$$P_t = \sum_i P_i = \text{const} \sum_i SQ = \text{const} \sum_i Q^{\eta} \qquad (17)$$

where η is the energy exponent equal to

$$\eta = \zeta + 1 \qquad (18)$$

According to the global optimal energy expenditure hypothesis, P_t should be minimum in the optimal network. Therefore the topological arrangement of network links that drains a given area with the lowest possible value of P_t is defined as the OCN for that drainage area [*Rodríguez-Iturbe et al.*, 1992]. There are many topological solutions to this minimization problem. The model in (17), together with the hypothesis of global optimality, has been used by many researchers to simulate OCNs with topological characteristics and properties very similar to natural river systems (drainage area is commonly used as a surrogate for discharge) [e.g., *Rinaldo et al.*, 1992; *Ijjász-Vásquez et al.*, 1993; *Rigon et al.*, 1993; *Sun et al.*, 1994].

The energy scaling exponent η connects local energy expenditure principles with the overall topological structure of the OCN [e.g., *Sun et al.*, 1994; *Troutman*, 1996]. On the basis of (15) and the hypothesis of local optimality, it can readily be shown that for the specific condition $b = f$,

$$\eta = 0.5 - m/2 = b \qquad (19)$$

The role of the energy exponent η has been intensively studied in the OCN context. The original OCN definition assumes constant velocity $m = 0$ and therefore $\eta = 0.5$, $b = f = 0.5$ and $\zeta = -0.5$. In the context of our study, $\eta = 0.5$ provides a threshold in the distribution of average flow velocity in the OCN. For $\eta < 0.5$, velocity increases downstream, and for $\eta > 0.5$, velocity decreases downstream.

Rinaldo et al. [1992] and *Rigon et al.* [1993] indirectly relax the assumption of constant velocity in a network by simulating optimal networks using exponents η different from 0.5. On the basis of numerical experiments, *Rinaldo et al.* [1992] conclude that OCNs with $0 < \eta < 1$ tend to be visually characterized by similar aggregation patterns. For $\eta \geq 1$ or $\eta \leq 0$ they did not observe Hortonian features in simulated OCNs. The limiting case of $\eta = 0$ does not give any preference to aggregation and minimizes total channel length in the OCN context. The case of $\eta = 1$, on the other hand, leads to directed networks and minimizes the mean link length to the outlet in the OCN context. From what we have shown in (18) and (19), this would require slopes independent of discharge and a decrease in flow velocity downstream. Such conditions are indicative of unchannelized hillslope runoff, which was also concluded by *Rigon et al.* [1993] from their simulations of OCNs.

Troutman and Karlinger [1994] estimated the exponent η from river network data using a two-parameter Gibbsian probability model to characterize the spatial behavior of river networks. Their results give an average $\eta = 0.75$, and the authors state that there seems to be a tendency for η to be greater than 0.5 even at scales greater than the hillslope scale. Analytical solutions were obtained for the global minima to equation (17) under specific conditions [*Maritan et al.*, 1996; *Colaiori et al.*, 1997]. Results show that natural river networks tend to be rather in states of local minima, and the proposition was made that nature chooses an approach of "feasible optimality" with a long memory in the evolution of its landscapes. Worse energetic performance, yet better representation of natural networks by OCNs in states of local optima, may imply a role of geologic and other constraints in the evolution of river networks [*Rodríguez-Iturbe and Rinaldo*, 1997, p. 353]. In any case, the energetic performance and structure of river networks remain interesting research topics.

4. Applying the Hypothesis of Local Optimality

We propose that in a river network, the local optimal energy expenditure hypothesis governs the average distribution of channel geometry. In this section we use the downstream hydraulic geometry relations of *Leopold and Maddock* [1953] to describe the variation of channel geometry and develop the relation for the energy dissipation rate per unit channel area P_a. We then use this relation to define the condition of local optimality in which P_a is constant throughout the network. In our analysis we do not constrain the downstream hydraulic geometry exponents for width and depth to be equal (i.e., in general, $b \neq f$).

Consider a river network composed of rectangular channel links, where the energy dissipation rate P_a in a given link is $P_a = P_i/P_w L$. Using (6) and (8) under local optimality, we get

$$P_a = g\rho_w[1 + C_V(G - 1)] \frac{SQ}{W + 2H} = \text{const} \quad (20)$$

The crucial problem in determining P_a throughout a river network is in evaluating the network variables in (20). These variables describe the driving conditions (discharge and sediment concentration), the channel geometry, and the topography of the watershed (channel slope). They can be related to the area draining to each point in the river network (drainage area is a quantity that is easily defined from DEMs).

4.1. Distribution of Driving Conditions and Channel Properties in a River Network

In hydrologic practice, flood quantiles are commonly related to drainage area. The flood quantile Q_p, with exceedance probability p, can be determined as [*Gupta et al.*, 1994]

$$Q_p = \psi_p A^{\theta(p)} \quad (21)$$

where A is the area draining to a point in the network, $\theta(p)$ is a scaling exponent dependent on p, and ψ_p is a network parameter dependent on p but constant throughout the network for a given discharge frequency. In the case of simple scaling, $\theta(p)$ will be independent of the exceedance probability [*Gupta and Dawdy*, 1995].

To describe the downstream variation of channel geometry, we used the downstream hydraulic geometry relations of *Leopold and Maddock* [1953]. These simple power laws relating channel geometry and characteristic discharge are to be seen as a representation of the average behavior of channel geometry:

$$W_p = a_p Q_p^b = a_p \psi_p^b A^{b\theta(p)}$$
$$H_p = c_p Q_p^f = c_p \psi_p^f A^{f\theta(p)} \quad (22)$$

where W_p and H_p are the flow width and depth of a channel at a point in the network corresponding to discharge Q_p, a_p and c_p are network constants dependent on discharge frequency, and b and f are channel flow width and depth scaling exponents (downstream hydraulic geometry exponents). The characteristic discharge has to be of channel-forming and -maintaining significance, with an equal frequency throughout the network.

Using the relation between channel bed slope and discharge in equation (16) we can express mean channel link slope as a function of drainage area:

$$S = sA^{\zeta\theta(p)} = sA^z \quad (23)$$

where s and z are network constants.

The spatial and temporal distributions of sediment concentration in a river network are probably the most difficult to estimate. One of the main reasons is that sediment load depends not only on the transport capacity of the stream, but also on sediment supply to the stream. A common technique (in supply-limited conditions) is to estimate the sediment concentration as a power function of discharge of a given frequency [e.g., *Leopold and Maddock*, 1953; *Julien*, 1995, p. 229]:

$$C_{Vp} = k_p Q_p^j = k_p \psi_p^j A^{j\theta(p)} \quad (24)$$

where C_{Vp} is the volumetric sediment concentration corresponding to discharge Q_p, k_p is a network constant dependent on discharge frequency, and j is a constant sediment scaling exponent. A list of all network parameters is provided in Table 1.

Table 1. List of River Network Parameters Used in the Energy Expenditure Analysis

River Network Characteristic	Parameters
Discharge	ψ_p and $\theta(p)$
Channel bed slope	s and z
Channel flow width	a_p and b
Channel flow depth	c_p and f
Sediment concentration	k_p and j

4.2. Rate of Energy Dissipation per Unit Channel Area

The rate at which energy is dissipated per unit channel area in any link in the network can then be obtained by combining (20) with the expressions for the individual network variables in (21) through (24). The resulting P_a is a function of the drainage area and all network parameters $P_a = P_a(A; b, f, \cdots)$, and can be divided into the energy dissipation component due to water P_a^w and sediment P_a^s,

$$P_a = P_a^w + P_a^s \quad (25)$$

where P_a^w is equal to

$$P_a^w = g\rho_w \frac{s\psi_p A^{\theta(p)+z}}{a_p \psi_p^b A^{b\theta(p)} + 2c_p \psi_p^f A^{f\theta(p)}} \quad (26)$$

and P_a^s is equal to

$$P_a^s = g\rho_w(G - 1) \frac{k_p s \psi_p^{j+1} A^{(j+1)\theta(p)+z}}{a_p \psi_p^b A^{b\theta(p)} + 2c_p \psi_p^f A^{f\theta(p)}} \quad (27)$$

The estimation of river network parameters is a cumbersome task and may not be possible for rivers with inadequate data. As was stressed earlier, equations (26) and (27) for P_a are valid for long-term averaged behavior, and they are not to be used for prediction in a condition where the joint distribution of the network variables may invalidate the operations used to develop the formulas [*Troutman*, 1996].

4.3. Minimization Aspects

In a network abiding by the local optimal energy expenditure hypothesis, the combination of all network parameters is such that P_a is constant throughout the network. Since P_a from (25) is a function of drainage area, local optimality will require

$$d(P_a)/dA = 0 \quad (28)$$

The above derivative will approach zero as a function of river network parameters. From the perspective of local optimality, the "best" set of network parameters is the one for which

Table 2. Values of River Network Parameters Calibrated With Goodwin Creek Data for Maximum Monthly and Daily Streamflow Conditions With Exceedance Probabilities p

Parameters	$p_m = 0.5$	$p_m = 0.1$	$p_m = 0.01$	$p_d = 0.5$
ψ_p	0.0675	0.1310	0.2247	0.8684
$\theta(p)$	0.949	0.966	0.979	0.950
c_p	0.3280	0.3550	0.3772	0.4337
f	0.317	0.321	0.324	0.320
k_p	1.401 (10^{-3})	1.507 (10^{-3})	1.457 (10^{-3})	3.0 (10^{-3})
j	0.249	0.339	0.409	0

In the calibration, discharge Q is in cubic meters per second, depth H is in meters, and volumetric sediment concentration C_V is in cubic meters per cubic meter. After *Molnár and Ramírez* [this issue].

$d(P_a)/dA$ is closest to zero throughout the network. Defining an optimality function $h(b, f, \cdots)$ as

$$h(b, f, \cdots) = \frac{1}{A_0}\int_{A_t}^{A_0}\left|\frac{d(P_a(A;\, b, f, \cdots))}{dA}\right| A\, dA \qquad (29)$$

the optimal set of network parameters $(b^*, f^*, \cdots)$ is obtained by minimizing h as,

$$h(b^*, f^*, \cdots) = \min_{\forall(b, f, \cdots)} h(b, f, \cdots)$$

$$= \min_{(b, f, \cdots)} \frac{1}{A_0}\int_{A_t}^{A_0}\left|\frac{d(P_a(A;\, b, f, \cdots))}{dA}\right| A\, dA \qquad (30)$$

The optimal set of network parameters $(b^*, f^*, \cdots)$ results in optimal channel characteristics of the river network. In (29) and (30), A_t is the drainage area associated with the channel initiation threshold, and A_0 is the drainage area at the outlet of the basin. The optimality function h weighs $d(P_a)/dA$ with the drainage area, thereby giving more weight to the mature downstream sections of the river network.

The derivative $d(P_a)/dA$ can be divided into its water and sediment components:

$$\frac{d(P_a)}{dA} = \frac{d(P_a^w)}{dA} + \frac{d(P_a^s)}{dA} \qquad (31)$$

where

$$\frac{d(P_a^w)}{dA} = g\rho_w s\psi_p \frac{[\theta(p) + z]A^{\theta(p)+z-1}K_1 - A^{\theta(p)+z}K_2}{K_1^2} \qquad (32)$$

$$\frac{d(P_a^s)}{dA} = g\rho_w(G-1)k_p s\psi_p^{j+1} \cdot \frac{[(j+1)\theta(p) + z]A^{(j+1)\theta(p)+z-1}K_1 - A^{(j+1)\theta(p)+z}K_2}{K_1^2} \qquad (33)$$

and where K_1 and K_2 are

$$K_1 = a_p\psi_p^b A^{b\theta(p)} + 2c_p\psi_p^f A^{f\theta(p)}$$
$$K_2 = a_p\psi_p^b b\theta(p)A^{b\theta(p)-1} + 2c_p\psi_p^f f\theta(p)A^{f\theta(p)-1} \qquad (34)$$

5. Optimal Channel Characteristics

On the basis of the theory in the preceding section, optimal channel characteristics can be defined for every river network. We found that these optimal channel characteristics closely resemble the properties of many natural river systems. This indicates that the hypothesis of local optimality may be a central principle that explains the average distribution of channel properties in river networks. In addition, in this section we provide examples of the interrelationships between the downstream hydraulic geometry exponents and the slope scaling exponent in an optimal river network. The analyses were conducted with data from Goodwin Creek, a small experimental watershed (21.4 km^2) located in Panola County, Mississippi. The watershed is well instrumented to measure streamflow, sediment load, channel properties, and other watershed parameters [*Blackmarr*, 1995] and is therefore ideal for an illustration of the application of energy expenditure hypotheses. This section contains the following investigations: (1) the optimal variation of channel width expressed by the exponent b in the downstream hydraulic geometry relation, (2) optimal combinations of channel width and depth expressed by the exponents b and f, and (3) the effect of channel topography expressed by the exponent z on optimal channel width.

In developing optimal downstream hydraulic geometry exponents for the Goodwin Creek network, we explored two different topographic conditions. For the first condition, the exponent z of the power function relating channel slopes to drainage area in (23) was set to $z = -0.5$ ($s = 0.0094$, $R^2 = 0.664$). This is the average value of the scaling exponent found by *Tarboton et al*. [1989] from numerous studies of DEM-extracted river networks and is also the value assumed in the OCN studies. The second condition was one where the best estimate of z fitted to Goodwin Creek data was used, $z = -0.622$ ($s = 0.0105$, $R^2 = 0.826$) [*Molnár and Ramírez*, this issue]. In addition, we studied four different characteristic streamflow-driving conditions. Maximum monthly discharges with exceedance probabilities $p_m = 0.5$, $p_m = 0.1$, and $p_m = 0.01$ represented channel-maintaining flow conditions, while mean maximum daily discharges with $p_d = 0.5$ represented channel-forming flow conditions. The values of the river network parameters calibrated for the different streamflow conditions at Goodwin Creek are in Table 2 [*Molnár and Ramírez*, this issue].

5.1. Width Exponent *b*

We determined the optimal downstream variation of channel flow width at Goodwin Creek under the local optimal energy expenditure hypothesis of equal P_a throughout the river network. This translates to finding an optimal width exponent b that minimizes the optimality function $h(b, f, \cdots)$ in (30), using the data-derived river network parameters from Table 2. In order to ensure that the results for different flow and topographic conditions are comparable, we maintained the same channel flow width at the outlet of the basin. Therefore when varying the exponent b in search for the optimum, we also continuously adjusted the coefficient a_p in (22) so that channel flow width at the watershed outlet corresponding to the discharge driving conditions was kept constant [*Molnár and Ramírez*, this issue].

Figure 1a shows the downstream variation of P_a for the optimal case and two suboptimal cases of the exponent b. The associated downstream variation of the derivative $d(P_a)/dA$ is shown in Figure 1b. The values of the optimal downstream hydraulic geometry exponent b and the associated values of constant P_a for every studied case are shown in Table 3. For the case $z = -0.5$, optimal b was between 0.488 and 0.506 (depending on the streamflow condition), which is close to the average value observed in many natural river systems [*Leopold and Maddock*, 1953; *Carlston*, 1969; *Park*, 1977; *Knighton*, 1984, p. 100]. For the case $z = -0.622$, optimal b was between 0.347 and 0.369, also within the range of observed values in natural river systems [*Park*, 1977]. The dependency of optimal b on the topography of the basin is clearly more significant than the dependency on discharge frequency. The optimal exponent also provides a threshold in the energy expenditure distribution throughout the basin. At values of b lower than optimal, upstream sections of the network dissipate energy per unit channel area at a lower rate than those downstream. At values of b higher than optimal, the process is reversed (Figure 1a).

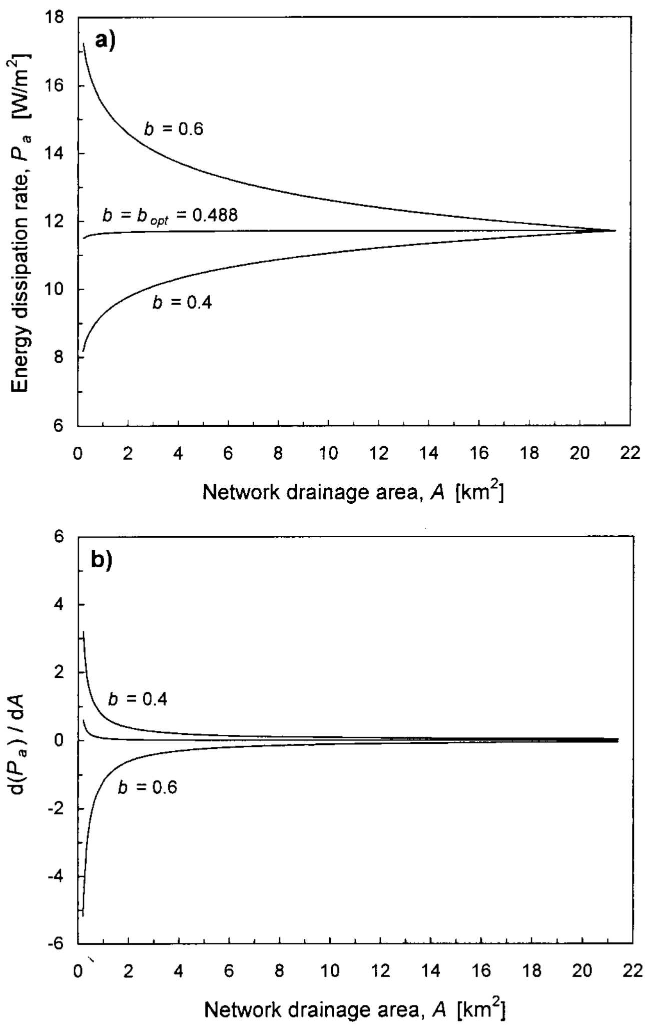

Figure 1. (a) Distribution of the energy dissipation rate P_a throughout Goodwin Creek for optimal and suboptimal values of the exponent b for the maximum daily streamflow condition and $z = -0.5$. (b) Distribution of the derivative $d(P_a)/dA$ with drainage area.

Table 3. Optimal Downstream Hydraulic Geometry Exponents b, f, and m; Constant Energy Dissipation Rate P_a at the Outlet of the Basin; and the Ratio P_a^s/P_a^w for Optimal Networks Developed for Maximum Monthly and Daily Streamflow Conditions With Exceedance Probabilities p and Topographic Conditions $z = -0.5$ and $z = -0.622$

	$p_m = 0.5$	$p_m = 0.1$	$p_m = 0.01$	$p_d = 0.5$
	Topographic Condition z = −0.5			
b optimal	0.488	0.498	0.506	0.488
f data-derived	0.317	0.321	0.324	0.320
m	0.195	0.181	0.170	0.192
P_a, W m^{-2}	2.72	4.09	5.70	11.71
P_a^s/P_a^w	0.244 (10^{-2})	0.340 (10^{-2})	0.445 (10^{-2})	0.495 (10^{-2})
	Topographic Condition z = −0.622			
b optimal	0.347	0.360	0.369	0.347
f data-derived	0.317	0.321	0.324	0.320
m	0.336	0.319	0.307	0.333
P_a, W m^{-2}	1.98	2.98	4.16	8.54
P_a^s/P_a^w	0.244 (10^{-2})	0.340 (10^{-2})	0.445 (10^{-2})	0.495 (10^{-2})

From the optimal hydraulic geometry of the Goodwin Creek network we can also determine other channel characteristics in the optimal network. For instance, in Figure 2 we show the variation of boundary shear stress and resistance to flow with drainage area. Average boundary shear stress τ_0 is computed from (5). Shear stress is higher in the upstream sections of the network but remains fairly constant once drainage areas exceed about 6 km^2 (mean drainage area at Goodwin Creek is 4.3 km^2). Bed shear is indicative of bed load sediment transport. *Kuhnle* [1992] reports that bed shear stress exceeding about 10 Pa is sufficient to induce bed load sediment transport at the Goodwin Creek watershed outlet. The computed boundary shear stress at that location in the optimal networks ranged between 5 Pa and 20 Pa (depending mostly on flow condition) and will therefore be sufficient to induce bed load sediment transport at higher flow stages [*Molnár and Ramírez*, this issue]. The dimensionless Darcy-Weisbach friction factor ff was used to describe resistance to flow throughout the optimal network:

$$ff = \frac{8}{\rho_m} \frac{\tau_0}{v^2} \tag{35}$$

Figure 2b shows a general downstream decrease in the friction factor, similar to many natural river systems. Sections of the optimal river network draining less than approximately 2.5 km^2 have ff in excess of 1.0 (for $p_m = 0.01$). This would correspond to fairly steep (S up to 0.005) and rough (d_{50} up to 100 mm) gravel bed channels [*Bathurst*, 1993]. The values of ff for drainage areas above 2.5 km^2 decrease from about 1.0 to 0.4 at the outlet of the basin. This would correspond to sand and gravel bed channels with moderate slopes [*Bathurst*, 1993]. Similar results were obtained for the other streamflow conditions. In Figure 2b the magnitude of ff increases with discharge at a given location in the network because of the large channel flow width increase between the two streamflow conditions. In a natural river system, higher resistance typical of overbank flow can increase total resistance to flow despite an increase in discharge [*Bathurst*, 1993].

The effect of sediment load in the form of added weight to

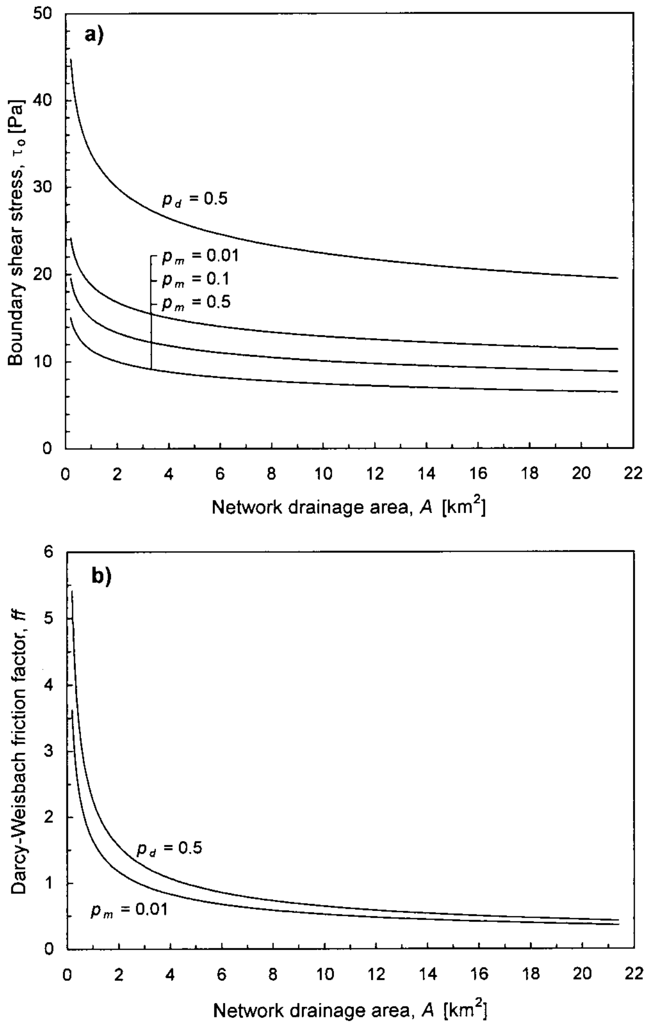

Figure 2. Distribution of (a) average boundary shear stress τ_0 in an optimal river network with $z = -0.5$, and (b) Darcy-Weisbach friction factor ff in an optimal river network with $z = -0.5$.

the fluid-sediment mixture on energy dissipation was found to be minimal at Goodwin Creek. The ratios of the energy dissipation rates due to transporting sediment and water P_a^s/P_a^w computed from (26) and (27) for each condition are given in Table 3. The values of this ratio are less than 0.5% for all studied conditions at Goodwin Creek.

5.2. Optimal Combination of *b* and *f*

In the analysis of optimal channel characteristics up to this point, we searched for the optimal width exponent b using the channel flow depth exponent f derived from Goodwin Creek data. In order to generalize our results, we explored the optimal combination of b and f that results in constant P_a. We used a constrained, bivariate minimization procedure to find the optimal combination of b and f that minimized the optimality function $h(b, f, \cdots)$. Again, to ensure comparability, we varied both constants a_p and c_p in the downstream hydraulic geometry relations in (22), so that a constant width and depth at the watershed outlet was maintained for the different conditions [*Molnár and Ramírez*, this issue]. The combinations of b and f that result in optimal networks for flow conditions $p_d = 0.5$ and $p_m = 0.01$ are shown in Figure 3. Only combinations that provide for an increasing width-depth ratio downstream (which is commonly the case in natural river systems) are shown in the figure $(b > f)$. Optimal combinations of the exponents can be found with b ranging between 0.472 and 0.545 for $z = -0.5$ and between 0.345 and 0.403 for $z = -0.622$. The corresponding exponent f varied between 0 and 0.491 for $z = -0.5$ and between 0 and 0.366 for $z = -0.622$. The circles in Figure 3 show the combinations with the data-derived exponent f from Table 2. In nature the observed value of the depth exponent f has been found to vary between approximately 0.1 and 0.5, with the most probable value between 0.3 and 0.4, while the width exponent b has been found to vary between approximately 0.2 and 0.7, with the most probable value between 0.4 and 0.6 [*Park*, 1977]. Our optimality analysis shows that within these ranges there can be numerous combinations of these exponents that satisfy the hypothesis of local optimality for a given river network, and that these combinations depend on the topography of the basin.

5.3. Optimal Combination of *b* and *z*

It is clear from the cases in Figure 3 that the value of the optimal exponent b is significantly affected by the topography of the basin represented by the slope scaling exponent z. We therefore analyzed the arrangement of topography and channel flow width throughout the river network on the basis of the local optimality hypothesis. Our results for the maximum daily streamflow condition are shown in Figure 4, where the relation between b and z represents an optimal condition with data-derived channel flow depth parameters from Table 2. Added to Figure 4 is a graph of the expression which describes the optimal arrangement between channel width and topography when the effect of channel depth on P_a as well as the sediment transport term are neglected. Then

$$P_a = \text{const}\ (SQ/W) = \text{const}\ A^{z+(1-b)\theta(p)} \qquad (36)$$

and for P_a to be constant,

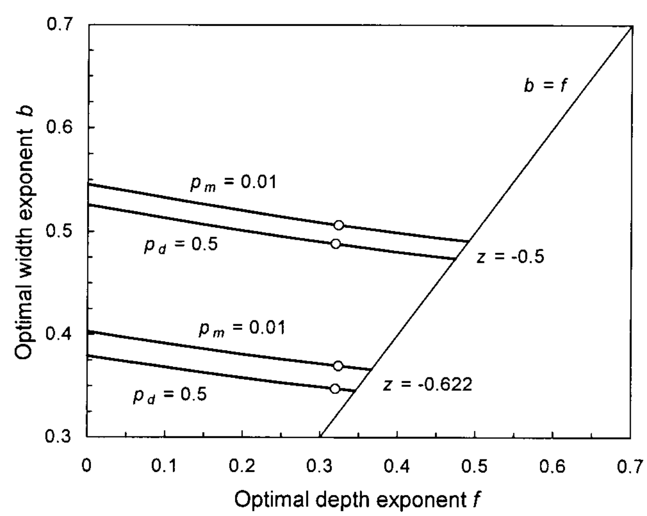

Figure 3. Optimal combination of channel downstream hydraulic geometry exponents for width b and depth f for topographic conditions $z = -0.5$ and $z = -0.622$, and average maximum daily and monthly streamflow conditions with exceedance probabilities p. Circles give the optimal conditions with data-derived values of f from Table 2.

$$b = 1 + z/\theta(p) \tag{37}$$

The difference between the optimal combinations and the expression in (37) in Figure 4 shows the effect of channel depth on optimal energy dissipation. Equation (37) can also be obtained by combining (18) and (19) and recognizing that $z = \zeta\theta(p)$ from (23), and it shows the connection between the energy exponent η and optimal topography and channel geometry.

As was pointed out previously, the channel width-depth ratio in our network consisting of links with rectangular cross sections increases downstream if $b > f$. In Figure 4 the width-depth ratio is constant throughout the river network (equal to 25.3) when $b = f = 0.320$, and the corresponding optimal z is then -0.646. For the exponent b greater than 0.320, the width-depth ratio increases in the downstream direction. The value of the ratio is 4 at the upstream end of the river network when $b = 0.736$, and $z = -0.288$ (the upstream end was defined by the channel initiation threshold $A_t = 0.18$ km^2). The channel width-depth ratio is a function of watershed properties such as silt and sand content, vegetation, etc., and therefore provides a natural constraint to the adjustment of the river system. The ratio commonly increases in the downstream direction, and natural river cross-sections rarely exhibit width-depth ratios below 4 [e.g., *Chang*, 1979; *Knighton*, 1984, p. 103]. This limits the range of optimal z value from Figure 4 to be between -0.288 and -0.646 (shown as the "range of optimality" in the figure). *Langbein and Leopold* [1964] argue that in nature a balance between constant P_a and minimum P_t leads to the scaling exponent z being somewhere in the range between -0.5 and -1. *Carlston* [1968] reports z (assuming $\theta(p) = 1$) to vary between -0.5 and -0.93 with an average of -0.65. The average value of the slope scaling exponent z observed from extensive DEM studies in the channeled fluvial zone was $z = -0.5$ [*Tarboton et al.*, 1989; *Ijjász-Vásquez and Bras*, 1995].

Our analysis shows that channel widths and bed slopes are closely linked in the context of the hypothesis of local optimality in energy expenditure. Furthermore, they are connected to the total rate of energy expenditure in the river network through the energy exponent η.

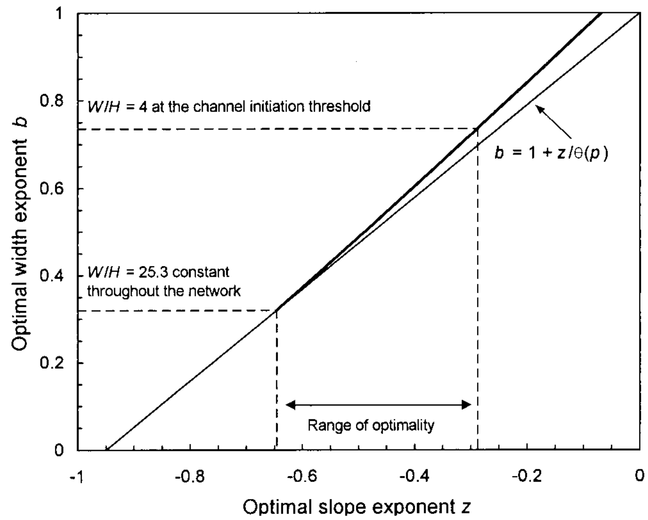

Figure 4. Optimal combination of the width downstream hydraulic geometry exponent b and the slope scaling exponent z for the maximum daily streamflow condition.

6. Conclusions

In this paper we explored the theory that river networks naturally evolve into structures with channel characteristics that are most efficient in the transport of water and sediment. The local optimal energy expenditure hypothesis of a constant energy dissipation rate per unit channel area P_a and its implications with regard to long-term average channel properties of a river system in equilibrium were investigated. We developed formulas for computing P_a throughout the river network and, using data from Goodwin Creek, showed the interrelationships between downstream hydraulic geometry of channels, basin topography, and discharge driving conditions in networks that satisfy local optimality.

We first showed that an expanded class of optimal river networks which includes drainage systems exhibiting trends in average flow velocity downstream can be defined. For these systems, channel resistance and flow velocity are not constant throughout the networks, but the product $ff\ v^3$ is constant. We developed an optimal scaling relationship between average channel geometry and discharge where the downstream hydraulic geometry exponents for width and depth are equal, and showed its effect on the energy exponent and the slope-discharge exponent.

On the basis of the optimal energy expenditure analysis conducted with Goodwin Creek data, we argued that the hypothesis of local optimality is a central principle that explains the average behavior of channel properties in river networks. Our findings were as follows:

1. The downstream hydraulic geometry exponents b and f required to maintain constant energy dissipation rate P_a throughout the river network are similar to those observed in many natural river systems [*Leopold and Maddock*, 1953; *Carlston*, 1969; *Park*, 1977].
2. Numerous optimal combinations of the exponents b and f can be found for different flow and topographic conditions.
3. The value of the optimal exponent b is predominantly influenced by the topography of the watershed represented by the slope scaling exponent z. A range of optimality for both exponents z (-0.29 to -0.65) and b (0.32 to 0.74) was found by analyzing width-depth ratios of the optimal channel networks.
4. Optimal channel characteristics of the developed optimal networks were found to resemble properties observed in many natural river systems. Average flow velocity increased in the downstream direction, channel resistance in the form of the Darcy-Weisbach friction factor, and boundary shear stress decreased in the direction of flow. The function of channel resistance and flow velocity $ff\ v^3$ was constant throughout the optimal network.
5. The effect of sediment load in the form of added weight to the fluid-sediment mixture on the rate of energy dissipation P_a was insignificant. It amounted to less than 0.5% in the studied cases at Goodwin Creek.

The results presented above should be seen from two perspectives. First, they describe long-term average behavior of channel geometry as expressed by the downstream hydraulic geometry relations of *Leopold and Maddock* [1953] and do not have any predictive meaning at a given location in a river network. Constant channel adjustment will prevent the hypothesis of local optimality to be satisfied at any given time and space in a river network. The variability of P_a in an optimal network is an ongoing subject of study by the authors. Second, both the global and local hypotheses of optimality assume that the watershed (river network) is not bound in its adjustment. However, the underlying geology, vegetation, human interference, and many other factors will also affect the evolution and adjustment of the channel system. Although topological adjustment of river networks and the adjustment of channel characteristics occur at different timescales, one can visualize their complementary effects in creating optimal combinations of topography and channel geometry in a river basin. There are other concerns regarding the process of energy dissipation. For instance, it is essentially impossible to budget energy dissipation into the individual processes that create it on a river network scale. The determination of the amount of energy spent in a river network on overcoming system and surface roughness, transporting sediment, creating and moving bedforms, internal turbulence, etc., remains an interesting research topic.

We conclude that efficiency in energy expenditure provides an appealing rationality to the behavior of natural river systems and that it can be a very useful tool for studying the structure and properties of river networks.

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