

# Climate, Soil, and Vegetation

## 5. A Derived Distribution of Storm Surface Runoff

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The Philip infiltration equation is integrated over the duration of a rainstorm of uniform intensity to give the depth of point surface runoff from such an event on a natural surface in terms of random variables defining the initial soil moisture, the rainfall intensity, and the storm duration. In a zeroth-order approximation the initial soil moisture is fixed at its climatic space and time average, whereupon by using exponential probability density functions for storm intensity and duration, the probability density function of point storm rainfall excess is derived. This distribution is used to define the annual average depth of point surface runoff and to derive the flood volume frequency relation, both in terms of a set of physically meaningful climate-soil parameters.

### INTRODUCTION

In seeking a physically based analytical description of the average annual water balance [Eagleson, 1978a], we must deal with the random variability of the various climatic variables involved in the physical processes defining the separate water balance components. Primary among these random variations are the alternate intervals of infiltration and evapotranspiration, the rates of rainfall and potential evapotranspiration, and the rate-controlling soil moisture concentration. Process physics can be introduced into the parameters of a statistical distribution of surface runoff by deriving this distribution from the known distributions of the independent climatic variables by means of an analytical relation between the storm properties and the surface runoff which represents the essential dynamics of the infiltration process.

This approach follows one taken earlier by the author [Eagleson, 1972] to incorporate runoff dynamics in a derivation of flood frequency.

### PROBLEM FORMULATION

Considering moisture fluxes at the soil surface during a rainstorm of duration  $t_r$ , we can write

$$\int_0^{t_r} [i(t) - f_i(t)] dt = \int_0^{t_r} [r_s(t) + v_{ss}(t)] dt \quad (1)$$

where  $i(t)$  is the rate of storm rainfall,  $f_i(t)$  is the infiltration rate,  $r_s(t)$  is the surface runoff rate, negative for surface inflows, and  $v_{ss}(t)$  is the rate of storage on the surface.

We will assume, as a first approximation, that there is no postprecipitation infiltration from the volumes on the right-hand side of (1) and that this process occurs uniformly over both bare soil and vegetated portions of the surface.

The depth of water held on the surface by surface forces is

$$E_r = \int_0^{t_r} v_{ss}(t) dt \quad (2)$$

which is called 'surface retention.' Whether it is completely evaporated in the subsequent interstorm period depends on the properties of that interval. The surface runoff is

$$R_s = \int_0^{t_r} r_s(t) dt \quad (3)$$

which is the surface runoff from the  $j$ th storm. Equation (1) can now be written

$$\int_0^{t_r} [i(t) - f_i(t)] dt = R_s + E_r \quad (4)$$

To evaluate the integral of (4), we will first eliminate from consideration any surface inflows from outside the region. Infiltration thus results solely from local precipitation.

### STORM SURFACE RUNOFF

Representing the precipitation climate by a train of randomly sized and spaced rectangular intensity pulses [Eagleson, 1978b], we have for any single storm,

$$i(t) = i = \text{const} \quad 0 \leq t \leq t_r \quad (5)$$

Beginning at  $t = 0$ , there is a sequence of surface states as is illustrated in Figure 1. First, there is a withdrawal of rainfall to satisfy the surface retention. For the case illustrated,  $t_r > h_0/i$ , and thus this surface retention reaches its capacity  $h_0$ . If on the other hand we had  $t_r \leq h_0/i$ , there would be no infiltration or surface runoff, and the surface retention  $E_r$  would equal the storm depth  $h$ . Returning to the case of Figure 1, infiltration will begin when  $t = h_0/i$ .

Using the Philip [1969] infiltration equation, Eagleson [1978c] has represented the infiltration rate  $f_i$  by

$$f_i = \frac{1}{2} S_i t^{-1/2} + A_0 \quad (6)$$

where gravitational infiltration and water table influence are included in the term  $A_0$ , while capillarity is embodied in the infiltration sorptivity  $S_i$ . When the surface is saturated (i.e., surface soil moisture concentration,  $s_1 = 1$ ),  $S_i$  and hence  $f_i$  are maximum for a given initial internal soil moisture concentration, and (6) defines the 'infiltration capacity'  $f_i^*$ . That is,

$$f_i = f_i^* \quad s_1 = 1, \quad 0 \leq t \leq t_r \quad (7)$$

For a saturated surface,

$$A_0 = \frac{1}{2} K(1)(1 + s_0^c) - w \quad (8)$$

and

$$S_i = 2(1 - s_0) \{ [5nK(1)\Psi(1)\phi_i(d, s_0)] / 3m\pi \}^{1/2} \quad (9)$$

where

$K(1)$  saturated effective hydraulic conductivity of soil, centimeters per second;

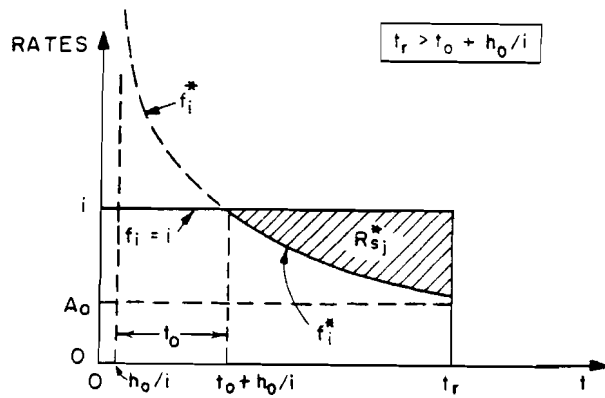


Fig. 1. Surface runoff generation during typical storm.

- $s_0$  initial (uniform) soil moisture concentration in the surface boundary layer;  
 $\Psi(l)$  saturated matrix potential of soil, centimeters (suction);  
 $\phi_i(d, s_0)$  dimensionless sorption diffusivity of soil;  
 $n$  effective porosity of soil;  
 $c$  pore disconnectedness index;  
 $m$  pore size distribution index;  
 $w$  apparent velocity of capillary rise from water table.

As will always be the case (for finite  $t$ ),  $f_i^*$  as given by (6) and (7) exceeds  $i$  at the start of infiltration, and the surface soil moisture will then adjust itself to that value, less than  $s_1 = 1$ , at which  $f_i = i$ . The capacity of the soil to infiltrate water will then decline with time by virtue of the water already infiltrated and the consequent increase in soil moisture concentration. This is illustrated in Figure 1. Accompanying this rise of internal moisture content is a rise in the surface moisture content in order that the moisture gradient at the surface can maintain infiltration at the rainfall rate  $i$ . At some time  $t_0 + h_0/i \leq t_r$ , the surface may reach saturation at which point, by definition,  $f_i = f_i^* = i$ .

For all  $t_0 + h_0/i < t \leq t_r$ ,  $i > f_i^*$ , and surface runoff is generated. This runoff is indicated by the shaded area of Figure 1.

To calculate the runoff depth  $R_{s_j}$ , we need first to assume that (6) and (7) apply in the period  $t_0 + h_0/i \leq t \leq t_r$ , even though the history of the infiltration process prior to  $t_0$  was not in accord with the conditions of their derivation. But how do we estimate  $t_0$ ?

It is customary [Linsley *et al.*, 1958, p. 179] to assume that

$$i = f_i^*$$

as given by (6) and (7) when

$$f_i^*(\nabla_i) = f_i^*(it_0) \quad (10)$$

From (6)

$$t = S_i^2 / [4(f_i^* - A_0)^2] \quad (11)$$

Integrating (6) gives the cumulative infiltrated depth

$$\nabla_i(t) = S_i t^{1/2} + A_0 t \quad (12)$$

Using (11) in (12) gives

$$\nabla_i = \frac{S_i^2}{2(f_i^* - A_0)} \left[ 1 + \frac{A_0}{2(f_i^* - A_0)} \right] \quad (13)$$

If we introduce the above assumption that  $f_i^* = i$  when  $\nabla_i = it_0$ , (13) becomes

$$t_0 = \frac{S_i^2}{2(i - A_0)} \left[ 1 + \frac{A_0}{2(i - A_0)} \right] \quad (14)$$

For  $i \gg A_0$  this reduces to

$$t_0 \approx \frac{S_i^2}{2(i - A_0)^2} \quad (15)$$

It should be noted that  $t_0$  may also be found by solving the linearized diffusion equation under constant flux boundary conditions to obtain the time at which the surface becomes saturated. The form of this solution is identical with (15) except that (using the same effective diffusivity) the constant coefficient is  $\pi^2/16$  rather than  $\frac{1}{2}$  [Carslaw and Jaeger, 1959, p. 75].

We can see that for the normal range of parameters,  $t_0 \gg h_0/i$ . We will therefore neglect the effect of  $h_0$  on the infiltration dynamics. It can be important volumetrically, however, and we will incorporate it by defining a rainfall excess  $R_{s_j}^*$  such that

$$R_{s_j}^* = R_{s_j} + E_r \quad (16)$$

where, in a refinement of (4), which neglects the time  $h_0/i$

$$R_{s_j}^* = \int_{t_0}^{t_r} (i - f_i^*) dt \quad (17)$$

Using (6) and (15), (17) becomes

$$\frac{R_{s_j}^*}{(i - A_0)t_r} = 1 - X \left[ 1 - \frac{2^{1/2} - 1}{2} X \right] \quad (18)$$

where

$$X = \frac{S_i}{(i - A_0)t_r^{1/2}} \quad (19)$$

We see from Figure 1 that

$$R_{s_j}^*/(i - A_0)t_r < 1 \quad (20)$$

Therefore from (18),

$$X < 1$$

and thus

$$[(2^{1/2} - 1)/2] X \ll 1 \quad (21)$$

To the first approximation, Therefore

$$\frac{R_{s_j}^*}{(i - A_0)t_r} \approx 1 - X \quad (22)$$

We should ask that this approximation be consistent with that for  $t_0$ , so that with  $t_r = t_0$  (using (15)),  $R_{s_j}^*(i, t_0) = 0$ . To achieve this, we must modify (22) slightly to the form

$$\frac{R_{s_j}^*}{(i - A_0)t_r} \approx 1 - X/2^{1/2} \quad (23)$$

which says the rainfall excess is

$$R_{s_j}^*(i, t_r, s_0) \approx (i - A_0)t_r - S_i(t_r/2)^{1/2} \quad (24)$$

Using (16), we have the desired result

$$R_{s_j}(i, t_r, h_0, s_0) \approx (i - A_0)t_r - S_i(t_r/2)^{1/2} - E_r \quad (25)$$

which is valid only where  $R_{s_j} \geq 0$ .

SURFACE RUNOFF PROBABILITY INTEGRAL

We can find the cumulative distribution function of the storm surface runoff  $R_s$ , according to

$$\text{Prob} [R_s < z] = F[R_s] = \iint_{R(z)} f(i, t_r, s_0 | h_0) dR \quad (26)$$

where  $f(i, t_r, s_0 | h_0)$  is the joint probability density function of storm intensity, duration, and initial soil moisture given the surface retention capacity  $h_0$ . In a zeroth-order approximation to (26) we will consider  $s_0$  to be a constant at its space and time average for the given climate soil system. This assumption [Eagleson, 1978a, c] forces all variability into the storm parameters, and (26) becomes

$$F[R_s] = \iint_{R(z)} f(i, t_r | s_0, h_0) dR = \iint_{R(z)} f(i, t_r) dR \quad (27)$$

Because of the approximation made in going from (14) to (15) it would be inconsistent (as well as analytically prohibitive) to consider the random variations of  $E_r$  in defining the region of integration  $R(z)$  in (27). We will thus replace  $E_r$  in (25) by its expected value  $E[E_r]$  to write the rainfall excess,

$$R_{s_j}^* = R_s + E[E_r] = (i - A_0)t_r - S_i(t_r/2)^{1/2} \quad (28)$$

in which [Eagleson, 1978d]

$$E[E_r] = (1 - M)E[E_{rs}] + ME[E_{rv}] \quad (29)$$

By neglecting carryover unevaporated retention the bare soil component is

$$\beta E[E_{rs}]/\bar{e}_p = 1 - e^{-\beta h_0/\bar{e}_p} \frac{\Gamma[\kappa, \lambda h_0]}{\Gamma(\kappa)} - \left[ 1 + \frac{\beta h_0/\bar{e}_p}{\lambda h_0} \right]^{-\kappa} \frac{\gamma[\kappa, (\lambda h_0 + \beta h_0/\bar{e}_p)]}{\Gamma(\kappa)} \quad (30)$$

Also to the first approximation the vegetal component is

$$\beta E[E_{rv}]/\bar{e}_p = k_v \left\{ 1 - e^{-\beta h_0/\bar{e}_p} \frac{\Gamma[\kappa, \lambda k_v h_0]}{\Gamma(\kappa)} - \left[ 1 + \frac{\beta h_0/\bar{e}_p}{\lambda k_v h_0} \right]^{-\kappa} \frac{\gamma[\kappa, (\lambda k_v h_0 + \beta h_0/\bar{e}_p)]}{\Gamma(\kappa)} \right\} \quad (31)$$

where

- $M$  vegetated fraction of land surface;
- $k_v$  vegetal surface retention amplification factor;
- $\beta^{-1}$  average duration of interstorm periods;
- $\bar{e}_p$  average rate of (soil surface) potential evaporation;
- $\kappa$  parameter of the assumed gamma distribution of storm depth  $h$  [Eagleson, 1978b];
- $\lambda$  parameter of assumed gamma distribution of storm depth  $h$ .

$$f_H(h) = \frac{\lambda(\lambda h)^{\kappa-1} e^{-\lambda h}}{\Gamma(\kappa)} \quad (32)$$

and the incomplete gamma function is defined by

$$\gamma[\alpha, x] = \Gamma(\alpha) - \Gamma[\alpha, x] = \int_0^x e^{-t} t^{\alpha-1} dt \quad (33)$$

The cdf of the rainfall excess is

$$\text{Prob} [R_{s_j}^* < z] \equiv F[R_{s_j}^*] = \iint_{R(z)} f(i, t_r) dR \quad (34)$$

and the region of integration is illustrated in Figure 2. Also shown in Figure 2 is the line representing (15), specialized for the case of  $t_0 = t_r$ . By neglecting  $E[E_r]$  this provides a limit below which no rainfall excess occurs. Thus integration of the joint distribution  $f(i, t_r)$  from the axes out to the line  $t_0 = t_r$  approximates the finite probability that a given storm will produce no rainfall excess. Integrating all the way out to  $R_{s_j}^* = z$  gives the probability that a given storm will produce  $R_{s_j}^* \leq z$ .

To carry out the integration of (34), we need the joint distribution  $f(i, t_r)$  of storm duration and average storm intensity.

Grayman and Eagleson [1969] demonstrated the stochastic dependence between the storm depth  $h$  and the duration  $t_r$ , and used a gamma distribution, with parameter dependence upon  $t_r$ , to fit the observed conditional distribution  $f(h | t_r)$  at Boston, Massachusetts. This leads to a joint distribution  $f(i, t_r)$  of sufficient mathematical complexity to preclude our desired analytical solution of (34). As a further expedient, therefore we will use the assumption that  $i$  and  $t_r$  are independent, whereupon [Eagleson, 1978b]

$$f(i, t_r) = \alpha \delta e^{-\alpha i - \delta t_r} \quad (35)$$

Where  $\alpha^{-1}$  is the mean storm intensity ( $m_i$ ),  $\delta^{-1}$  is the mean storm duration ( $m_{t_r}$ ), and, because of the independence assumption,  $\alpha^{-1}\delta^{-1}$  is the mean storm depth (equal to  $\eta^{-1} = m_H$ ).

Using (35), we can prepare a three-dimensional view of the probability calculation of (34) as shown in Figure 3.

Substituting (35) into (34) and using (28) gives

$$\text{Prob} [R_{s_j}^* < z] = \alpha \delta \int_0^\infty e^{-\delta t_r} dt_r \int_0^{z/t_r + S_i/(2t_r)^{1/2} + A_0} e^{-\alpha i} di \quad (36)$$

or

$$\text{Prob} [R_{s_j}^* < z] = 1 - \delta e^{-G} I^* \quad (37)$$

where

$$G = \alpha A_0 \quad (38)$$

and

$$I^* = \int_0^\infty \exp\{-[\delta t_r + \alpha z/t_r + \alpha S_i/(2t_r)^{1/2}]\} dt_r \quad (39)$$

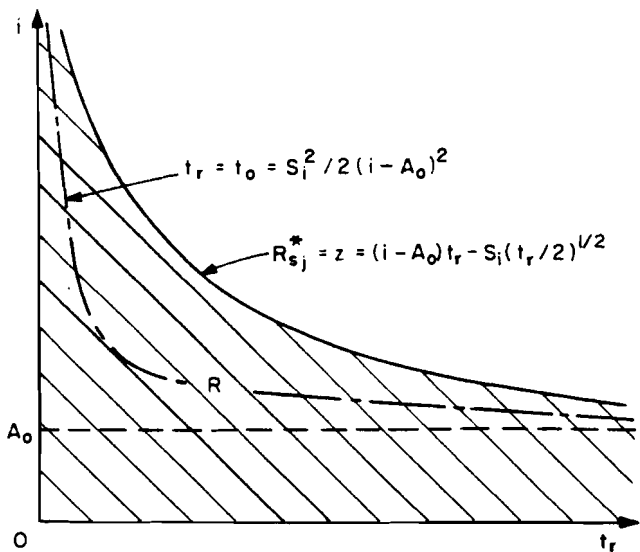


Fig. 2. Integration region for probability of rainfall excess.

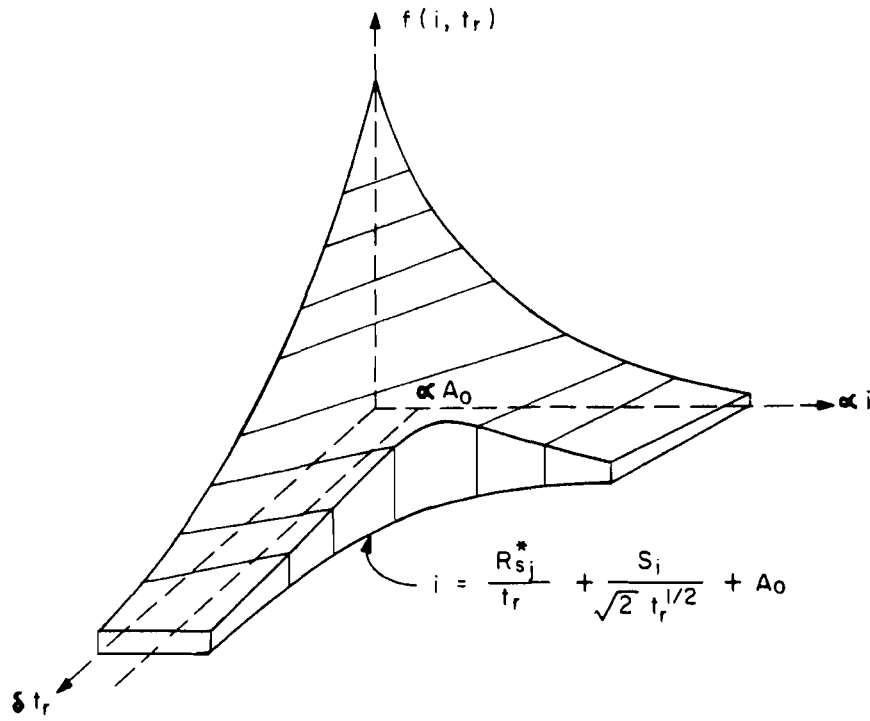


Fig. 3. Calculation of rainfall excess distribution.

How do we integrate this? First, we rewrite (39) as

$$I^* = \int_0^\infty e^{-|a/x+bx+c/x^{1/2}|} dx \quad (40)$$

where

$$\begin{aligned} a &= \alpha z \\ b &= \delta \\ c &= \alpha S_i / 2^{1/2} \end{aligned} \quad (41)$$

It does not appear to be possible to integrate (40) exactly, thus approximate methods must be sought.

**Probability of zero-rainfall excess.** Let us first look at the probability of zero-rainfall excess. Letting  $z = 0$  in (39) gives

$$I^* = \int_0^\infty e^{-|bx+c/x^{1/2}|} dx \quad (42)$$

which can be approximated [Eagleson, 1978e] by

$$I^* = e^{-2\sigma} \Gamma(\sigma + 1) / \delta \sigma^\sigma \quad (43)$$

where

$$\sigma = b(c/2b)^{2/3} = (\delta \alpha^2 S_i^2)^{1/3} / 2 \quad (44)$$

Using (43) in (37) gives

$$\text{Prob}[R_{s_j}^* = 0] = 1 - e^{-G-2\sigma} \Gamma(\sigma + 1) / \sigma^\sigma \quad (45)$$

The desired probability density function  $f(R_{s_j}^*)$  must therefore be a compound distribution having an impulse given by (45) at the origin with a continuous portion of area

$$\text{Prob}[R_{s_j}^* > 0] = e^{-G-2\sigma} \Gamma(\sigma + 1) / \sigma^\sigma \quad (46)$$

**Approximation for nonzero-rainfall excess.** If we next consider the case of large  $R_{s_j}^*$ , we can again simplify (40). Using

the representative soil properties given in Tables 1a and 1b and the representative climate properties tabulated below, we can show that the third term in the exponent of (39) is negligible with respect to the other two.

Property	Value
$m_H$	2.54 cm
$m_T$	365 d
$m_v$	75 events
$T_a$	15°C
$\bar{e}_p$	0.2 cm/d
$\alpha$	$1.5 \times 10^8$ s/cm
$\delta$	$10^{-1}$ h <sup>-1</sup>
$\beta$	$7 \times 10^{-3}$ h <sup>-1</sup>
$h_0$	$10^{-1}$ cm
$\kappa$	0.5

In this extreme, (40) reduces to

$$I^* = \int_0^\infty e^{-|a/x+bx|} dx \quad (47)$$

which has the exact value [Gradshteyn and Ryzhik, 1965, p. 307]

$$I^* = 2(a/b)^{1/2} K_1[2(ab)^{1/2}] \quad (48)$$

in which  $K_n[ ]$  is the modified Bessel function of order  $n$ . If we use the definition of (41), (48) can be substituted in (37) to get

TABLE 1a. Independent Soil Properties [Eagleson, 1978c]

Property	Clay	Clay Loam	Silty Loam	Sandy Loam
$k(1)$ , cm <sup>2</sup>	$1 \times 10^{-10}$	$2.8 \times 10^{-10}$	$1.2 \times 10^{-9}$	$2.5 \times 10^{-9}$
$n$	0.45	0.35	0.35	0.25
$c$	12	10	6	4
$Z$ , m	$\infty$	$\infty$	$\infty$	$\infty$

TABLE 1b. Derived Soil Parameters [Eagleson, 1978c]

Derived Parameter	Clay	Clay Loam	Silty Loam	Sandy Loam
<i>m</i>	0.222	0.286	0.667	2
<i>d</i>	6.5	5.5	3.5	2.5
$\Psi(1)$ , cm	25	19	166	200
<i>K</i> (1), cm/s	$8.28 \times 10^{-6}$	$2.32 \times 10^{-5}$	$9.94 \times 10^{-5}$	$2.08 \times 10^{-4}$
$\phi_i(d, 0)$	0.122	0.140	0.194	0.240
$\phi_i(d, 1)$	0.6	0.6	0.6	0.6
<i>G</i> (0)	0.0621	0.174	0.746	1.560
<i>G</i> (1)	0.124	0.348	1.490	3.120
$\sigma(0)$	0.432	0.482	1.340	1.220
$\sigma(1)$	0	0	0	0

Bare soil surface retention parameter is  $E[E_r]/m_H = 0.03$ .

$$\text{Prob}[R_{s_j}^* < z] = 1 - 2(\alpha\delta z)^{1/2} e^{-G} K_1[2(\alpha\delta z)^{1/2}] \quad (49)$$

where *z* is large. Differentiating to get the probability density function for large  $R_{s_j}^*$ , we get

$$f(R_{s_j}^*) = 2\alpha\delta e^{-G} K_0[2(\alpha\delta R_{s_j}^*)^{1/2}] \quad (50)$$

where  $R_{s_j}$  is large. From the above approximation the area of the continuous part of the density function is

$$\int_0^\infty f(R_{s_j}^*) dR_{s_j}^* = e^{-G} \quad (51)$$

which is different by the factor  $e^{-2\sigma}\Gamma(\sigma + 1)/\sigma^\sigma$ , from the true value given by (46).

*Distribution of rainfall excess.* We will now approximate the continuous portion of the probability density function of rainfall excess over its full range by (50) rescaled through multiplication by the above factor in order to give it the proper area. That is,

$$f(R_{s_j}^*) = 2\alpha\delta e^{-G-2\sigma} K_0[2(\alpha\delta R_{s_j}^*)^{1/2}] \cdot \Gamma(\sigma + 1)/\sigma^\sigma \quad R_{s_j}^* > 0 \quad (52)$$

The mean value of the complete distribution is then [GradshTEYN and RYZHIK, 1965, p. 684]

$$E[R_{s_j}^*] = \int_0^\infty R_{s_j}^* f(R_{s_j}^*) dR_{s_j}^* = e^{-G-2\sigma}\Gamma(\sigma + 1)/\alpha\delta\sigma^\sigma \quad (53)$$

and the variance is

$$\text{Var}[R_{s_j}^*] = \int_0^\infty (R_{s_j}^* - E[R_{s_j}^*])^2 f(R_{s_j}^*) dR_{s_j}^* \quad (54)$$

which gives

$$\text{Var}[R_{s_j}^*] = E[R_{s_j}^*]\{4/\alpha\delta - E[R_{s_j}^*] + \alpha\delta E^2[R_{s_j}^*]\} \quad (55)$$

Because we are assuming independence of *i* and *t<sub>r</sub>* in this development, we can replace  $\alpha\delta$  by  $m_H^{-1}$  or its equivalent,  $\eta$ . This allows us to write the complement of the cdf in the dimensionless form

$$\text{Prob}[\eta R_{s_j}^* > z] = 2z^{1/2} e^{-G-2\sigma} K_1(2z^{1/2}) \Gamma(\sigma + 1)/\sigma^\sigma \quad (56)$$

which is plotted as the continuous line of Figure 4 for  $\sigma = 0.5$ . To compare the approximation of (56) with the true value as given by (37)–(39), we rewrite the latter as

$$\text{Prob}[\eta R_{s_j}^* > z] = e^{-G} \int_0^\infty e^{-[v+z/y+2\sigma^{2/2}/y^{1/2}]} dy \quad (57)$$

Equation (57) has been evaluated numerically for  $\sigma = 0.5$  and is also presented in Figure 4 as the plotted points.

Considering all the approximations made in arriving at (57), the agreement with the 'true' value is remarkable except at very small values of  $\eta R_{s_j}^*$ . We thus suggest (45) and (52) as the distribution of rainfall excess from an individual storm.

*Distribution of surface runoff.* Using (16), we can transform the rainfall excess relationships of the previous section into their equivalents for surface runoff  $R_{s_j}$ .

Equation (52) becomes

$$f(R_{s_j}) = 2\eta e^{-G-2\sigma} K_0[2(\eta(R_{s_j} + E[E_r]))^{1/2}] \cdot \Gamma(\sigma + 1)/\sigma^\sigma \quad R_{s_j} > -E[E_r] \quad (58)$$

where the unrealistic lower limit results from our earlier expedient approximation of the lower limit of the rainfall excess by zero instead of by  $E[E_r]$ . With negligible loss of total probability, we can replace this limit by the physically realistic  $R_{s_j} > 0$ .

Equation (53) transforms to

$$E[R_{s_j}] = e^{-G-2\sigma}\Gamma(\sigma + 1)/\eta\sigma^\sigma - E[E_r] \quad (59)$$

Equation (55) becomes

$$\text{Var}[R_{s_j}] = E[R_{s_j}^*]\{4/\eta - E[R_{s_j}^*] + \eta E^2[R_{s_j}^*]\} \quad (60)$$

while (56) gives the cdf complement

$$\text{Prob}[R_{s_j} > z] = 2(z + \eta E[E_r])^{1/2} e^{-G-2\sigma} \cdot K_1[2(z + \eta E[E_r])^{1/2}] \Gamma(\sigma + 1)/\sigma^\sigma \quad (61)$$

ANNUAL AVERAGE SURFACE RUNOFF

Remembering from (16) that

$$E[R_{s_j}] = E[R_{s_j}^*] - E[E_r]$$

and replacing  $\eta$  by its equivalent,  $m_H^{-1}$ , (59) gives

$$E[R_{s_j}] = m_H e^{-G-2\sigma}\Gamma(\sigma + 1)/\sigma^\sigma - E[E_r] \quad E[R_{s_j}^*] > E[E_r] \quad (62a)$$

and

$$E[R_{s_j}] = 0 \quad \text{otherwise} \quad (62b)$$

Multiplying both sides of (62a) by the average number of storms per year  $m_v$ , we obtain

$$m_v E[R_{s_j}] = m_v m_H e^{-G-2\sigma}\Gamma(\sigma + 1)/\sigma^\sigma - m_v E[E_r] \quad (63)$$

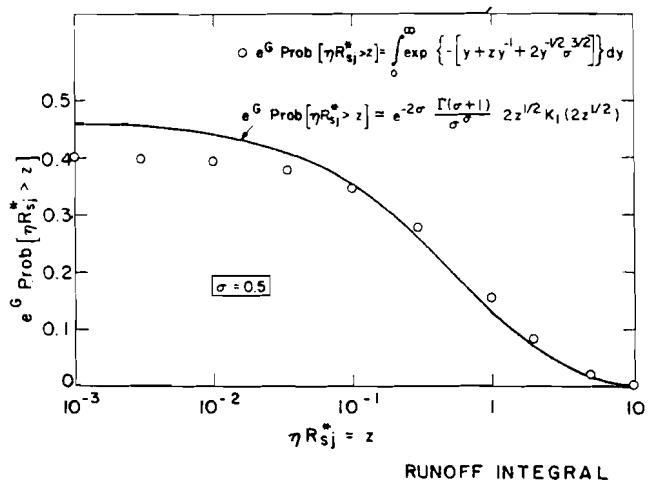


Fig. 4. Approximation to rainfall excess integral.

Summing the random variable  $R_s$  over the random number of storms per year defines the annual surface runoff  $R_{s_A}$  to be

$$R_{s_A} = \sum_{j=1}^v R_{s_j} \quad (64)$$

from which we find the mean annual surface runoff  $E[R_{s_A}]$  to be

$$E[R_{s_A}] = m_r E[R_{s_j}] \quad (65)$$

Similarly, we can write the annual precipitation in terms of the individual storm depths as

$$P_A = \sum_{j=1}^v h_j \quad (66)$$

which gives the mean annual precipitation  $E[P_A]$  as

$$E[P_A] = m_r m_H \quad (67)$$

Using (65) and (67), we can write (63) as

$$\frac{E[R_{s_A}]}{E[P_A]} = e^{-G-2\sigma} \Gamma(\sigma + 1) / \sigma^\sigma - E[E_r] / m_H \quad (68)$$

for positive values of the right-hand side. Otherwise  $E[R_{s_A}] / E[P_A] = 0$ . Using (7) and (44), we see that

$$\sigma = \left[ \frac{5n\eta^2 K(1)\Psi(1)(1-s_0)^2 \phi_i(d, s_0)}{6\pi \delta m} \right]^{1/3} \quad (69)$$

and, from (8) and (38), that

$$G = [\alpha K(1)/2][1 + s_0^c] - \alpha w \quad (70)$$

We will refer to (68) as the 'surface runoff function,' and it is plotted in Figures 5 and 6 for the special case of negligible surface retention. In this case this function gives the average surface runoff generating capability of a natural soil surface in terms of two parameters,  $G$  and  $\sigma$ . For small  $\sigma$  the soil behaves as though wet ( $s_0 \rightarrow 1$ ) and the mean annual surface runoff approaches  $e^{-G} E[P_A]$  in the limit. As  $\sigma$  increases, the soil becomes effectively drier and the mean annual surface runoff becomes a decreasing fraction of the mean annual rainfall. The function is seen to have its primary sensitivity at small values of both  $\sigma$  and  $G$ .

We will assume that the presence of vegetation makes no difference in the rate or areal distribution of precipitation at the soil surface other than to give a larger effective surface retention capacity. The above development is then applicable to both vegetated and bare soil surfaces.

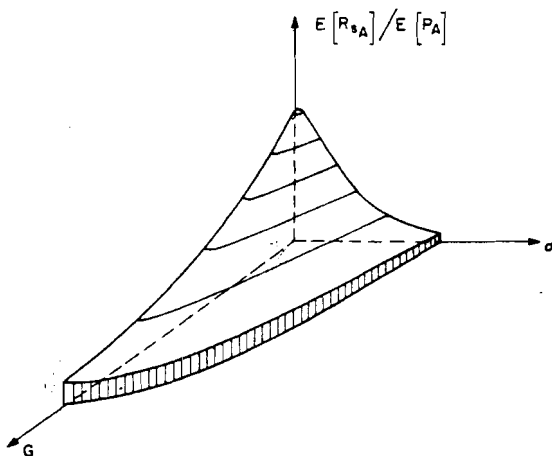


Fig. 5. Shape of surface runoff function ( $h_0 = 0$ ).

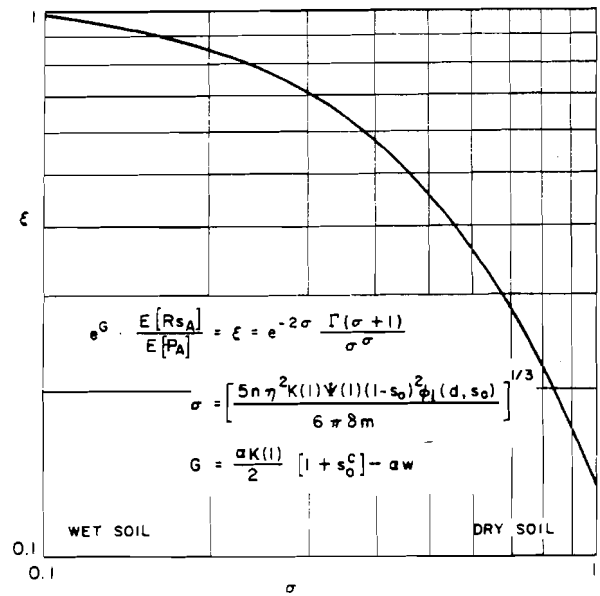


Fig. 6. Plot of surface runoff function ( $h_0 = 0$ ).

Impermeable surface fractions such as water surfaces and paved surfaces may be handled in the approximate fashion,

$$E[R_{s_A}] / E[P_A] = M_I + (1 - M_I)$$

$$\cdot (e^{-G-2\sigma} \Gamma(\sigma + 1) / \sigma^\sigma - E[E_r] / m_H) \quad (71)$$

for positive values of the bracketed quantity, where  $M_I$  is the fraction of area which is impermeable.

FREQUENCY OF FLOOD VOLUME

An immediate result of hydrologic utility results from transforming the partial duration series of (61) into an annual exceedance series in order to obtain the recurrence interval, in years, for a dimensionless storm surface runoff of magnitude  $z$  [see Eagleson, 1972]. This is the 'frequency of flood volume' and is given by

$$T_E^{-1} = m_r \text{Prob} [\eta R_{s_A} > z] = 2m_r (z + \eta E[E_r])^{1/2} e^{-G-2\sigma} \cdot K_1 [2(z + \eta E[E_r])^{1/2}] \Gamma(\sigma + 1) / \sigma^\sigma \quad (72)$$

where  $T_E$  is the recurrence interval in years.

COMPARISON WITH OBSERVATIONS

In order to verify (71) and (72) through comparisons with observation, it is necessary to know the average soil moisture  $s_0$  and the fraction of impervious area  $M_I$ . The former value can be determined only by solution of the entire water balance equation, of which the surface runoff is but a single term, and most observations do not mention  $M_I$ . We can, however, at this stage of the development, evaluate the range of (68) as given by the extreme values  $s_0 = 0$  and  $s_0 = 1$ . This range can be compared with typical values of the ratio  $E[R_{s_A}] / E[P_A]$  as given by the U.S. Geological Survey. This comparison is made in Table 2a and 2b by using typical soil and climate parameters as given in Tables 1a and 1b. The comparison gives us confidence that (68) yields reasonable values for the surface runoff component of annual precipitation.

AVERAGE ANNUAL INFILTRATION

Equation (4) specifies the rainfall excess to be the difference between storm precipitation and storm infiltration. Taking

TABLE 2a. Observations of the Range of Values of the Surface Runoff Function [Hoyt et al., 1936]

Location	$E[R_{s_0}]/E[P_A]$
Red River above Grand Forks, North Dakota	0.02
Mississippi River above Keokuk, Iowa	0.12
Neosho River above Iola, Kansas	0.12
James River above Cartersville, Virginia	0.19
Chattahoochee River above West Point, Georgia	0.20
Miami River above Dayton, Ohio	0.21
Merrimac River above Lawrence, Massachusetts	0.24
Pomeraug River above Bennett's Bridge, Connecticut	0.27
Tennessee River above Chattanooga, Tennessee	0.31

expected values of (4), term-by-term, gives the average storm infiltration  $E[I_j]$  as

$$E[I_j] = E[h_j] - E[R_{s_j}^*] \quad (73)$$

Multiplying by  $m_v$ , the average number of storms per year, gives the average annual infiltration,

$$E[I_A] = E[P_A] - E[R_{s_A}^*] \quad (74)$$

Equation (53) can be used to rewrite (74) as

$$\frac{E[I_A]}{(1 - M_I)E[P_A]} = 1 - e^{-G-2\sigma\Gamma(\sigma+1)/\sigma^\sigma} \quad (75)$$

Otherwise,

$$\frac{E[I_A]}{(1 - M_I)E[P_A]} = 1 - E[E_r]/m_H \quad (76)$$

#### SUMMARY

Use of the Philip infiltration equation to derive the distribution of surface runoff volume from the distributions of rain-storm intensity and duration leads to a flood volume frequency relation and to an expression for the fraction of mean annual precipitation becoming mean annual surface runoff. This fraction is sensitive to the gravitational and capillary infiltration potentials  $G(s_0 = 0)$  and  $\sigma(s_0 = 0)$ , respectively, and to the average soil moisture  $s_0$ . Values of the fraction evaluated for typical climate and soil properties compare favorably with observations.

Both the flood volume frequency relation and the expected annual surface runoff volume are expressed in terms of physically meaningful climate-soil parameters so that sensitivity to physical changes can be explored.

#### NOTATION

- $A_0$  gravitational infiltration rate as modified by capillary rise from water table, centimeters per second.
- $a$  coefficient in surface runoff integral, seconds.
- $c$  pore disconnectedness index; coefficient in surface runoff integral,  $s^{1/2}$ .
- $d$  diffusivity index.
- $\bar{e}_p$  average rate of (soil surface) potential evaporation, centimeters per second.
- $E_r$  storm surface retention, centimeters.
- $E_{r_0}$  bare soil component of surface retention, centimeters.
- $E_{r_v}$  vegetal component of surface retention, centimeters.

TABLE 2b. Calculated Range of Values of the Surface Runoff Function

Runoff Function	Clay	Clay Loam	Silty Loam	Sandy Loam
$E[R_{s_0}]/E[P_A]$ for $s_0 = 0$	0.49	0.37	0	0
$E[R_{s_0}]/E[P_A]$ for $s_0 = 1$	0.85	0.68	0.20	0.01

Equation (68), table 1, and in-text tabulation were used in calculations.

- $f_i$  infiltration rate, centimeters per second.
- $f_i^*$  infiltration capacity, centimeters per second.
- $G$  gravitational infiltration parameter, equal to  $\alpha A_0$ .
- $G(0)$  dry soil gravitational infiltration parameter.
- $G(1)$  saturated soil gravitational infiltration parameter.
- $h$  storm depth, centimeters.
- $h_0$  surface retention capacity, centimeters.
- $I_A$  annual infiltration, centimeters.
- $I^*$  surface runoff integral.
- $i$  precipitation rate, centimeters per second.
- $K(1)$  saturated effective hydraulic conductivity, centimeters per second.
- $k(1)$  saturated intrinsic permeability, square centimeters.
- $k_v$  effective transpiring leaf area per unit of land surface.
- $M$  vegetated fraction of land surface.
- $M_I$  fraction of area which is impermeable.
- $m$  pore size distribution index.
- $m_H$  mean storm depth, centimeters.
- $m_i$  mean storm intensity, centimeters per second.
- $m_t$  mean storm duration, days.
- $m_v$  mean number of storms per year.
- $m_r$  mean length of rainy season, days.
- $n$  medium effective porosity, which equals volume of active voids divided by total volume.
- $P_A$  annual precipitation, centimeters.
- $R$  region of integration.
- $R_{s_j}$  storm surface runoff, centimeters.
- $R_{s_j}^*$  storm rainfall excess, centimeters.
- $r_s$  surface runoff rate, centimeters per second.
- $S_i$  infiltration sorptivity,  $\text{cm}/s^{1/2}$ .
- $s$  degree of effective medium saturation (i.e., effective soil moisture concentration), which equals volume of active soil moisture divided by effective volume of voids.
- $s_0$  time and spatial average effective soil moisture concentration in surface boundary layer.
- $s_1$  degree of effective saturation at surface of medium.
- $\bar{T}_a$  normal annual temperature, degrees Centigrade.
- $T_E$  recurrence interval, years.
- $t$  time, seconds.
- $t_r$  storm duration, days.
- $t_0$  time at which surface reaches saturation during precipitation, seconds.
- $\nabla_t$  cumulative depth of storm infiltration, centimeters.
- $v_{00}$  rate of surface storage, centimeters per second.
- $w$  upward apparent pore fluid velocity representing capillary rise from the water table, centimeters per second.
- $X$  soil-storm infiltration parameter.

- $Z$  depth to water table, centimeters.  
 $z$  value of storm surface runoff, centimeters.  
 $\alpha$  reciprocal of average rainstorm intensity, equal to  $m_i^{-1}$ , seconds per centimeter.  
 $\beta$  reciprocal of average time between storms, equal to  $m_t^{-1}$ , days $^{-1}$ .  
 $\delta$  reciprocal of average storm duration, equal to  $m_d^{-1}$ , days $^{-1}$ .  
 $\eta$  reciprocal of mean storm depth, equal to  $m_H^{-1}$ , cm $^{-1}$ .  
 $\kappa$  parameter of gamma distribution of storm depth.  
 $\lambda$  parameter of gamma distribution of storm depth, cm $^{-1}$ .  
 $\sigma$  capillary infiltration parameter.  
 $\sigma(0)$  dry soil capillary infiltration parameter.  
 $\sigma(1)$  saturated soil capillary infiltration parameter.  
 $\phi_i$  dimensionless infiltration diffusivity.  
 $\Psi(1)$  saturated soil matrix potential, centimeters (suction).  
 $E[ ]$  expected value of [ ].  
 $F[ ]$  cumulative distribution function.  
 $f( )$  probability density function of ( ).  
 $K_0[ ]$  Bessel function of order zero.  
 $K_1[ ]$  Bessel function of order one.  
 $\text{Var}[ ]$  variance of [ ].  
 $\gamma[ ]$  incomplete gamma function.  
 $\Gamma[ ]$  gamma function.

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