Climate, Soil, and Vegetation
4. The Expected Value of Annual Evapotranspiration

PETER S. EAGLESON

*Department of Civil Engineering, Massachusetts Institute of Technology
Cambridge, Massachusetts 02139*

The depth of interstorm evapotranspiration from natural surfaces is composed (by proportion to vegetal canopy density) of evaporation from bare soil and transpiration from vegetation. The former is obtained in terms of random variables describing initial soil moisture, time between storms, and potential rate of evapotranspiration from an exfiltration analogy to the Philip infiltration equation modified to incorporate moisture extraction by plant roots. The latter is assumed to occur at the potential rate for natural vegetal systems. In a zeroth-order approximation the initial soil moisture is fixed at its climatic space and time average whereby using an exponential distribution of time between storms and a constant potential rate of evapotranspiration the expected value of interstorm evapotranspiration is derived. This mean value is used to obtain the annual average point evapotranspiration as a fraction of the potential value and as a function of dimensionless parameters defining the climate-soil-vegetation system.

INTRODUCTION

In seeking a physically based analytical description of the average annual water balance [*Eagleson*, 1978*a*] we must deal with the random variability of the various climatic variables involved in the physical processes defining the separate water balance components. Primary among these random variations are the alternate intervals of infiltration and evapotranspiration, the rates of rainfall and potential evapotranspiration, and the rate-controlling soil moisture concentration. Process physics can be introduced into the parameters of a statistical distribution of evapotranspiration by deriving this distribution from the known distributions of the independent climatic variables by means of an analytical relation between the interstorm period, the potential rate of evapotranspiration, and the volumes of exfiltrated and transpired soil moisture. We will find only the first moment of this distribution.

This approach follows one taken earlier by the author [*Eagleson*, 1972] to incorporate catchment-stream dynamics into a derivation of flood frequency.

POTENTIAL EVAPOTRANSPIRATION

The exchange of water mass between soil and atmosphere across the land surface is driven by the solar energy which is available at the surface for evaporation. This atmospheric evaporative capacity is measured by the potential rate of evaporation e_p. It is most commonly defined in terms of the residual of an energy budget computation in which, for practical applications, advection and storage are usually neglected [*Penman*, 1948].

It is important to remember throughout this work that e_p represents the rate of evaporation which is expected from a particular surface under conditions of unlimited moisture supply at that surface. This means that e_p will be a function not only of the climatic energy input (insolation, cloud cover, etc.) but also of the surface itself through its albedo, through its effective evaporation area per unit of land surface, and through the parameters governing the vapor transport in the atmospheric boundary layer (i.e., area size, surface roughness, wind speed, and ambient vapor concentration). We need to know all these factors before e_p can be specified. Since e_p will influence the soil moisture available for actual evapotranspiration and hence for vegetal growth, it will affect the albedo of the composite soil-vegetation surface, the aerodynamic properties of the vegetation (density and height), and the vapor content of the ambient near-surface atmosphere. This feedback and its climatic significance have been pointed out by many investigators, including *Bouchet* [1963], *Morton* [1965, 1968], and more recently *Charney* [1975].

In this work we will decouple the atmosphere-land surface feedback at this point and treat e_p as an independent variable, remembering, however, that in application we must compute its value for the particular climate-soil-vegetation system being considered. To facilitate this computation, which at best can only be approximate, we have formulated the problem using separate potential rates for the bare soil surface and for the vegetated surface. The fact that these may have different numerical values results primarily from the different effective elevations for the evaporation and transpiration, the effective transpiration area per unit of land surface, the configuration (with respect to insolation and wind) of the effective transpiration surfaces, and the shielding of the bare soil component by the vegetal component.

PROBLEM FORMULATION

We will consider permeable natural surfaces which are a homogeneous mixture of vegetated and bare soil fractions. Allowing only vertical moisture fluxes, we will subdivide the evapotranspiration process into three separate elements.

1. Surface retention loss E_r is the depth of free-standing water left on all surfaces at the conclusion of precipitation and surface runoff. This retention will be assumed to be removed by evaporation at the wet surface potential rate e_p from the bare soil surface and at the vegetal potential rate e_{p_v} from the vegetation.
2. Bare soil evaporation E_s is the depth of soil moisture evaporated from the bare soil fraction of the surface. This volume is brought to the surface against gravity by capillarity in the soil moisture movement process called exfiltration. It takes place at rate f_e.
3. Transpiration E_v is the depth of soil moisture evaporated metabolically by plants from the vegetated fraction of the surface. This process takes place at rate e_v.

Paper number 8W0187.
0043-1397/78/058W-0187$01.00

The material, including vegetation, and slope of any surface will determine the maximum depth of water h_0, equal to surface retention capacity, which can be held there by surface forces against the forces of capillarity and gravity. Any rainstorm must exceed this depth before infiltration and/or surface runoff can occur. At the conclusion of each storm the surface retention will be evaporated at the potential rate, being completely exhausted provided the interstorm period is sufficiently long. With this concept and the rainfall event series modeled in Figure 1, we will now look at the physics of these three components.

BARE SOIL EVAPORATION

At the beginning of each interstorm period there will be a thin surface retention film of depth E_{rs} on all soil surfaces. This film will be evaporated at the soil surface potential rate e_p and must be exhausted before the soil can begin delivering moisture to the surface for evaporation. We can express this surface retention loss as

$$
\begin{aligned}
E_{rs} &= h \qquad h < h_0 \qquad t_b \geq h/e_p \\
E_{rs} &= h_0 \qquad h \geq h_0 \qquad t_b \geq h_0/e_p \qquad (1)\\
E_{rs} &= e_p t_b \qquad \text{otherwise}
\end{aligned}
$$

where h equals storm depth, which has the probability density function [*Eagleson*, 1978*b*]

$$
f_H(h) = \frac{\lambda(\lambda h)^{\kappa-1} e^{-\lambda h}}{\Gamma(\kappa)} \qquad h \geq 0 \qquad (2)
$$

And t_b is the time between storms, which has the probability density function [*Eagleson*, 1978*b*]

$$
f_T(t_b) = \beta e^{-\beta t_b} \qquad t_b \geq 0 \qquad (3)
$$

where

$$
\beta^{-1} = m_{t_b}
$$

is the average time between storms.

The time t^* at which the surface retention is completely evaporated is

$$
t^* = E_{rs}/e_p \qquad (4)
$$

As can be seen in Figure 2, for storm duration $t_b > t^*$ the actual rate of evaporation from the soil surface depends upon the relative magnitudes of the potential rate of evaporation e_p and the potential rate of exfiltration (called the 'exfiltration capacity') f_e^*. This latter quantity represents the rate at which the soil can deliver moisture to a dry surface and is a function of the internal soil moisture concentration s, the rate of moisture extraction by vegetation, and the rate of moisture supply from the water table [*Eagleson*, 1978*c*]. Early in an interstorm period while the internal soil moisture is high (see Figure 2), this potential delivery rate is high and normally exceeds the rate e_p at which the climate can remove water from the surface. In response to this inequality the surface soil moisture s_1 takes on a value of $0 < s_1 < 1$ such that the actual delivery rate f_e exactly matches e_p. As time goes on, moisture is evaporated, and the internal soil moisture falls. The surface soil moisture must also fall to maintain $f_e = e_p$. This continues until time $t^* + t_0$, when $s_1 = 0$ and $f_e = f_e^*$. For $t > t^* + t_0$, $f_e^* < e_p$, and evaporation from the soil surface proceeds at the rate f_e^*. The process ceases either when $t = t_e$, at which time $f_e^* \equiv 0$, or at $t = t_b$, when the next rainfall begins, whichever occurs first.

The relationship among these rates and times is illustrated for a typical interstorm period in Figure 2. In this figure we indicate our assumption that the randomly variable e_p may be replaced by its long-term average value; that is,

$$
e_p(t) \simeq \bar{e}_p \qquad (5)
$$

Extending the infiltration equation of *Philip* [1969], *Eagleson* [1978*c*] has represented the exfiltration rate by

$$
f_e = \tfrac{1}{2} S_e t^{-1/2} - M e_v + w \qquad (6)
$$

where

- M vegetated fraction of surface;
- e_v vegetal transpiration rate (a function of the spatial average soil moisture concentration in the root zone), centimeters per second;
- w apparent velocity of capillary rise from water table, centimeters per second.

The exfiltration 'desorptivity' S_e is defined for a dry surface by

$$
S_e = 2 s_0^{1+d/2} \left[\frac{n K(1) \Psi(1) \phi_e(d)}{\pi m} \right]^{1/2} \qquad s_1 = 0 \qquad (7)
$$

where

- n effective porosity of soil;
- $K(1)$ saturated effective hydraulic conductivity of soil, centimeters per second;
- $\Psi(1)$ saturated matrix potential of soil, centimeters (suction);
- $\phi_e(d)$ dimensionless desorption diffusivity of soil;
- d diffusivity index of soil;
- m pore size distribution index of soil;
- s_0 initial soil moisture which is assumed constant throughout the surface boundary layer.

Using (7) in (6) gives the exfiltration capacity

$$
f_e^* = s_0^{1+d/2} \left[\frac{n K(1) \Psi(1) \phi_e(d)}{\pi m t} \right]^{1/2} - M e_v + w \qquad (8)
$$

We will assume it to apply in the interval $t^* \leq t \leq t_b$, even though the history of the exfiltration process was not in accord with the conditions of its derivation.

But what are t_0 and t_e? To estimate t_0, we will use the customary assumption [*Linsley et al.*, 1958, p. 179] that

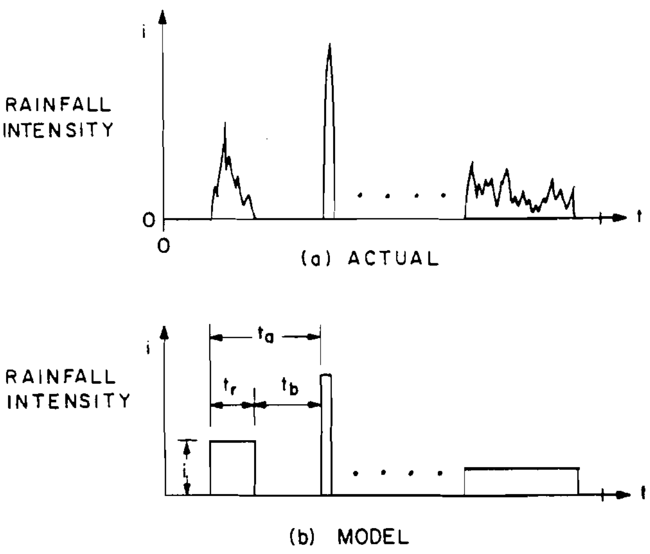

Fig. 1. Model of precipitation event series.

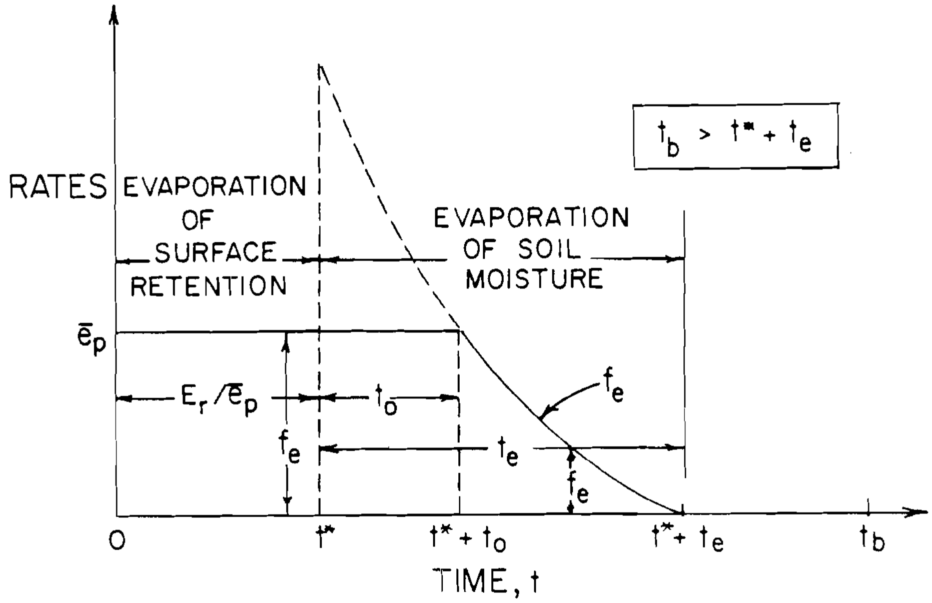

Fig. 2. Interstorm evaporation from bare soil.

$$\bar{e}_p = f_e^* \tag{9a}$$

as given by (6) and (7) when

$$f_e^*(\nabla_e) = f_e^*(\bar{e}_p t_o) \tag{9b}$$

where $\nabla_e(t)$ is the cumulative volume of moisture extractions at time $t \geq t^*$.

To calculate ∇_e, we must consider all processes removing soil moisture through the soil-atmosphere interface. This requires that we first examine transpiration by vegetation.

TRANSPIRATION

In a steady state approximation to plant moisture movement, *van den Honert* [1948] drew a physiological analogy to Ohm's law to write the volume rate of transpiration per unit width of plant community as

$$q_T = (1)e_v = \left[\frac{\Psi_s - \Psi_l}{r_s + r_p}\right]\gamma_w^{-1} \tag{10}$$

where

q_T transpiration rate per unit width, $cm^2\ s^{-1}$;
Ψ_s soil moisture potential, bars;
Ψ_l leaf moisture potential, bars;
r_s resistance to moisture flow in soil, $s\ cm^{-1}$;
r_p resistance to moisture flow in plants, $s\ cm^{-1}$;
γ_w specific weight of soil water, $dyn\ cm^{-3}$.

Under conditions of unlimited water supply to the plant roots we expect the transpiration rate e_v to equal the potential rate of transpiration $\bar{e}_{pv}$ for the particular vegetal type. As the soil moisture is progressively reduced, the soil moisture potential becomes larger (negatively). That is [*Eagleson*, 1978*c*],

$$\Psi_s = \Psi_s(s) = \Psi(1)s^{-1/m} \tag{11}$$

If we assume that the resistances r_s and r_p remain constant and remember that Ψ_l is negative also, e_v will tend to be reduced and thus the plant will be put in a state of stress. The plant

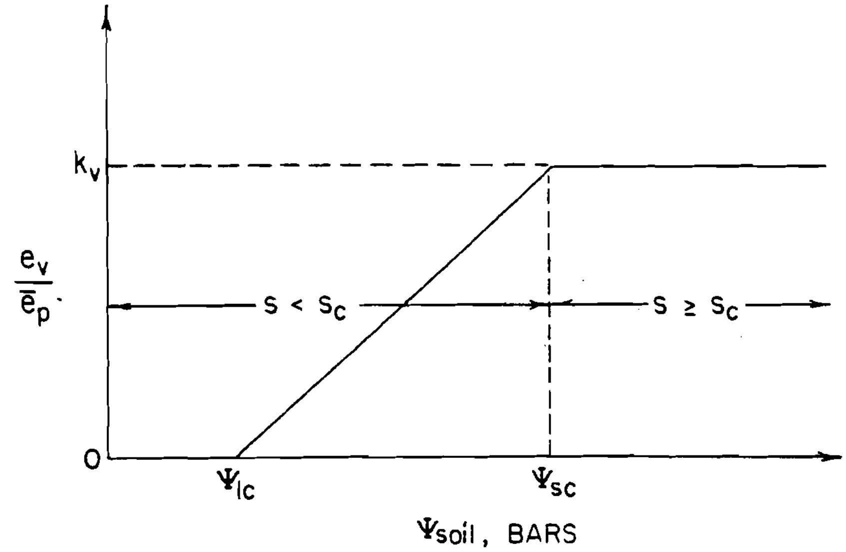

Fig. 3. Steady state interstorm transpiration.

responds by closing its stomata sufficiently to make Ψ_l larger (negatively) and thus to maintain the difference $\Psi_s - \Psi_l$. As the soil drying progresses, the plant eventually reaches its limit of stomatal control at which point $\Psi_l = \Psi_{lc}$, the critical leaf moisture potential. For still lower soil moistures, Ψ_s becomes a larger negative number, while Ψ_l remains at Ψ_{lc}. To the extent that the resistances remain constant, we can thus expect e_v to decline linearly with Ψ_s as the soil is dried beyond this point. Using (10), we can find this critical soil moisture potential Ψ_{sc} to be

$$\Psi_{sc} = \gamma_w \bar{e}_{pv}(r_s + r_p) + \Psi_{lc} \qquad (12)$$

In a manner similar to that suggested by *Cowan* [1965] our transpiration model is then written for a unit area of vegetated surface,

$$W(s) = e_v/\bar{e}_p = \frac{\Psi(1)s^{-1/m} - \Psi_{lc}}{\bar{e}_p(r_s + r_p)\gamma_w} \qquad \Psi_{lc} \le \Psi_s \le \Psi_{sc} \qquad (13a)$$

and

$$W(s) = k_v \qquad \Psi_s \ge \Psi_{sc} \qquad (13b)$$

where

$$k_v = \bar{e}_{pv}/\bar{e}_p \qquad (14)$$

This model is illustrated in Figure 3. Typical values of the resistances and critical potentials are given in the plant physiological literature, including the works of *Evans* [1963], *Cowan* [1965], *Slatyer* [1967], and *Kramer* [1969].

To use (13) in the calculation of soil moisture extraction requires specification of the species-dependent critical leaf potential Ψ_{lc} and transpiration efficiency k_v as well as the species- and time-dependent plant-soil resistance $r_s + r_p$. Generalization of these vegetal parameters is beyond the scope of this work, and an expedient simplification will be introduced.

We will neglect the time variation of $W(s)$ during the interstorm period, replacing $W(s)$ by its value at the space and time average soil moisture. That is,

$$W(s) \equiv W(s_0) \qquad (15)$$

Confining our attention to natural vegetal systems (i.e., no irrigation or artificial fertilization), we hypothesize that through natural selection, all species will operate under average soil moisture conditions for which $\Psi_s \ge \Psi_{sc}$, since under these conditions they are unstressed. It is suggested that the stressed condition is unstable in the long term due to increased plant susceptibility to drought and to disease.

Under this hypothesis, a unit area of vegetation will transpire at the potential rate $\bar{e}_{pv}$ and give

$$W = W(s_0) = \bar{e}_{pv}/\bar{e}_p = k_v \qquad (16)$$

The numerical value of k_v is a matter of some controversy.

Slatyer [1967, p. 53] states that because the actual area of evaporating surface may be much greater than the equivalent land area, the total evapotranspiration from a plant community, per unit of land area, may exceed that from a similar area of bare wet soil. *Linacre et al.* [1970] found values of k_v for water plants which ranged from 0.6 to 2.5 depending upon species. *Penman* [1963, Table 18, p. 52], however, shows peak water use to be independent of crop type in a given climate, and *Kramer* [1969, p. 338] states that evaporation from a plant community never exceeds that from a similar area of wet soil with the same exposure.

Under the assumption that k_v reflects the effective area of transpiring leaf surface per unit of vegetated land surface, we will use it as an amplification factor to approximate the surface retention loss E_{rv} from the vegetation.

This film will be evaporated at the vegetal potential rate $k_v\bar{e}_p$ and must be exhausted before transpiration can begin. We express this surface retention loss as

$$\begin{aligned} E_{rv} &= h & h < k_v h_0 \quad t_b > h/k_v\bar{e}_p \\ E_{rv} &= k_v h_0 & h \ge k_v h_0 \quad t_b > k_v h_0/k_v\bar{e}_p \qquad (17) \\ E_{rv} &= k_v\bar{e}_p t_b & \text{otherwise} \end{aligned}$$

EVAPOTRANSPIRATION

Considering only plant communities composed of homogeneous mixtures of vegetation and bare soil and assuming the plant growing season to be coincident with the rainy season, we will proportion the total evapotranspiration E_T from a unit land surface according to

$$E_T = (1 - M)E_S + ME_V \qquad (18)$$

in which

$$\begin{aligned} E_S &= E_s + E_{rs} \\ E_V &= E_v + E_{rv} \end{aligned} \qquad (19)$$

And as was mentioned before M equals the canopy density, which is the fraction of land area covered by leaves.

INTERSTORM EVAPOTRANSPIRATION

We are now ready to calculate the interstorm volume of evapotranspiration as indicated by Figure 2. This first requires estimation of the times t_0 and t_e.

If we begin at $t = t^*$, soil moisture will be exhausted by both exfiltration and transpiration. Assuming both processes draw from the same soil moisture reservoir, we can use (6) and (18) to write the volume $\forall_e$ of extracted soil moisture as

$$\forall_e = \{(1 - M)[S_e(t - t^*)^{1/2} + (w - Mk_v\bar{e}_p)(t - t^*)]\} + Mk_v\bar{e}_p(t - t^*) \qquad t - t^* \le t_e \qquad (20)$$

while from (6),

$$t - t^* = \{S_e/[2(f_e^* - w + Mk_v\bar{e}_p)]\}^2 \qquad (21)$$

Substituting (21) in (20) and using (9), we obtain

$$t_0 = \frac{S_e^2}{2\bar{e}_p^2(1 + Mk_v - w/\bar{e}_p)} \cdot \left(1 - M + \frac{M^2k_v + (1 - M)w/\bar{e}_p}{2(1 + Mk_v - w/\bar{e}_p)}\right) \qquad (22)$$

Dividing by the average interstorm duration m_{t_b}, we obtain the evapotranspiration effectiveness

$$t_0/m_{t_b} = \left(\frac{1 - M}{1 + Mk_v - w/\bar{e}_p} + \frac{M^2k_v + (1 - M)w/\bar{e}_p}{2(1 + Mk_v - w/\bar{e}_p)^2}\right)E \qquad (23)$$

in which

$$E = \beta S_e^2/2\bar{e}_p^2 \qquad (24)$$

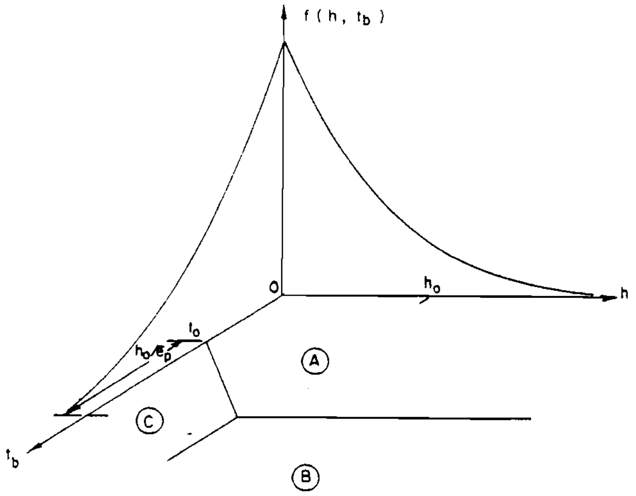

Fig. 4. Integration regions for calculation of expected value of interstorm evapotranspiration.

is the bare soil ($M = 0$) evaporation effectiveness.

We can also find t_0 (and thus E) by solving the linearized diffusion equation under a constant flux boundary condition to obtain the time at which $s_1 = 0$. The form of this solution is the same as that given in (24) except that (by using the same effective diffusivity) the constant coefficient is $\pi^2/16$ rather than $\frac{1}{2}$ [*Carslaw and Jaeger*, 1959, p. 75].

Because $E \sim t_0/m_{t_b}$, E represents the relative importance of soil properties in the dynamics of exfiltration. Referring to Figure 2, we see that for large t_0 relative to storm duration, evapotranspiration will occur predominately at the potential rate, and the climate-soil-vegetation system is largely under climate control insofar as evapotranspiration is concerned. When t_0 is small relative to t_b, the evapotranspiration is controlled by the soil properties. The ratio $E \sim t_0/m_{t_b}$ measures the average state of this relationship for the given climate-soil-vegetation system.

From (23) we find that $t^* = E_r/\bar{e}_p$ and t_0 are of the same order of magnitude. The surface retention will thus have a significant effect on the exfiltration dynamics, and we must incorporate it in our further calculations.

To calculate the volume E_{T_j} of evapotranspiration during the jth interstorm period, we will treat the bare soil evaporation E_{S_j} and the transpiraton E_{V_j} separately. Look first at the bare soil evaporation. We can use Figure 2 to identify three regimes of behavior:

Regime A

$$E_{S_j} = \bar{e}_p t_b \qquad 0 \le t_b \le t^* + t_0 \tag{25a}$$

Regime B

$$E_{S_j} = h_0 + \bar{e}_p t_0 + \int_{t_0}^{t_b - h_0/\bar{e}_p} f_e^* \, dt \tag{25b}$$

$$t_e \ge t_b > t_0 + h_0/\bar{e}_p \qquad h \ge h_0$$

and

$$E_{S_j} = h_0 + \bar{e}_p t_0 + \int_{t_0}^{t_e - h_0/\bar{e}_p} f_e^* \, dt \tag{25c}$$

$$t_b > t_e + t^* \quad h \ge h_0$$

Regime C

$$E_{S_j} = h + \bar{e}_p t_0 + \int_{t_0}^{t_b - h/\bar{e}_p} f_e^* \, dt \tag{25d}$$

$$t_e \ge t_b > t_0 + h/\bar{e}_p \qquad h < h_0$$

and

$$E_{S_j} = h + \bar{e}_p t_0 + \int_{t_0}^{t_e} f_e^* \, dt \tag{25e}$$

$$t_b > t_e + t^* \quad h < h_0$$

Here for $t = t_e$, $f_e^* = 0$. From (6) this gives

$$t_e = \{S_e/[2\bar{e}_p(Mk_v - w/\bar{e}_p)]\}^2 \tag{26}$$

Again dividing by m_{t_b},

$$t_e/m_{t_b} = E/2(Mk_v - w/\bar{e}_p)^2 \tag{27}$$

The expected value of interstorm evaporation is given by the integral under the joint probability density function

$$E[E_{S_j}] = \iint E_{S_j} f(h, t_b) \, dh \, dt_b \tag{28}$$

If we assume h and t_b are independent and use (2) and (3), the joint distribution is written

$$f(h, t_b) = f_H(h) f_T(t_b) = \frac{\beta\lambda(\lambda h)^{\kappa-1} e^{-\lambda h - \beta t_b}}{\Gamma(\kappa)} \tag{29}$$

Equations (25a)-(25e) divide the integration region for bare soil into the three regimes separated by the solid lines in Figure 4. Equation (28) can now be written as follows:

In regime A

$$E_A[E_{S_j}] = \int_0^{h_0} f(h) \, dh \int_0^{t_0 + h/\bar{e}_p} \bar{e}_p t_b f(t_b) \, dt_b + \int_{h_0}^{\infty} f(h) \, dh \int_0^{t_0 + h_0/\bar{e}_p} \bar{e}_p t_b f(t_b) \, dt_b \tag{30}$$

In regime B

$$E_B[E_{S_j}] = \int_{h_0}^{\infty} f(h) \, dh \int_{t_0 + h_0/\bar{e}_p}^{t_e + h_0/\bar{e}_p} \left[h_0 + \bar{e}_p t_0 + \int_{t_0}^{t_b - h_0/\bar{e}_p} f_e^* \, dt \right] f(t_b) \, dt_b + \int_{h_0}^{\infty} f(h) \, dh \int_{t_e + h_0/\bar{e}_p}^{\infty} \left[h_0 + \bar{e}_p t_0 + \int_{t_0}^{t_e - h_0/\bar{e}_p} f_e^* \, dt \right] f(t_b) \, dt_b \tag{31}$$

In regime C

$$E_C[E_{S_j}] = \int_0^{h_0} f(h) \, dh \int_{t_0 + h/\bar{e}_p}^{t_e + h/\bar{e}_p} \left[h + \bar{e}_p t_0 + \int_{t_0}^{t_b - h/\bar{e}_p} f_e^* \, dt \right] f(t_b) \, dt_b$$

$$+ \int_0^{h_0} f(h)\,dh \int_{t_e + h/\bar{e}_p}^{\infty} \left[h + \bar{e}_p t_0 + \int_{t_0}^{t_e} f_e^* \, dt \right] f(t_b)\,dt_b \quad (32)$$

The average interstorm evaporation from a unit area of bare soil is then given by adding the above three components to get

$$E[E_{S_j}] = E_A[E_{S_j}] + E_B[E_{S_j}] + E_C[E_{S_j}] \quad (33)$$

Using (6) and (29) to integrate (30), (31), and (32) and adding according to (33) gives for bare soil evaporation,

$$\frac{\beta}{\bar{e}_p} E[E_{S_j}] = \frac{\gamma[\kappa, \lambda h_0]}{\Gamma(\kappa)} - \left[1 + \frac{\beta h_0/\bar{e}_p}{\lambda h_0}\right]^{-\kappa} \frac{\gamma[\kappa, \lambda h_0 + \beta h_0/\bar{e}_p]}{\Gamma(\kappa)} e^{-BE} + \left\{1 - \frac{\gamma[\kappa, \lambda h_0]}{\Gamma(\kappa)}\right\} \{1 - e^{-BE - \beta h_0/\bar{e}_p}$$
$$\cdot [1 + Mk_v + (2B)^{1/2}E - w/\bar{e}_p] + e^{-CE - \beta h_0/\bar{e}_p}[Mk_v + (2C)^{1/2}E - w/\bar{e}_p] + (2E)^{1/2} e^{-\beta h_0/\bar{e}_p} [\gamma(\tfrac{3}{2}, CE) - \gamma(\tfrac{3}{2}, BE)]\}$$
$$+ \left[1 + \frac{\beta h_0/\bar{e}_p}{\lambda h_0}\right]^{-\kappa} \frac{\gamma[\kappa, \lambda h_0 + \beta h_0/\bar{e}_p]}{\Gamma(\kappa)} \{(2E)^{1/2}[\gamma(\tfrac{3}{2}, CE) - \gamma(\tfrac{3}{2}, BE)] + e^{-CE}[Mk_v + (2C)^{1/2} E - w/\bar{e}_p]$$
$$- e^{-BE}[Mk_v + (2B)^{1/2} E - w/\bar{e}_p]\} \quad (34)$$

Here

$$B = \frac{1 - M}{1 + Mk_v - w/\bar{e}_p} + \frac{M^2 k_v + (1 - M)w/\bar{e}_p}{2(1 + Mk_v - w/\bar{e}_p)^2} \quad (35)$$

and

$$C = \tfrac{1}{2}(Mk_v - w/\bar{e}_p)^2 \quad (36)$$

For a vegetal surface we are assuming that transpiration is always at the potential rate $k_v \bar{e}_p$, thus we have simply

$$(\beta/\bar{e}_p)\, E[E_{V_j}] = k_v \quad (37)$$

Finally, we obtain the expected interstorm evapotranspiration from a homogeneous composite surface by weighting (34) and (41) according to the canopy density. By following (18) we get

$$E[E_{T_j}] = (1 - M)E[E_{S_j}] + ME[E_{V_j}] \quad (38)$$

In (34) we see two new dimensionless parameters λh_0 and $\beta h_0/\bar{e}_p$. The first of these, λh_0, appears when we account for those storms which do not fill the surface retention capacity. The evapotranspiration will thus vary inversely with this parameter. The second parameter, $\beta h_0/\bar{e}_p$, can be rewritten

$$\beta h_0/\bar{e}_p = (h_0/\bar{e}_p)/m_{t_b}$$

From Figure 2, with $\dot{E}_r = h_0$, we see that this parameter measures the potential effect of surface retention on the dynamics of the exfiltration process.

ANNUAL EVAPOTRANSPIRATION

To expand (38) to the desired mean annual (seasonal) evapotranspiration requires that we recognize that for ν, the number of interstorm periods in a year (season), the annual evapotranspiration is

$$E_{T_A} = \sum_{j=1}^{\nu} E_{T_j} \quad (39)$$

the expectation of which is given by

$$E[E_{T_A}] = m_\nu E[E_{T_j}] \quad (40)$$

where

$$m_\nu \equiv E[\nu] \quad (41)$$

Following our assumption of a constant potential evapotranspiration rate the weighted mean annual (seasonal) potential evapotranspiration is

$$E[E_{P_A}] = m_\nu m_{t_b} \bar{e}_p^* = m_\nu \bar{e}_p^*/\beta \quad (42)$$

where $\bar{e}_p^*$, the weighted average potential evapotranspiration rate, is given by

$$\bar{e}_p^* = (1 - M)\bar{e}_p + M\bar{e}_{pv} = [1 - M(1 - k_v)]\bar{e}_p \quad (43)$$

Dividing (40) by (42) gives

$$J(E, M, k_v, h_0) = \frac{E[E_{T_A}]}{E[E_{P_A}]} = \frac{(1 - M)\beta E[E_{S_j}]/\bar{e}_p + M\beta E[E_{V_j}]/\bar{e}_p}{(1 - M + Mk_v)} \quad (44)$$

where $\beta E[E_{S_j}]/\bar{e}_p$ and $\beta E[E_{V_j}]/\bar{e}_p$ are given by (34) and (37), respectively.

Equation (44) is evaluated as a function of E for representative values of the parameters κ, k_v, λh_0, $\beta h_0/\bar{e}_p$, and M and is presented in Figures 5 and 6 under the usual condition that $w/\bar{e}_p \ll 1$.

Bare soil. By letting $M = 0$, $w/\bar{e}_p \ll 1$, and $h_0 = 0$ in (44), the bare soil evaporation function is

$$E[E_{T_A}]/E[E_{P_A}] = J(E) = 1 - [1 + 2^{1/2} E]e^{-E} + (2E)^{1/2}\, \Gamma[\tfrac{3}{2}, E] \quad (45)$$

where

$$\Gamma[a, x] = \int_x^{\infty} e^{-t} t^{a-1}\, dt \quad (46)$$

and E is given by (24) and (7) as

$$E = [2\beta n K(1)\Psi(1)/\pi m \bar{e}_p^2]\phi_e s_0^{d+2} \quad (47)$$

Equation (45) is plotted in Figure 5 along with its asymptotes. Let us look at these asymptotes to see what (46) tells us physically.

As the precipitation events occur more frequently, as the soil sorptivity increases, and/or as the potential rate of evaporation decreases (such as in a cold moist climate), the parameter E, as given by (47), increases. These conditions all indicate an increase in relative evaporation, and hence for a constant value of the climatic parameter $\bar{e}_p$ we expect the actual annual evaporation to approach the potential in the limit. Indeed, if

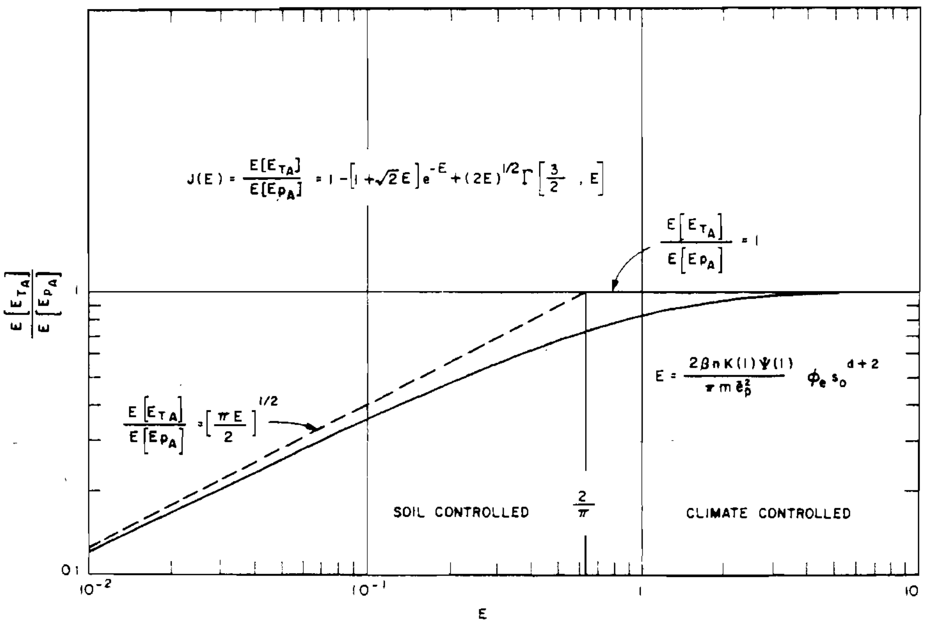

Fig. 5. Bare soil evaporation function ($w/\bar{e}_p \ll 1$).

we let $E \to \infty$ in (45) and recognize that $\Gamma[\frac{3}{2}, \infty] = 0$, we have

$$\lim_{E\to\infty} E[E_{T_A}]/E[E_{P_A}] = 1 \tag{48}$$

The actual evaporation is thus controlled primarily by the climate for large E through the potential rate of evaporation.

At the other extreme, when precipitation events are few, with long times between, when the soil sorptivity is low, and/or when the potential rate of evaporation is high (such as in hot dry climates), the parameter E decreases. These conditions all indicate a decrease in relative evaporation. Taking the limit of (45) as $E \to 0$, we have

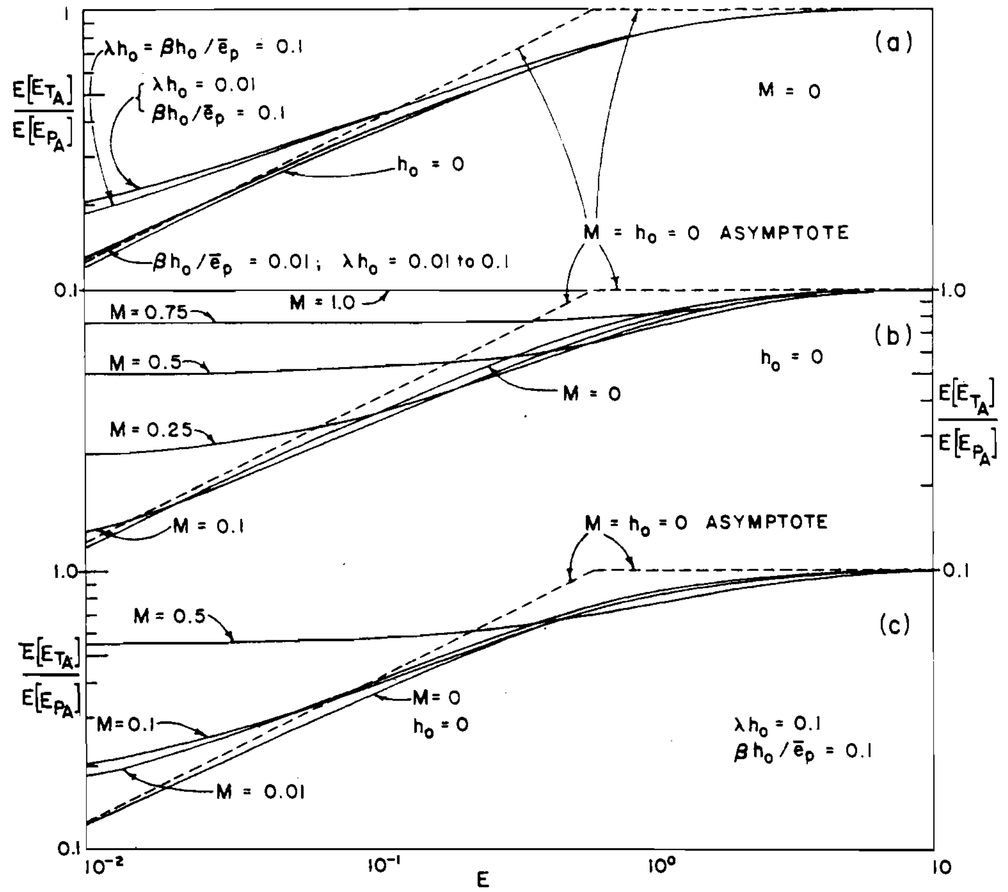

Fig. 6. Evapotranspiration function ($\kappa = 0.5$, $k_v = 1$, and $w/\bar{e}_p \ll 1$).

$$\lim_{E \to 0} E[E_{T_A}]/E[E_{p_A}] = [\pi E/2]^{1/2} \qquad (49)$$

Under these arid conditions we see that the soil sorptivity becomes the limiting factor in controlling the actual evaporation. For constant soil properties the sorptivity is directly related to the soil moisture s_0 and hence to the availability of precipitation.

The intersection of these two asymptotes is given by setting (49) equal to 1, from which we obtain

$$E_c = 2/\pi \qquad (50)$$

which may be used as a rational criterion for the classification of climate-bare soil systems as either soil controlled ($E < E_c$), or climate controlled ($E > E_c$), insofar as relative evaporation is concerned.

We must remember that this development assumes $e_p >> w$, which, although it is the 'normal' condition, will break down in a cold wet climate with a high water table.

It is interesting to note that (45) gives a physically rational analytical basis for the form of the natural surface evapotranspiration function developed empirically by earlier investigators such as *Budyko* [1956] and *Pike* [1964].

Bare soil with surface retention. Using empirical expressions for interception [*Wigham*, 1973, p. 4.6] to estimate surface retention capacity, we find that $h_0 = 0(1)$ mm. Observations of climate properties [*Eagleson*, 1978*b*] give $\beta = 0(10^{-1} - 10^{-2})$ d^{-1}, $\lambda = 0(10^{-1} - 10^{-2})$ mm^{-1}, and $\bar{e}_p = 0(1)$ mm/d. Therefore $\lambda h_0 = 0(10^{-1} - 10^{-2})$ and $\beta h_0/\bar{e}_p = 0(10^{-1} - 10^{-2})$.

Equation (44) is evaluated over this range of surface retention in Figure 6a under bare soil conditions, $M = 0$ and $w/\bar{e}_p << 1$. As could be expected, surface retention makes an appreciable difference in the annual evaporation only in the very arid situations where E is small. It can also be seen in this figure that $J(E)$ is very insensitive, even at small E, to variations in the relative surface retention λh_0 but is quite sensitive there to changes in $\beta h_0/\bar{e}_p$.

Mixed vegetation and bare soil. In Figures 6*b* and 6*c*, (44) is plotted for various values of M from bare soil ($M = 0$) to complete vegetal cover ($M = 1$) both without surface retention and with representative values of these parameters. In all cases, $w/\bar{e}_p << 1$. With $k_v = 1$, vegetal cover apparently has a strong effect on the annual evapotranspiration, particularly for small E. However, in interpreting these figures we must keep in mind that M and E are related under natural conditions through the soil moisture s_0. For large E the system has excess moisture and can support much vegetation, thus M will tend to unity. For small E the reverse is true, and M will tend to zero. Specification of the anticipated functional relationship between E and M will transform $J(E, M, k_v, h_0)$ into $J(E, k_v, h_0)$. This important task is accomplished in a later paper [*Eagleson*, 1978*d*] through introduction of a natural selection hypothesis. Its existence will make $J(E, M) \to J(E)$, which further emphasizes the fundamental importance of the bare soil evaporation function $J(E)$ in controlling the evapotranspiration from vegetated surfaces.

EVAPORATION FROM SURFACE RETENTION

In a manner completely analogous to the above derivation we may isolate that portion E_r of the total evapotranspiration which comes from surface retention. This is

$$\frac{E[E_{r_A}]}{E[E_{p_A}]} = \frac{(1 - M)\beta E[E_{r_{s_j}}]/\bar{e}_p + M\beta E[E_{r_{v_j}}]/\bar{e}_p}{(1 - M + Mk_v)} \qquad (51)$$

If we neglect carryover of unevaporated retention, the bare soil component is

$$\beta E[E_{r_{s_j}}]/\bar{e}_p = 1 - e^{-\beta h_0/\bar{e}_p} \frac{\Gamma[\kappa, \lambda h_0]}{\Gamma(\kappa)} - \left[1 + \frac{\beta h_0/\bar{e}_p}{\lambda h_0}\right]^{-\kappa} \frac{\gamma[\kappa, (\lambda h_0 + \beta h_0/\bar{e}_p)]}{\Gamma(\kappa)} \qquad (52)$$

Also to the first approximation, the vegetal component is

$$\beta E[E_{r_{v_j}}]/\bar{e}_p = k_v \left\{1 - e^{-\beta h_0/\bar{e}_p} \frac{\Gamma[\kappa, \lambda k_v h_0]}{\Gamma(\kappa)} - \left[1 + \frac{\beta h_0/\bar{e}_p}{\lambda k_v h_0}\right]^{-\kappa} \frac{\gamma[\kappa, (\lambda k_v h_0 + \beta h_0/\bar{e}_p)]}{\Gamma(\kappa)}\right\} \qquad (53)$$

This component is useful in calculating surface runoff [*Eagleson*, 1978*e*].

SUMMARY AND CONCLUSIONS

Bare soil evaporation and vegetal transpiration are calculated for a typical interstorm period as functions of properties of the storm sequence, the surface, the soil, and the average rate of potential evapotranspiration. The expected value of annual interstorm evapotranspiration is then derived by using observed distributions of the random climatic variables. The normalized average annual evapotranspiration is defined in terms of the independent variable E, which is a function of the space and time average soil moisture within the surface boundary layer. The evapotranspiration function for vegetated surfaces appears to conform closely to that defined by bare soil conditions.

NOTATION

- E evaporation effectiveness.
- E_{p_A} average annual potential evapotranspiration, millimeters.
- E_r surface retention loss, millimeters.
- E_{rs} surface retention loss from bare soil fraction, millimeters.
- E_{rv} surface retention loss from vegetated fraction, millimeters.
- E_s soil moisture evaporation from bare soil fraction of surface, millimeters.
- E_S total evaporation from bare soil fraction of surface, millimeters.
- E_T total evapotranspiration, millimeters.
- E_{T_A} annual evapotranspiration, millimeters.
- E_{S_j} interstorm bare soil evaporation, millimeters.
- E_{T_j} interstorm evapotranspiration, millimeters.
- E_v transpiration from vegetated fraction of surface, millimeters.
- E_V total evapotranspiration from vegetated fraction of surface, millimeters.
- E_{V_j} interstorm evapotranspiration from vegetated fraction of surface, millimeters.
- e_p potential (soil surface) evaporation rate, millimeters per day.
- $\bar{e}_p$ long-term time average rate of potential (soil surface) evaporation, millimeters per day.
- $\bar{e}_{pv}$ long-term time average potential rate of transpiration, millimeters per day.
- e_p^* potential evapotranspiration rate, millimeters per day.

- e_v rate of transpiration, millimeters per day.
- f_e exfiltration rate, millimeters per day.
- f_e^* exfiltration capacity, millimeters per day.
- h storm depth, millimeters.
- h_0 surface retention capacity, millimeters.
- j counting variable.
- k_v ratio of potential rates of transpiration and soil surface evaporation.
- M vegetal canopy density.
- m pore size distribution index.
- m_H mean storm depth, millimeters.
- m_{t_b} mean time between storms, days.
- m_v mean number of storms per year.
- q_T transpiration rate per unit width, square millimeters per second.
- r_p resistance to moisture flow in plant, seconds per centimeter.
- r_s resistance to moisture flow in soil, seconds per centimeter.
- S_e exfiltration sorptivity, cm/s$^{1/2}$.
- s degree of medium saturation (i.e., soil moisture concentration), which equals volume of water divided by volume of voids.
- s_1 degree of saturation at surface of medium.
- t time, seconds.
- t_0 duration of exfiltration at the rate $\bar{e}_p$, days.
- t^* time at which surface retention is exhausted, days.
- t_b time between storms, days.
- t_r storm duration, days.
- ∇_e cumulative volume of interstorm exfiltration, millimeters.
- $W(s_0)$ relative rate of transpiration.
- w upward apparent pore fluid velocity representing capillary rise from the water table, centimeters per second.
- β reciprocal of average time between storms, equal to $m_{t_b}^{-1}$, days^{-1}.
- γ_w specific weight of liquid, dynes per cubic centimeter.
- κ parameter of gamma distribution of storm depth.
- λ parameter of gamma distribution of storm depths, equal to κ/m_H, mm^{-1}.
- ν number of interstorm periods in a year.
- ϕ_e dimensionless exfiltration diffusivity.
- Ψ_{lc} critical leaf moisture potential, centimeters (suction).
- Ψ_s soil moisture potential, centimeters (suction).
- Ψ_{sc} critical soil moisture potential, centimeters (suction).
- $\Psi(1)$ saturated soil matrix potential, cm (suction).
- Ψ_l leaf moisture potential, centimeters (suction).
- $E[\ ]$ expected value of [ ].
- $f(\ )$ probability density function of ( ).
- $J(\ )$ evapotranspiration function.
- $0(\ )$ of the order ( ).
- $\Gamma(\ )$ gamma function.
- $\gamma[a, x]$ incomplete gamma function.

Acknowledgments. This work was performed in part while the author was a visiting associate in the Environmental Quality Laboratory of the California Institute of Technology on sabbatical leave from MIT. Publication has been made possible by a grant from the Sloan Basic Research Fund of MIT.

REFERENCES

Bouchet, R. J., Evapotranspiration réelle et potentielle signification climatique, *Publ. 62*, Int. Ass. of Sci. Hydrol., Gentbrugge, Belgium, 1963.

Budyko, M. I., The heat balance of the earth's suface (in Russian), Leningrad, 1956. (Translated from Russian by N. A. Stepanova, Office of Technical Services, U.S. Department of Commerce, Washington, D. C., 1958).

Carslaw, H. S., and J. C. Jaeger, *Conduction of Heat in Solids*, 2nd ed., Oxford University Press, London, 1959.

Charney, J. G., Dynamics of deserts and droughts in the Sahel, *Quart. J. Roy. Meteorol. Soc.*, *101*(428), 193-202, 1975.

Cowan, J. R., Transport of water in the soil-plant-atmosphere system, *J. Appl. Ecol.*, *2*, 221-239, 1965.

Eagleson, P. S., Dynamics of flood frequency, *Water Resour. Res.*, *8*(4), 878-898, 1972.

Eagleson, P. S., Climate, soil, and vegetation, 1, Introduction to water balance dynamics, *Water Resour. Res.*, *14*, this issue, 1978*a*.

Eagleson, P. S., Climate, soil, and vegetation, 2, The distribution of annual precipitation derived from observed storm sequences, *Water Resour. Res.*, *14*, this issue, 1978*b*.

Eagleson, P. S., Climate, soil, and vegetation, 3, A simplified model of soil moisture movement in the liquid phase, *Water Resour. Res.*, *14*, this issue, 1978*c*.

Eagleson, P. S., Climate, soil, and vegetation, 6, Dynamics of the annual water balance, *Water Resour. Res.*, *14*, this issue, 1978*d*.

Eagleson, P. S., Climate, soil, and vegetation, 5, A derived distribution of storm surface runoff, *Water Resour. Res.*, *14*, this issue, 1978*e*.

Evans, L. T. (Ed.), *Environmental Control of Plant Growth*, Academic Press, New York, 1963.

Kramer, P. J., *Plant and Soil Water Relationships—A Modern Synthesis*, McGraw-Hill, New York, 1969.

Linacre, E. T., B. B. Hicks, G. R. Sainty, and G. Grauze, The evaporation from a swamp, *Agr. Meteorol.*, *7*, 375-386, 1970.

Linsley, R. K., Jr., M. A. Kohler, and J. L. H. Paulhus, *Hydrology for Engineers*, McGraw-Hill, New York, 1958.

Morton, F. I., Potential evaporation and river basin evaporation, *J. Hydrol. Div. Amer. Soc. Civil Eng.*, *HY6*(4534), 67-97, 1965.

Morton, F. I., Evaporation and climate—A study in cause and effect, *Sci. Ser. 4*, Inland Waters Branch, Dep. of Energy, Mines, and Resour., Ottawa, Canada, 1968.

Penman, H. L., Natural evaporation from open water, bare soil and grass, *Proc. Roy. Soc. of London, Ser. A*, *193*(1032), 120-145, 1948.

Penman, H. L., Vegetation and hydrology, *Tech. Communic. 53*, Commonwealth Bur. of Soils, Harpenden, 1963.

Philip, J. R., Theory of infiltration, in *Advances in Hydroscience*, vol. 5, edited by V. T. Chow, pp. 215-296, Academic Press, New York, 1969.

Pike, J. G., The estimation of annual runoff from meteorological data in a tropical climate, *J. Hydrol.*, *2*, 116-123, 1964.

Slatyer, R. O., *Plant-Water Relationships*, Academic Press, New York, 1967.

van den Honert, T. H., Water transport as a catenary process, *Trans. Faraday Soc.*, *3*, pp. 146-153, 1948.

Wigham, J. M., Interception, in *Principles of Hydrology*, edited by D. M. Gray, Water Information Center, Inc., Port Washington, N.Y., 1973.

(Received August 19, 1977;
revised January 3, 1978;
accepted February 16, 1978.)