Climate, Soil, and Vegetation
3. A Simplified Model of Soil Moisture Movement in the Liquid Phase

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Natural soil systems are modeled one dimensionally from the surface to a stationary water table by a homogeneous medium defined by three independent parameters. Four varieties of soil moisture movement are analyzed separately, and their effects are linearly superimposed. Infiltration and exfiltration are described by the Philip equation, which assumes the medium to be effectively semiinfinite and the internal soil moisture at the beginning of each storm and interstorm period is assumed to be uniform at its long-term space-time average. The exfiltration equation is modified for the presence of natural vegetation through the approximate introduction of a distributed sink representing the moisture extraction by plant roots. Gravitational percolation to groundwater is assumed to be steady throughout the rainy season at a rate determined by the long-term space-time average soil moisture. Capillary rise from the water table is assumed to be steady throughout the year and to take place to a dry surface.

INTRODUCTION

The vertical movement of soil moisture (in the liquid phase) between the surface and the water table can be subdivided into three categories according to the predominant forces involved.

1. *Infiltration and exfiltration.* Alternate wetting and drying of the soil surface during consecutive storm and interstorm periods will cause a penetration of the medium by an unsteady wavelike diffusion of liquid soil moisture into the soil during wet surface (storm) periods under the complementary effects of capillarity and gravity and out of the soil during dry surface (interstorm) periods when capillarity opposes gravity. With increasing depth of penetration, diffusion reduces the soil moisture gradients and thus reduces the effect of capillarity until moisture movement becomes dominated by gravity. The depth at which surface-induced capillary forces become negligible determines the 'penetration depth,' or boundary layer thickness, of the surface process and is used to define the thickness of the zone of soil moisture. The presence of transpiring vegetation adds another mechanism for moisture extraction distributed over a depth which is related to root structure.

2. *Percolation.* Liquid soil moisture moves out of the bottom of the zone of soil moisture and percolates downward under the domination of gravity forces until it encounters the increasing soil moisture gradients lying above the water table. At some depth the capillary forces (here upward at all times) will again merit consideration, thus defining the bottom of this 'intermediate zone.'

3. *Capillary rise.* Between the water table and the intermediate zone is a 'capillary fringe' in which gravity and capillarity again jointly govern the liquid soil moisture movement.

These zones are, of course, artificial, there being a continuous variability of soil moisture and the resulting fluid forces. For homogeneous bare soil this is expressed by the one-dimensional concentration dependent diffusion equation [*Philip*, 1960]

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z}\right] - \frac{\partial K(\theta)}{\partial z} \tag{1}$$

where θ is the effective volumetric moisture content, which is equal to the volume of active soil moisture divided by the total volume, t is the time in seconds, and $K(\theta)$ is the effective hydraulic conductivity in centimeters per second. Also $D(\theta)$ is the diffusivity, (in square centimeters per second),

$$D(\theta) = K(\theta)\ \partial\Psi(\theta)/\partial\theta \tag{2}$$

where $\Psi(\theta)$ is the soil matrix potential in centimeters. Here t and z represent the time and vertical space coordinates, respectively (z positive downward from surface).

For a partially (but homogeneously) vegetated surface we must incorporate the internal extraction of soil moisture by the plant roots. The local extraction rate will be a function of the plant species through root structure, effective leaf area, and the moisture potential differences between and across these surfaces. It will be a function of the climate through the potential rate of evaporation and will be sensitive to the soil moisture content which it helps to determine. We will represent the local strength of this distributed moisture 'sink' by the function $g_r(z, \theta)$, in which species dependency is implicit.

Equation (1) can now be written

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z}\right] - \frac{\partial K(\theta)}{\partial z} - g_r(z, \theta) \tag{3}$$

For the problem at hand, the solution to (3) is governed by general initial $\theta_0(z)$ and boundary θ_1 and θ_2 conditions such as are shown in Figure 1, and no exact analytical solution has yet been found.

To pursue an analytical balance of the annual water budget, we will decompose the liquid soil moisture movement into the separate processes defined above. Use the solutions of (1) and (3) under simplified boundary and initial conditions to approximate the moisture flux of each, and superimpose the separate fluxes as though the problem were linear. This decomposition is shown schematically in Figure 2. Movement of soil moisture in the vapor phase will not be considered nor will the presence of ice or snow.

SOIL PROPERTIES

To obtain analytical solutions of (3) under even simplified conditions, we must define the functions $K(\theta)$ and $\Psi(\theta)$ in forms which are mathematically tractable. Many functional forms have been suggested [see *Dooge*, 1973, p. 256]. Among others, *Brooks and Corey* [1966] demonstrate the utility of the relation

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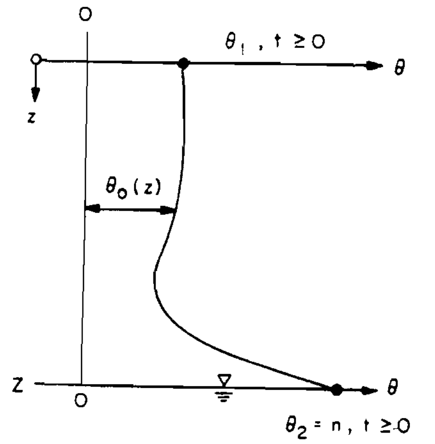

Fig. 1. Complete boundary value problem.

$$\Psi(s) = \Psi(1)s^{-1/m} \quad (4)$$

where m is called the pore size distribution index. In this equation s is the effective degree of medium saturation, or the volume of active soil moisture divided by the effective volume of voids, equal to θ/n, where n is the effective medium porosity, or the effective volume of voids divided by the total volume. And $\Psi(1)$ is the matrix potential at effective saturation $s = 1$.

This value of Ψ is called the 'bubbling pressure head,' p_b/γ_w, by *Brooks and Corey* [1966], since it represents that value of the suction head at which, during dewatering of a sample, gas is first drawn through the sample.

Defining the effective intrinsic permeability (in square centimeters) as

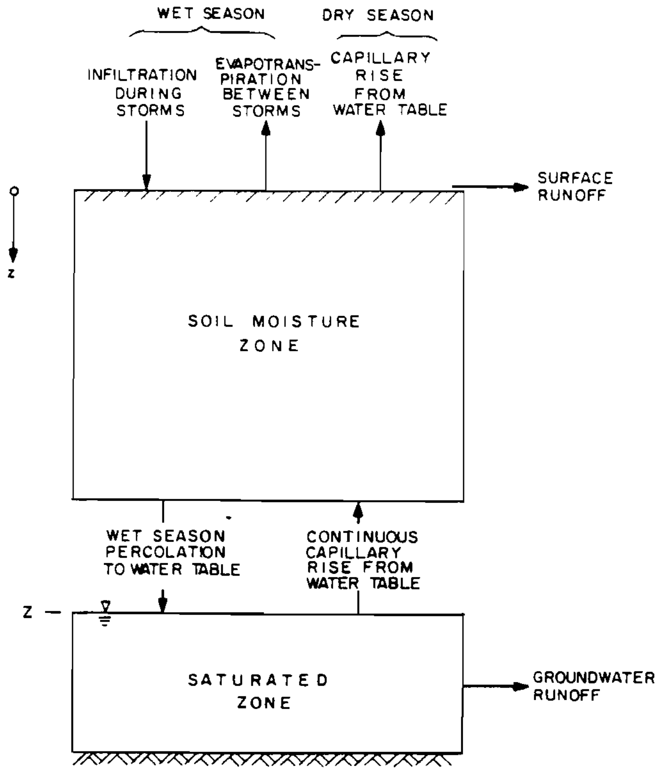

Fig. 2. Schematic representation of soil column.

$$k(s) = (\mu/\gamma_w)K(s) \quad (6)$$

where μ is the dynamic viscosity of pore water in poises and γ_w is the specific weight of pore water in dynes per cubic centimeter, *Brooks and Corey* [1966] show that during wetting the functional form

$$k(s) = k(1)s^c \quad (7)$$

allows integration of the (simplified) *Burdine* [1958] equations governing the relationship between effective permeability and capillary pressure in irregular pore structures to obtain

$$c = (2 + 3m)/m \quad (8)$$

The validity of the representations given by (4) and (7) is demonstrated for both a cohesive and a cohesionless soil in Figures 3 and 4. Data are from the work of *Talsma* [1974] and *Moore* [1939]. We will neglect any hysteretic differences between wetting and drying processes. Note that the value $\Psi(1)$ must be obtained by extrapolation of the fitted function, which is not a good representation near $s = 1$.

Equation (8), which gives an interrelation of the two soil indices c and m, is compared with observations from a range of soils (including those of Figures 3 and 4) in Figure 5.

Using the definition $\theta = ns$, we can write, from (4),

$$\Psi(\theta) \equiv \Psi(\theta/n) = \Psi(1)(\theta/n)^{-1/m} = \Psi(1)s^{-1/m} \quad (9)$$

and from (6) and (7),

$$K(\theta) \equiv K(\theta/n) = K(1)(\theta/n)^c = K(1)s^c \quad (10)$$

From the definition of diffusivity, (2), we can then write

$$D(\theta) = \frac{|\Psi(1)|K(1)}{nm} s^{c-(1/m)-1} \quad (11)$$

Omitting the absolute value symbols for convenience and defining

$$d = c - (1/m) - 1 \quad (12)$$

we have

$$D(\theta) = \frac{\Psi(1)K(1)}{nm} s^d \quad (13)$$

From empirical studies on the effects of pore shape on flow in porous media, *Carman* [1937] found for granular media that

$$\left[\frac{\sigma_w}{\gamma_w}\right]^2 \frac{n}{k(1)\Psi^2(1)} = 5\left[\frac{2+m}{m}\right] \quad (14)$$

where σ_w is the surface tension, or the density of interfacial free energy, in dynes per centimeter.

Defining the pore shape parameter as

$$\Phi = \left[\frac{\sigma_w}{\gamma_w}\right]^2 \frac{n}{k(1)\Psi(1)} \quad (15)$$

we can use observations of n, $k(1)$, and $\Psi(1)$ to define $\Phi(m)$ over the full range of soil types as is shown in Figure 6. This gives the empirical relation

$$\Phi = 10^{0.66+0.55/m+0.14/m^2} \quad (16)$$

and serves, along with (8) and (12), to reduce the set of six soil parameters, $K(1)$, $\Psi(1)$, c, d, m, and n to three independent parameters.

Since m and c are functionally related through (8), we can rewrite (15) in the form

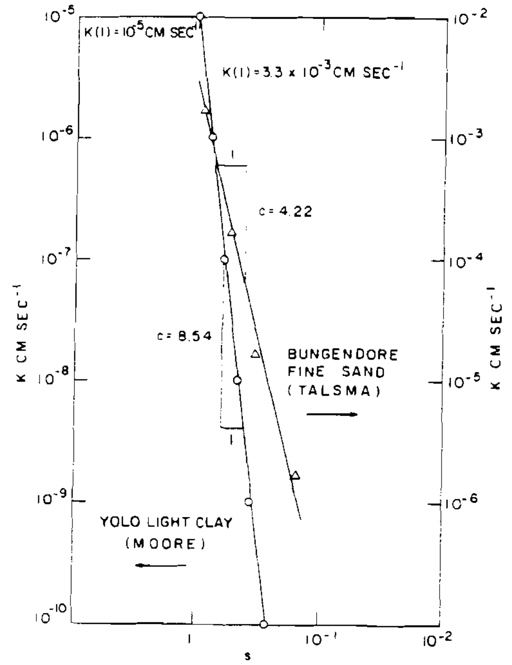

Fig. 3. Hydraulic conductivity (water at 15°C).

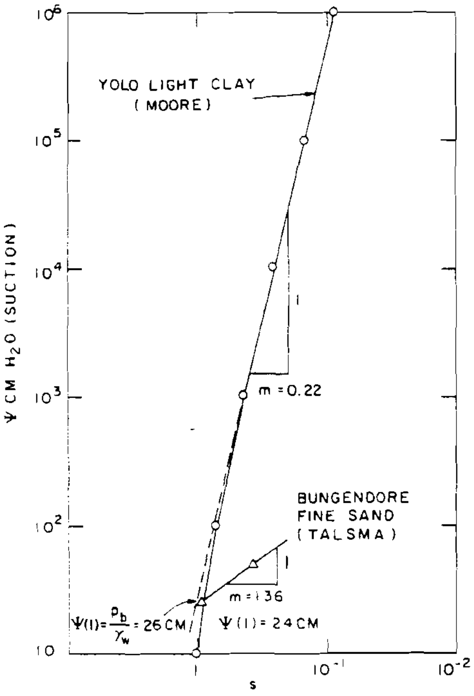

Fig. 4. Matrix potential (water at 15°C).

$$\Psi(1) = \frac{\sigma_w}{\gamma_w}\left[\frac{n}{k(1)\Phi(c)}\right]^{1/2} \qquad (17)$$

If we use (9), this becomes

$$\Psi(\theta) = \frac{\sigma_w}{\gamma_w}\left[\frac{n}{k(1)\Phi(c)}\right]^{1/2}\left[\frac{\theta}{n}\right]^{-1/m} = \frac{\sigma_w}{\gamma_w}\psi(\theta) \qquad (18)$$

where, in the manner of (6),

$$\psi(\theta) \equiv \left[\frac{n}{k(1)\Phi(c)}\right]^{1/2}\left[\frac{\theta}{n}\right]^{-1/m} \qquad (19)$$

which is the intrinsic matrix potential. Just as in the case of permeability, this definition allows separation of the fluid properties from those of the medium alone. It is thus the intrinsic values at saturation $k(1)$ and $\psi(1)$ which, along with the exponents c and m, allow us to define $K(\theta)$ and $\Psi(\theta)$ for any fluid, temperature, and pressure.

Considering the relative ease of making permeability measurements, it seems most convenient to define the soil behavior in terms of the three independent parameters n, $k(1)$, and c.

INFILTRATION AND EXFILTRATION

To obtain an analytical approximation of a solution of (3) or of its special case, (1), we will first make the following major simplifications in the applicable initial and boundary conditions.

1. The water table depth $z = Z$ is much greater than the larger of the penetration depth $z = z_{max}$ of the surface-induced capillary transient and the root depth Z_r. This allows us to consider the medium to be effectively semiinfinite.

2. The soil moisture throughout the surface boundary layer (i.e., for $Z_r \geq z \leq z_{max}$) is spatially uniform at the start of each storm period and at the start of each interstorm period at the value $s = s_0$.

3. The vegetation, although covering only a fraction M of the soil surface, is distributed uniformly, and its roots extend into the entire volume of soil, $z \leq Z_r$. We are thus implicitly assuming that the plant species have adapted by natural selection to a density and root structure which in combination is in balance with the available soil moisture.

With these simplifications, we will define s_1 as the degree of saturation at the surface of the medium, f_i as the infiltration rate, f_i^* as the infiltration capacity (i.e., the infiltration rate with $s_1 = 1$), i as the rainfall rate, f_e as the exfiltration rate, f_e^* as the exfiltration capacity (i.e., the exfiltration rate with $s_1 = 0$), e_T as the actual evaporation rate, and e_p as the potential (soil surface) evaporation rate. Considering what happens at the soil-atmosphere interface, we allow only the following boundary conditions.

For infiltration

$$f_i = f_i^* \qquad s_1 = 1 \qquad i > f_i^*$$

$$f_i = i \qquad s_1 < 1 \qquad i \leq f_i^*$$

For exfiltration

$$f_e = e_T = f_e^* \qquad s_1 = 0 \qquad e_p > f_e^*$$

$$f_e = e_T = e_p \qquad s_1 > 0 \qquad e_p \leq f_e^*$$

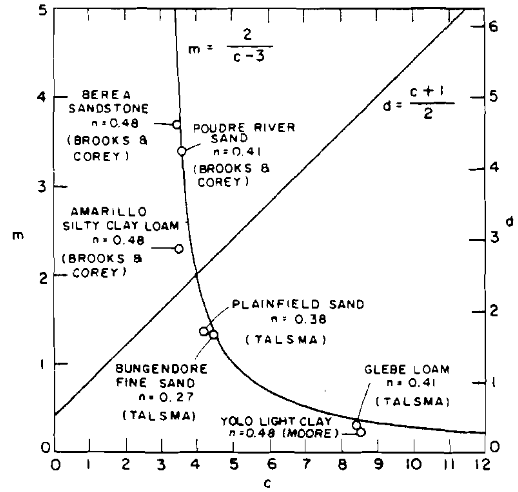

Fig. 5. Interrelation of soil indices.

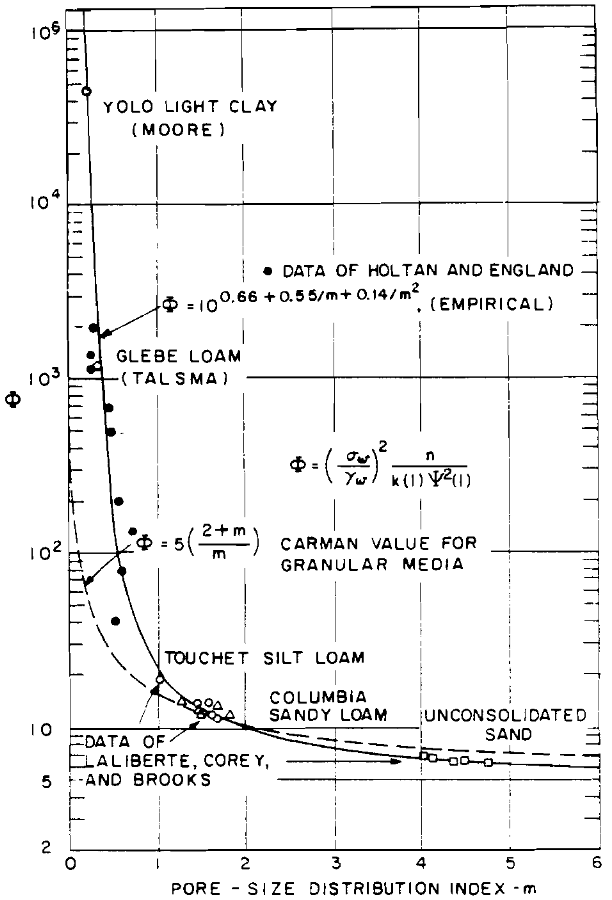

Fig. 6. Pore shape parameter.

These conditions are illustrated in Figure 7, where potential rates are shown as dashed lines and actual rates as solid lines. The shaded areas indicate the volumes of moisture exchanged between soil and atmosphere in each time interval. Analytical expression of these volumes is a key to water balance computations.

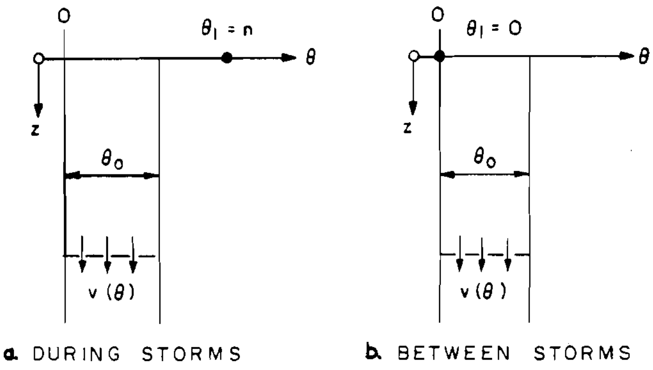

Fig. 8. Simplified boundary and initial soil moisture conditions.

Since for $i \leq f_i{}^*$ and for $e_p \leq f_e{}^*$ the actual moisture exchange rates are fixed by the climatic variables i and e_p, respectively, we only need to solve the soil moisture equations for one infiltration condition (saturated surface) and one exfiltration condition (dry surface). The boundary and initial conditions for these solutions are shown in Figure 8, in which $\theta_0 = ns_0$ represents the uniform initial soil moisture.

In the manner of *Philip* [1969], we can now write the total increase of moisture content in the semiinfinite column during infiltration:

$$\int_0^\infty (\theta - \theta_0)\, dz = \int_{\theta_0}^{\theta_1} z\, d\theta = F_i(t) - K(\theta_0)t \qquad (20)$$

where $F_i(t)$ is the cumulative infiltration in centimeters and where we assume that the extraction of soil moisture by vegetation ceases during storm (i.e., infiltration) periods due to cessation of transpiration.

During interstorm periods we have a similar relationship,

$$\int_0^\infty (\theta_0 - \theta)\, dz = \int_{\theta_1}^{\theta_0} z\, d\theta = F_e(t) + [K(\theta_0) + Me_v]t \quad (21)$$

where $F_e(t)$ is the cumulative exfiltration in centimeters, M is the fraction of the surface which is vegetated, and e_v is the rate of transpiration by vegetation. Also here we assume that the

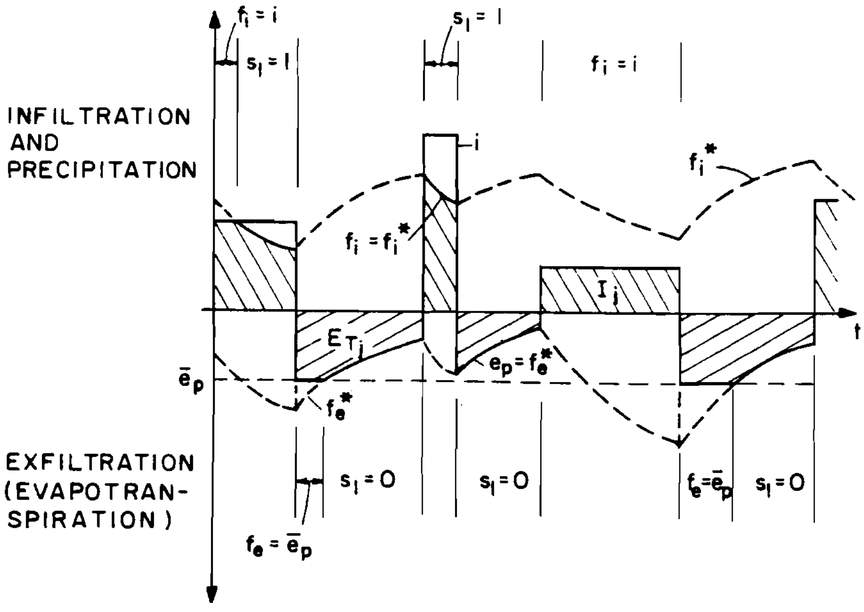

Fig. 7. Surface boundary conditions for idealized storm and interstorm periods.

rate of moisture extraction by the roots is in equilibrium with the transpiration rate by the leaves.

The integrals on the left-hand side of (20) and (21) are represented by *Philip* [1960] in terms of the expansion

$$z(\theta, t) = \phi_1 t^{1/2} + \phi_2 t + \phi_3 t^{3/2} + \phi_4 t^2 + \cdots \quad (22)$$

Using (22) in (20) and differentiating, we have the (apparent) velocity of infiltration

$$f_i(t) = \tfrac{1}{2}A_1 t^{-1/2} + A_2 + K(\theta_0) + \tfrac{3}{2}A_3 t^{1/2} + 2A_4 t + \cdots \quad (23)$$

in which

$$A_n = \int_{\theta_0}^{\theta_1} \phi_n \, d\theta \quad (24)$$

Philip [1960] rewrote the governing differential equation, (1), in the form

$$-\frac{\partial}{\partial t}\int_{\theta_0}^{\theta} z \, d\theta = D\frac{\partial\theta}{\partial z} - [K(\theta) - K(\theta_0)] \quad (25)$$

Using (22) and the simple initial and boundary conditions of Figure 8*a*, he found

$$A_1 = S_i = 2(\theta_1 - \theta_0)[D_i/\pi]^{1/2} \quad (26)$$

the infiltration sorptivity, where D_i is the effective infiltration diffusivity over the range θ_1–θ_0. This makes the first term of (23) identical with the well-known solution for linearized sorption (i.e., zero-gravity infiltration). For short times he showed that

$$A_2 \approx \tfrac{1}{2}[K(\theta_1) - K(\theta_0)] \quad (27)$$

Neglecting higher-order terms, we have the desired (apparent) infiltration velocity

$$f_i(t) \simeq \tfrac{1}{2}S_i t^{-1/2} + \tfrac{1}{2}[K(\theta_1) + K(\theta_0)] \quad (28)$$

For exfiltration we use (22) in (21) and differentiate to obtain the (apparent) exfiltration velocity:

$$f_e(t) = \tfrac{1}{2}A_1 t^{-1/2} + A_2 - K(\theta_0) - Me_v + A_3 t^{1/2} + \cdots \quad (29)$$

Following *Philip* [1960], we can rewrite the governing differential equation, (3), as

$$-\frac{\partial}{\partial t}\int_{\theta_0}^{\theta} z \, d\theta = D\frac{\partial\theta}{\partial z} - [K(\theta) - K(\theta_0)] - \int_{\theta_0}^{\theta} g_r(z, \theta)\frac{\partial z}{\partial\theta} \, d\theta \quad (30)$$

To obtain the coefficients A_n in (29), we need a series solution of (30) under the simple initial and boundary conditions of Figure 8*b*. This in turn requires specification of the root extraction function $g_r(z, \theta)$, which opens the whole issue of plant physiology and is beyond the scope of this work. For expedience we will infer values of A_1, A_2, and A_3 from known solutions of similar problems.

For desorption (i.e., zero-gravity exfiltration) under the conditions of Figure 8*b* and with a steady sink of strength Me_v/Z_r distributed uniformly over the root zone $0 \le z \le Z_r$, the desorption rate is given [see *Carslaw and Jaeger*, 1959, p. 80] by

$$f_e = \tfrac{1}{2}S_e t^{-1/2} - [2Me_v/Z_r][D_e t/\pi]^{1/2} \quad (31)$$

where $S_e = 2(\theta_0 - \theta_1)[D_e/\pi]^{1/2}$ (32)

the exfiltration sorptivity and D_e is the effective exfiltration diffusivity over the range θ_0–θ_1. For reasonable values of the variables and parameters of (31) the first term is of the order 10^{-6} cm/s, and the second term is of the order 10^{-8} cm/s. The last term is thus negligible.

By analogy with the similarities in the sorption and infiltration solutions we compare (29) and (31) to state

$$A_1 = S_e \qquad A_3 = 0$$

and by analogy with the infiltration solution (allowing for the opposite sense of the surface flux) for short times we assume

$$A_2 = \tfrac{1}{2}[K(\theta_0) - K(\theta_1)] \quad (33)$$

Finally, our approximate exfiltration rate is

$$f_e(t) \simeq \tfrac{1}{2}S_e t^{-1/2} - \tfrac{1}{2}[K(\theta_1) + K(\theta_0)] - Me_v \quad (34)$$

For the sorption and desorption problems (infiltration and exfiltration without gravity and vegetation) under analogous boundary and initial conditions and in a medium for which $D(\theta)$ increases monotonically with θ, *Crank* [1956, p. 256] found that D_i and D_e are approximated well by

$$D = D_i = \tfrac{5}{3}(\theta_1 - \theta_0)^{-5/3}\int_{\theta_0}^{\theta_1}(\theta - \theta_0)^{2/3}D(\theta) \, d\theta \quad (35)$$

and

$$D = D_e = 1.85(\theta_0 - \theta_1)^{-1.85}\int_{\theta_1}^{\theta_0}(\theta_0 - \theta)^{0.85}D(\theta) \, d\theta \quad (36)$$

We can now use (13) in (35) and (36) to evaluate the effective diffusivities in terms of s_0 for the two surface boundary conditions of interest, $s_1 = 1$ for infiltration and $s_1 = 0$ for exfiltration. Changing variables and using $\theta = ns$, we obtain the dimensionless sorption diffusivity

$$\frac{3mnD_i}{5K(1)\Psi(1)} = \phi_i(d, s_0) = (1 - s_0)^{-5/3}\int_{s_0}^{1} s^d[s - s_0]^{2/3} \, ds \quad (37)$$

which becomes, for integer d only,

$$\phi_i(d, s_0) = (1 - s_0)^d \left[\frac{1}{d + 5/3} + \sum_{n=1}^{d} \frac{1}{d + [(5/3) - n]}\binom{d}{n}\left(\frac{s_0}{1 - s_0}\right)^n\right] \quad (38)$$

Similarly, we obtain the dimensionless desorption diffusivity

$$\frac{mnD_e}{K(1)\Psi(1)} = s_0^{\,d}\phi_e(d) = 1.85 s_0^{-1.85}\int_0^{s_0} s^d[s_0 - s]^{0.85} \, ds \quad (39)$$

which becomes, for integer d only,

$$\phi_e(d) = \left[1 + 1.85 \sum_{n=1}^{d} (-1)^n \binom{d}{n} \frac{1}{1.85 + n}\right] \quad (40)$$

These functions have been evaluated and are plotted in Figures 9 and 10. It is useful to note that for $s_0 = 0$,

$$\phi_i(d, 0) = 1/(d + \tfrac{5}{3}) \quad (41)$$

If we use (5), (6), (7), (25), and (37), then (28) can be written for our chosen boundary condition to give the infiltration capacity f_i^*:

$$\frac{f_i^*(t, s_0)}{K(1)} = (1 - s_0)\left[\frac{5n\Psi(1)\phi_i(d, s_0)}{3\pi m t K(1)}\right]^{1/2} + \tfrac{1}{2}[1 + s_0^{\,c}] \quad (42)$$

Before performing the same transformation in (34) we note that in this case,

$$K(\theta_1 = 0) = 0$$

and that for $t = m_{t_b}$, the average time between storms, we can expect

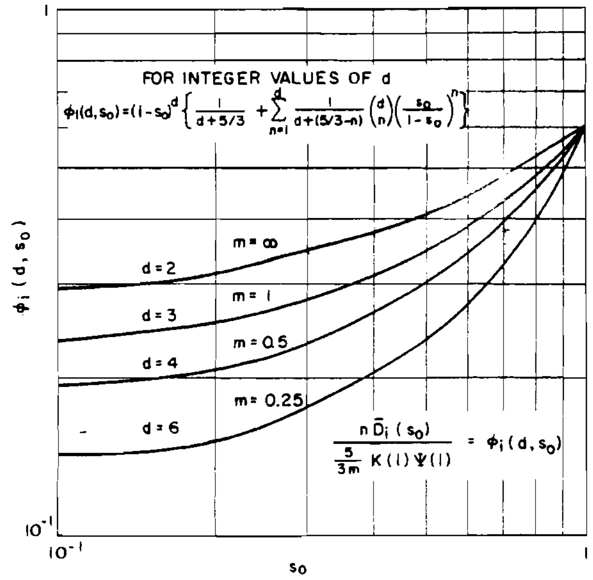

Fig. 9. Dimensionless infiltration diffusivity.

$$K(\theta_0) << S_e t^{-1/2}$$

With these simplifications, we get the exfiltration capacity f_e^*:

$$f_e^* \simeq \tfrac{1}{2}S_e t^{-1/2} - Me_v \tag{43}$$

which by using (5) and (39) becomes

$$\frac{f_e^*(t, s_0)}{K(1)} = s_0^{\,1+d/2}\left[\frac{n\Psi(1)\phi_e(d)}{\pi m t K(1)}\right]^{1/2} - \frac{Me_v}{K(1)} \tag{44}$$

To apply (42) and (44) to the calculation of climatic exchanges of soil moisture, we need to know the value of the initial soil moisture s_0. This is certainly a function of the immediate historical sequence of storm and interstorm events, as is shown schematically by the solid line in Figure 11 and is consequently difficult to relate generally to climate in any but a probabilistic fashion. The initial soil moisture values for use in (42) are represented by the 'troughs' of the solid line in Figure 11, while those for use in (44) are represented by the crests. A first-order probabilistic approximation [see *Benjamin and Cornell*, 1970, p. 180] to the statistics of storm infiltration volume and of interstorm exfiltration volume would call for the substitution of the mean values of these troughs and crests into (42) and (44), respectively. Since these amplitudes depend upon the solution being sought, we will use a 'zeroth-order'

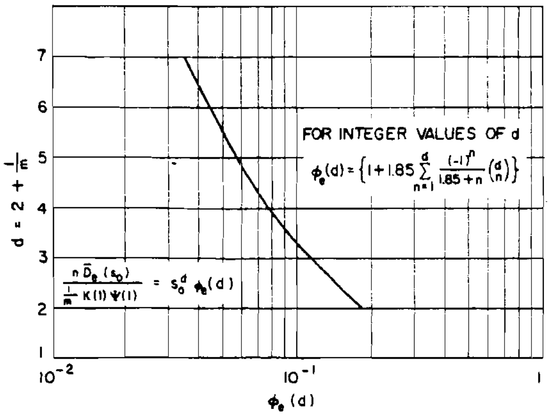

Fig. 10. Dimensionless Exfiltration diffusivity.

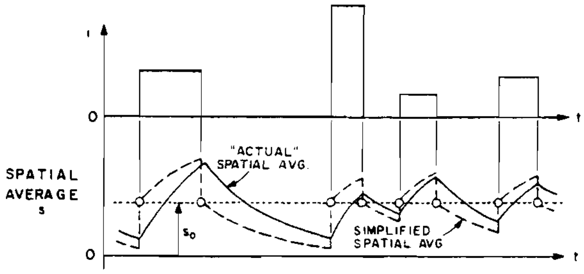

Fig. 11. Soil moisture representation.

approximation which fixes s_0 for all infiltration and exfiltration events at the space-time average soil moisture within the soil surface boundary layer, a value which provides a significant characterization of soil climate [*Eagleson*, 1978]. The resulting approximation is shown by the dashed line of Figure 11.

Using the results of this analysis, we can estimate the penetration depth z_{max} of a step change in surface soil moisture as a check on our assumption that $z_{max} << Z$. There will be two components to z_{max}, a diffusive component z_1, given by *Eagleson* [1970, p. 293] as

$$z_1 \simeq 4(Dt)^{1/2} \tag{45}$$

and a gravitational seepage component z_2, given by the product of the seepage velocity and time. The appropriate moisture content for evaluation of this seepage velocity is that at the initial moisture content θ_0, i.e.,

$$z_2 = \frac{K(\theta_0)}{n}t \tag{46}$$

When we sum

$$z_{max} = z_1 + z_2 = 4(Dt)^{1/2} + tK(\theta_0)/n \tag{47}$$

the characteristic time is $t = m_{t_r}$, the average storm duration, for $D = D_i$, and is $t = m_{t_b}$, the average interstorm period, for $D = D_e$. For typical climate-soil systems, z_{max} will be of the order of 1 m, which is also the order of vegetal root depth.

PERCOLATION

At the lower extremity of the soil moisture zone the soil moisture may be assumed to be constant during the wet season at its climatic average value ns_0. This elevation is, by definition, at the limit of surface-induced capillary effects; thus we may formulate the apparent percolation velocity v as a steady gravitational seepage which is simply

$$v = K(\theta = ns_0) = K(s_0) \tag{48}$$

During the dry season we assume that $s_0 = 0$, and therefore $v = 0$.

CAPILLARY RISE FROM WATER TABLE

Considering the isolated problem of steady capillary rise from a water table at $z = Z$ to the surface, we can simplify the governing diffusion equation, (1), to write

$$0 = \frac{d}{dz}\left[K(\theta)\frac{d\Psi(\theta)}{dz} + K(\theta)\right] \tag{49}$$

Since the bracketed term represents the apparent upward fluid velocity $-w$, (49) can be integrated to give

$$K(\theta)\left[\frac{d\Psi(\theta)}{dz} + 1\right] = \text{const} = -w \tag{50}$$

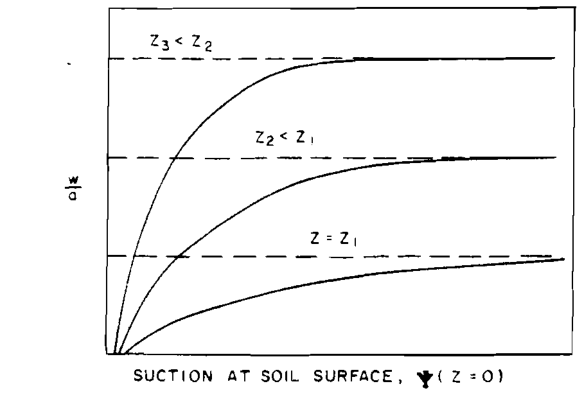

Fig. 12. Steady state capillary rise from water table [*Gardner*, 1958].

Integrating this from the surface $z = 0$ to the water table $z = Z$ gives

$$Z = \int_{\Psi(1)}^{\Psi(z=0)} \frac{d\Psi(\theta)}{1 + w/K(\theta)} \qquad (51)$$

Gardner [1958], in a classic study of this problem, assumed $K(\theta)$ and $\Psi(\theta)$ to be related for vertical flow by

$$K(\theta) = a\Psi(\theta)^{-b} \qquad (52)$$

This relation allowed him to integrate (51) analytically for certain values of b. The nature of this result is shown qualitatively in Figure 12.

If the surface is dry (as we are assuming here), that is, if $w < e_p$ (or as long as the relative humidity of the atmosphere is less than about 99% [*Philip*, 1957]), the absolute value of the surface matrix potential $|\Psi(z = 0)|$ will be very large. At the same time the saturated value $|\Psi(1)|$ will be quite small by comparison and the integral of (51) may be taken over the range $0-\infty$. This gives us the asymptotic value for w (see Figure 12). According to *Gardner*'s [1958] analytical solution this is

$$w = w_{max} = Ba/Z^b \qquad (53)$$

Using (9) and (10), we can write

$$\frac{K(s)}{K(1)} = \left[\frac{\Psi(1)}{\Psi(s)}\right]^{mc} \qquad (54)$$

By comparison with (52),

$$a = K(1)[\Psi(1)]^{mc} \qquad (55)$$

and

$$b = mc \qquad (56)$$

Then (53) gives the desired relation

$$\frac{w}{K(1)} = B\left[\frac{\Psi(1)}{Z}\right]^{mc} \qquad (57)$$

where the coefficient B is given as a function of mc for the discrete values used by *Gardner* [1958] and listed in the following tabulation.

$b = mc$	B
$\frac{3}{2}$	3.77
2	2.46
3	1.76
4	1.52

These values are plotted in Figure 13, where they are fitted with the continuous relation

$$B = 1 + [\tfrac{3}{2}/(mc - 1)] \qquad (58)$$

in order to assist in interpolation and extrapolation over the full continuous range of the soil index mc.

SUPERPOSITION OF SOIL MOISTURE FLOWS

We have chosen not to solve the complete boundary value problem containing the water table, as is shown in Figure 1, because of the necessity, barring unacceptable approximation, of abandoning our analytical approach. Instead we have analyzed independently the three primary mechanisms of soil moisture movement including the approximate influence of vegetation. We will now combine these mechanisms in an approximate fashion by linear superposition of the apparent velocities derived above in order to derive the water balance fluxes indicated in Figure 2. This superposition should give a good approximation as long as

$$Z \geq z_{max}$$

which is not a serious restriction except in near-swamp situations. The superposition is presented in the following section, with the assumption that the plant growing season is coincident with the rainy season.

SUMMARY OF SOIL MOISTURE FLUXES

Dry season. Capillary rise from the water table is the only operative process. Equations (57) and (58) give the velocity

$$\frac{w}{K(1)} = \left[1 + \frac{3/2}{mc - 1}\right]\left[\frac{\Psi(1)}{Z}\right]^{mc} \qquad w/e_p < 1 \qquad (59)$$

Rainy season. The following equations apply to the rainy season.

Infiltration during storms

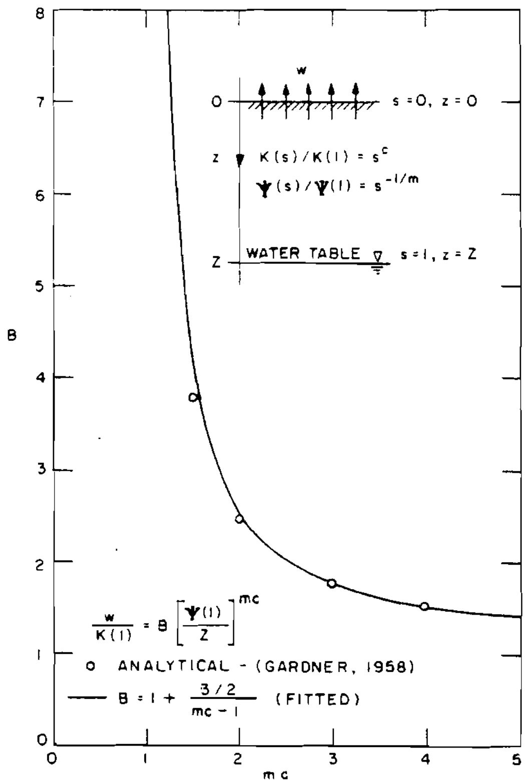

Fig. 13. Capillary rise from water table (steady state solution).

$$\frac{f_i^*(t, s_0)}{K(1)} = (1 - s_0)\left[\frac{5n\Psi(1)\phi_i(d, s_0)}{3\pi m t K(1)}\right]^{1/2} + \frac{(1 + s_0^c)}{2} - \frac{w}{K(1)} \quad (60)$$

Exfiltration between storms

$$\frac{f_e^*(t, s_0)}{K(1)} = s_0^{1+d/2}\left[\frac{n\Psi(1)\phi_e(d)}{\pi m t K(1)}\right]^{1/2} - \frac{Me_v}{K(1)} + \frac{w}{K(1)} \quad (61)$$

Net percolation to water table

$$\frac{v(s_0)}{K(1)} = s_0^c - \frac{w}{K(1)} \quad (62)$$

To obtain the special cases $Z = \infty$ and $s_0 = 1$, we must set $w \equiv 0$ in the above equations.

CONCLUSIONS

Vertical movement of soil moisture in the unsaturated zone of a homogeneous medium is decomposed into storm infiltration, interstorm exfiltration including the extraction of moisture by plant roots, percolation to the water table, and capillary rise from the water table to the surface. The soil moisture within the surface capillary boundary layer is assumed to be spatially uniform at the beginning of each storm and interstorm period with a common concentration s_0, given by the long-term average. Classical solutions of the concentration dependent diffusion equation are then used to define each moisture movement component in terms of s_0 and of four physical parameters: the effective porosity n, the effective intrinsic permeability at saturation $k(1)$, the slope c of the log $k(1)$ − log s relation, and the depth to the water table, Z.

Linear superposition of the appropriate components provides a zeroth-order approximation to the storm and interstorm net soil moisture fluxes, thereby opening the way to an analytical expression of the average annual water balance in terms of the physical parameters of both climate and soil.

NOTATION

- a coefficient relating soil properties, cm^{b+1}/s.
- B index relating soil properties.
- b index relating soil properties, equal to mc.
- c pore disconnectedness index.
- D soil moisture diffusivity, square centimeters per second.
- D_e desorption diffusivity, square centimeters per second.
- D_i sorption diffusivity, square centimeters per second.
- d diffusivity index.
- e_p potential evaporation rate from a bare soil surface, centimeters per second.
- e_T evaporation rate, centimeters per second.
- e_v transpiration rate, centimeters per second.
- f_e exfiltration rate, centimeters per second.
- f_e^* exfiltration capacity, centimeters per second.
- f_i infiltration rate, centimeters per second.
- f_i^* infiltration capacity, centimeters per second.
- i precipitation rate, centimeters per second.
- $K(\theta)$ effective hydraulic conductivity, centimeters per second.
- $K(1)$ saturated effective hydraulic conductivity, centimeters per second.
- k effective intrinsic permeability, square centimeters.
- $k(1)$ saturated effective intrinsic permeability, square centimeters.
- M vegetated fraction of land surface (i.e., 'canopy density').
- m pore size distribution index.
- m_{t_b} mean time between storms, days.
- m_{t_r} mean storm duration, days.
- n effective medium porosity, which is effective volume of voids divided by total volume.
- S_e exfiltration sorptivity, $cm/s^{1/2}$.
- S_i infiltration sorptivity, $cm/s^{1/2}$.
- s degree of effective medium saturation (i.e., effective soil moisture concentration), which equals volume of active soil moisture divided by effective volume of voids.
- s_0 initial degree of saturation in surface boundary layer.
- s_1 degree of saturation at surface of medium.
- t time, seconds.
- v apparent fluid velocity out of lower boundary of soil moisture zone due to gravitational percolation, centimeters per second.
- w upward apparent pore fluid velocity representing capillary rise from the water table, centimeters per second.
- Z depth to water table, centimeters.
- Z_r root depth, centimeters.
- z vertical coordinate, centimeters.
- z_{max} penetration depth of transient surface moisture flux process, centimeters.
- γ_w specific weight of liquid, dynes per cubic centimeter.
- θ effective volumetric soil moisture content, which is volume of active soil moisture divided by total volume.
- θ_0 initial effective volumetric moisture content.
- θ_1 effective volumetric moisture content at surface of medium.
- μ dynamic viscosity of fluid, poises.
- σ_w surface tension of pore liquid, dynes/per centimeter.
- Φ pore shape parameter.
- ϕ_e dimensionless exfiltration diffusivity.
- ϕ_i dimensionless infiltration diffusivity.
- Ψ soil matrix potential, centimeters (suction).
- $\Psi(1)$ saturated soil matrix potential, centimeters (suction).
- ψ intrinsic matrix potential, cm^{-1}.
- $g_r(Z, \theta)$ root extraction function.

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