

# Climate, Soil, and Vegetation

## 2. The Distribution of Annual Precipitation Derived From Observed Storm Sequences

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Point precipitation is represented by Poisson arrivals of rectangular intensity pulses that have random depth and duration. By assuming the storm depths to be independent and identically gamma distributed, the cumulative distribution function for normalized annual precipitation is derived in terms of two parameters of the storm sequence, the mean number of storms per year and the order of the gamma distribution. In comparison with long-term observations in a subhumid and an arid climate it is demonstrated that when working with only 5 years of storm observations this method tends to improve the estimate of the variance of the distribution of the normalized annual values over that obtained by conventional hydrologic methods which utilize only the observed annual totals.

### INTRODUCTION

Point precipitation is a random time series of discrete storm events which under certain restrictions may be assumed to be mutually independent and in which the average time between events is large with respect to the average duration of the events themselves.

For reasons of physical validity an engineering model of this process must retain those natural features which are important to the problem at hand, while for economy of computation and for clarity of behavior it should omit those features deemed unessential. Maximum understanding of hydrologic variability would be achieved with a model of the precipitation process which incorporates the geophysical dynamics through which atmospheric disturbances are generated and propagated and produce local precipitation. Unfortunately, the base of scientific knowledge does not yet exist for this model. Instead we will take the phenomenological approach, which represents the essential observed features of the precipitation time series by a stationary stochastic generating process that is analytically tractable.

The essential features of precipitation in terms of a physically oriented model of water balance processes are (1) the time between storms, since this is the 'window' for evapotranspiration, (2) the duration of the storms, since this starts and stops the infiltration process, and (3) the storm depths, since these determine the water availability for infiltration.

Various models have been used for this process (see the reviews by *Grace and Eagleson* [1966] and *Gupta* [1973]), but the Poisson process [*Benjamin and Cornell*, 1970, pp. 236-249] seems to provide the best compromise between the conflicting demands of simplicity and generality.

Perhaps the most general development of the Poisson precipitation models is that of *Todorovic* [1968], in which he considers the individual storm depths to be distributed exponentially with a mean depth which is annually periodic. The Poisson arrival rate of these events is also considered to be annually periodic. In a subsequent summary of this work, *Todorovic and Yevjevich* [1969] apply it with some success to the description of the probability density functions of seasonal precipitation.

*Benjamin and Cornell* [1970, p. 310] simplify this approach

by considering the processes to be stationary rather than periodic.

In this work we will continue the simplifying assumption of stationarity but will replace the exponential distribution of individual storm depths by the more general two-parameter gamma distribution, as was done earlier by *Todorovic and Yevjevich* [1967] and more recently by *Ison et al.* [1971]. We will extend these developments to the case of a randomly variable length of the rainy season and will demonstrate the value of the technique to the derivation of the frequency of annual rainfall from only a few years of storm observations.

### THE POISSON ARRIVAL PROCESS

The classical derivation of the Poisson distribution of storm arrivals at a point assumes successive events to be independent. The probability of one arrival in time interval  $\Delta t$  is  $p$  and that of more than one arrival in this interval is negligible. This leads to the binomial distribution for the probability  $p_\theta(\nu)$  of having  $\theta = \nu$  storms in total time  $t = m\Delta t$ :

$$p_\theta(\nu) = \binom{m}{\theta} p^\theta (1-p)^{m-\theta} \quad (1)$$

which has mean

$$E[\theta] = mp \quad (2)$$

and variance

$$\text{Var}[\theta] = mp(1-p) \quad (3)$$

It should be noted that the independence assumption is strictly applicable only for instantaneous arrivals—in this case, storms of zero duration. A storm of finite duration occurring in interval  $\Delta t$ , may overlap into the subsequent interval  $\Delta t_{j+1}$ , thereby decreasing the effective length of that interval and hence the probability of obtaining another storm arrival therein. A measure of the validity of this assumption in application to an idealized storm series such as is shown in Figure 1 will be the 'smallness' of the ratio of the average storm duration to the average time between storms,  $m_s/m_t$ . If we let  $\Delta t \rightarrow 0$  without otherwise changing this stationary process,  $p \rightarrow 0$  and  $m \rightarrow \infty$ , thus giving

$$mp = \omega t \quad (4)$$

where  $\omega$  is the average arrival rate of the storm events and  $t$  is

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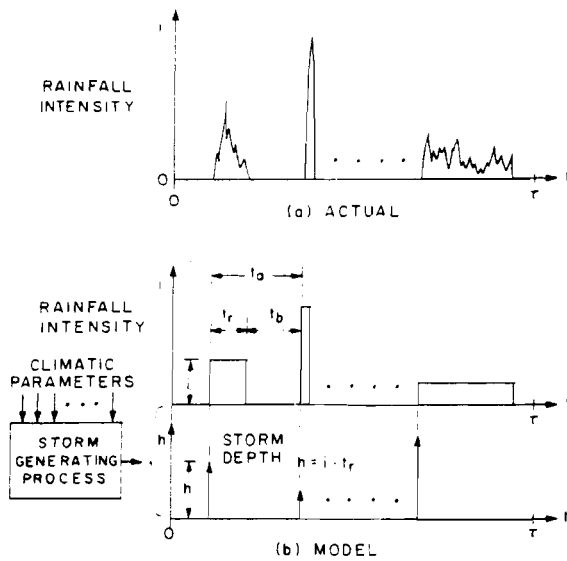


Fig. 1. Model of precipitation event series.

the time period of interest. Equation (1) becomes the Poisson distribution

$$p_{\theta,t}(\nu) = \frac{(\omega t)^\nu e^{-\omega t}}{\nu!} \quad \nu = 0, 1, 2, \dots \quad (5)$$

for the probability of obtaining exactly  $\nu$  events in time  $t$ . From this distribution we can find the mean

$$E[\theta | t] \equiv m_\nu = \omega t \quad (6)$$

and the variance

$$\text{Var} [\theta | t] \equiv \sigma_\nu^2 = \omega t \quad (7)$$

INTERARRIVAL TIMES

If  $T$  is a random variable representing the time until the first storm arrives,

$$\text{Prob} [T > t_a] = 1 - F_T(t_a) = p_{\theta,t_a}(0) \quad (8)$$

By using (5) this becomes

$$F_T(t_a) = 1 - e^{-\omega t_a} \quad (9)$$

which is the probability that the first storm will arrive after elapsed time  $t_a$ . Since the Poisson process is assumed to be stationary, the origin of the time interval  $t_a$  is arbitrary and may be thought of as coinciding with one of the instantaneous occurrences shown in Figure 1b.  $F_T(t_a)$  then becomes the cumulative distribution function (cdf) of the continuous random variable:  $t_a$  is the interarrival time. Differentiating (9) gives the exponential distribution for interarrival times of the Poisson process:

$$f_T(t_a) = \omega e^{-\omega t_a} \quad (10)$$

which has mean and variance

$$m_{t_a} = \omega^{-1} \quad (11)$$

and

$$\sigma_{t_a}^2 = \omega^{-2} \quad (12)$$

TOTAL PRECIPITATION IN A GIVEN TIME

The total precipitation delivered by  $\nu$  events is

$$P(\nu) = \sum_{j=1}^{\nu} h_j \quad (13)$$

where  $h$  is the total precipitation (depth) from a single storm. We define  $f_{P(\nu)}(y)$  as the probability density function of  $P(\nu)$ .

The probability density of the total precipitation  $P$  in time  $t$  is then given by summing the probability densities  $f_{P(\nu)}(y)$  for each of the (mutually exclusive and collectively exhaustive) number of storms  $\nu$  which could deliver  $y$  precipitation in this time, each weighted by the discrete probability  $P_{\theta,t}(\nu)$  that exactly that number of storms will occur. Analytically, this gives a compound distribution composed of the continuous density function

$$f_{P,t}(y) = \sum_{\nu=1}^{\infty} f_{P(\nu)}(y) P_{\theta,t}(\nu) \quad y > 0 \quad (14a)$$

plus the impulse

$$P_{\theta,t}(0) = e^{-\omega t} \quad y = 0 \quad (14b)$$

To complete this derivation, we need the density function  $f_{P(\nu)}(y)$ , where  $P(\nu)$  is the sum of random variables  $h_j$ .

We will assume these  $h_j$  to be independent, and, although in many natural cases, seasonal differences are present, we will assume the  $h_j$  to be identically distributed. This assumption may be removed (at considerable computational cost, however) should the circumstances so dictate.

Having made the independence assumption, we will select a distribution for the storm depths which meets the following criteria: (1) it provides a good fit with observations over a range of climatic types and yet (2) it is simple enough to be analytically tractable. Both of these conditions are met by the gamma distribution.

$$f_H(h) = G(\kappa, \lambda) = \frac{\lambda(\lambda h)^{\kappa-1} e^{-\lambda h}}{\Gamma(\kappa)} \quad (15)$$

which has mean and variance given by

$$m_H = \kappa / \lambda \quad (16)$$

and

$$\sigma_H^2 = \kappa / (\lambda)^2 \quad (17)$$

Defining the mean storm depth alternatively as

$$m_H = \eta^{-1} \quad (18)$$

we have

$$\lambda = \eta \kappa \quad (19)$$

Another advantage of the gamma distribution is that it is 'self-preserving,' or 'regenerative.' This means in our case [Benjamin and Cornell, 1970, p. 292] that the sum of  $\nu$  independent  $G(\kappa, \lambda)$  gamma variables with common  $\lambda$  and  $\kappa$  is also gamma distributed, with

$$\kappa_\nu = \sum_{j=1}^{\nu} \kappa_j = \nu \kappa \quad (20)$$

and

$$\lambda_\nu = \lambda \quad (21)$$

Thus considering (13) and assuming independence and identical distribution for the  $h_j$ , we have from (15),

$$f_{P(\nu)}(y) = G(\nu \kappa, \lambda) = \frac{\lambda(\lambda y)^{\nu \kappa - 1} e^{-\lambda y}}{\Gamma(\nu \kappa)} \quad (22)$$

which is the density function of total precipitation from  $\nu$  storms. The mean and variance of this distribution are

$$m_{P_A} = \nu\kappa/\lambda \tag{23}$$

and

$$\sigma_{P_A}^2 = \nu\kappa/\lambda^2 \tag{24}$$

Substituting (5) and (22) in (14) and using (19), we have the (compound) distribution of cumulative point precipitation:

$$f_{P_A}(y) = \sum_{\nu=1}^{\infty} \frac{\eta\kappa(\eta\kappa y)^{\nu\kappa-1} e^{-\eta\kappa y}}{\Gamma(\nu\kappa)} \frac{(\omega\tau)^\nu e^{-\omega\tau}}{\nu!} \quad y > 0$$

$$P_{\theta_T}(0) = e^{-\omega\tau} \quad y = 0 \tag{25}$$

where  $\tau$  is the length of the rainy season.

Equation (25) assumes a constant value for the length of the rainy season, which is reasonable for humid regions where  $\tau$  is often 365 days. In arid regions, however,  $\tau$  will be a random variable, and (25) gives the (conditional) distribution of annual rainfall given the season length. That is,

$$f_{P_A}(y) = f_{P_A,T}(y|\tau)$$

We find the desired marginal distribution of annual rainfall depth  $f_P(y)$  by integrating the joint distribution of this depth and the season length over all values of the latter variable:

$$f_P(y) = \int_0^{365} f_{P_A,T}(y, \tau) d\tau = \int_0^{365} f_{P_A,T}(y|\tau) f_T(\tau) d\tau$$

Although this integration is readily performed for simple  $f_T(\tau)$  (such as the uniform distribution), we will omit this refinement here, assuming the coefficient of variation of  $\tau$  to be small. We will thus replace the random variable  $\tau$  in (25) by its mean value  $m_\tau$ .

Taking the expected value of (13) gives

$$E[P_A] = m_{P_A} = E \left[ \sum_{j=1}^{\theta} h_j \right] = E[\theta m_H] = m_\theta m_H \tag{26}$$

If we use (6) and (11), this gives the mean annual precipitation as

$$m_{P_A} = E[\omega\tau] m_H = \omega m_\tau m_H \tag{27}$$

The variance can be written

$$\text{Var} [P_A] \equiv \sigma_{P_A}^2 = E[P_A^2] - E^2[P_A] \tag{28}$$

which can be reduced, in the manner of (26), to

$$\sigma_{P_A}^2 = m_\nu \sigma_H^2 + m_H^2 \sigma_\nu^2 \tag{29}$$

By using (6), (7), (16), (17), and (19) the variance of the mean annual precipitation becomes

$$\sigma_{P_A}^2 = (m_{P_A}^2/\omega m_\tau)(1 + \kappa^{-1}) \tag{30}$$

The cumulative distribution function of the annual point precipitation is of primary interest here. This is defined by

$$F_{P_A}(y) = \text{Prob} [P_A(\tau) < y] = \int_0^y f_{P_A}(x) dx \tag{31}$$

By using (25), letting  $z$  equal  $\eta\kappa x$ , and replacing  $\tau$  by its mean value  $m_\tau$ ,

$$F_{P_A}(y) = e^{-\omega m_\tau} \left( 1 + \sum_{\nu=1}^{\infty} \frac{(\omega m_\tau)^\nu}{\Gamma(\nu\kappa)\nu!} \int_0^{\eta\kappa y} e^{-z} z^{\nu\kappa-1} dz \right) \tag{32}$$

The integral is [Gradshteyn and Ryzhik, 1965, p. 317]

$$\int_0^{\eta\kappa y} e^{-z} z^{\nu\kappa-1} dz = \gamma[\nu\kappa, \eta\kappa y] \tag{33}$$

where  $\gamma[a, x]$  is the incomplete gamma function. Equation (32) is now

$$F_{P_A}(y) = \text{Prob} [P_A < y] = e^{-\omega m_\tau} \cdot \left\{ 1 + \sum_{\nu=1}^{\infty} \frac{(\omega m_\tau)^\nu}{\nu!} P[\nu\kappa, \eta\kappa y] \right\} \tag{34}$$

where  $P[\nu\kappa, \eta\kappa y]$  is Pearson's incomplete gamma function; that is,

$$P[a, x] \equiv \gamma[a, x]/\Gamma(a) \tag{35}$$

It will be convenient to have (34) in dimensionless form. The natural choice for a normalizing parameter is the climatic variable  $m_{P_A}$ . By using (18) and (27) and letting  $z$  equal  $y/m_{P_A}$ ,

$$\text{Prob} \left[ \frac{P_A}{m_{P_A}} < z \right] = e^{-\omega m_\tau} \cdot \left\{ 1 + \sum_{\nu=1}^{\infty} \frac{(\omega m_\tau)^\nu}{\nu!} P[\nu\kappa, \omega m_\tau \kappa z] \right\} \tag{36}$$

This gives the cdf for annual point precipitation under the following simplifying assumptions: (1) The point storm precipitation is a (stationary) Poisson arrival process, (2) individual storm depths are identically distributed according to  $G(\kappa, \lambda)$ , and (3) the coefficient of variation of the season length is small. Note that the normalized distribution is specified in terms of only two parameters, the mean number of independent storms per year  $\omega m_\tau$ , and the order  $\kappa$  of the gamma distribution of storm depths.

We will now verify (36), using storm observations from both moist and dry climates.

### OBSERVED DISTRIBUTIONS OF STORM PROPERTIES

To assemble observed distributions of storm properties, we first need a criterion for separating the time series into independent events. *Grace and Eagleson* [1966] used the rank correlation coefficient to test for linear dependence among successive 10-min rainfall depths. They found for two locations in New England that linear dependence was insignificant (at the 5% level) between two rainy 10-min periods separated by 2 h or more. Although this test does not assure independence in any but the linear sense, a dry interval of  $t_0 = 2$  h was adopted here as the criterion for distinguishing independent consecutive events at Boston.

From an analysis of 5 years of hourly precipitation data at Boston, Massachusetts, *Grayman and Eagleson* [1969] found the following to be true.

1. Storm duration is distributed exponentially at Boston according to

$$f_{T_r}(t_r) = \delta e^{-\delta t_r} \quad t_r \geq 0 \tag{37}$$

where

$$\delta = m_{t_r}^{-1} \tag{38}$$

The observed and fitted distributions are shown in Figure 2.

2. The time between storms  $t_b$  is distributed exponentially at Boston according to

$$f_{T_b}(t_b) = \beta e^{-\beta(t_b - t_0)} \quad t_b \geq t_0$$

Since  $\beta t_0 \ll 1$ , we will use the approximation

$$f_{T_b}(t_b) = \beta e^{-\beta t_b} \quad t_b \geq 0 \tag{39}$$

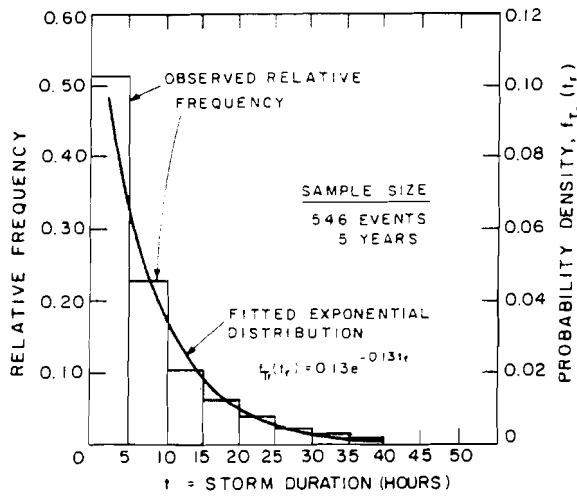


Fig. 2. Distribution of storm durations at Boston, Massachusetts.

where

$$\beta = m_{t_r}^{-1} \quad (40)$$

The observed and fitted distributions are shown in Figure 3.

3. Storm intensity  $i$  is distributed exponentially at Boston according to

$$f_i(i) = \alpha e^{-\alpha i} \quad i \geq 0 \quad (41)$$

where

$$\alpha = m_i^{-1} \quad (42)$$

The observed and fitted distributions are shown in Figure 4.

4. The storm interarrival time may be written (see Figure 1b)

$$t_a = t_r + t_b \quad (43)$$

Assuming that  $t_r$  and  $t_b$  are independent (this has been found to be true by *Grace and Eagleson* [1966]), we can write their joint distribution,

$$f_{T_r, T_b}(t_r, t_b) = f_{T_r}(t_r) f_{T_b}(t_b) \quad (44)$$

which, using (37) and (39), is

$$f_{T_r, T_b}(t_r, t_b) = \beta \delta e^{-\beta t_r - \delta t_b} \quad (45)$$

We can now derive the cdf of storm interarrival times  $t_a$  according to

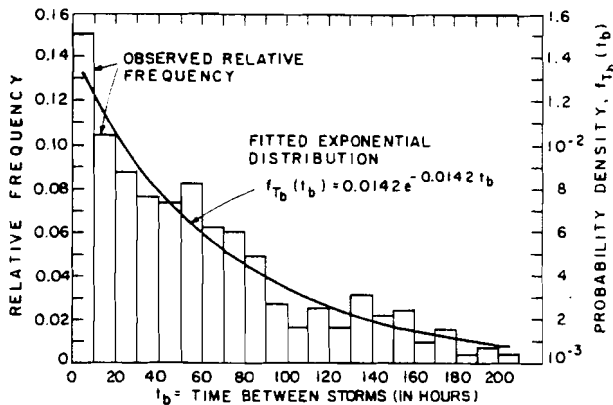


Fig. 3. Distribution of time between storms at Boston, Massachusetts.

$$F_T(t_a) = \int \int f_{T_r, T_b}(t_r, t_b) dR = \int_0^{t_a} dt_b \int_0^{t_a - t_b} \beta \delta e^{-\beta t_r - \delta t_b} dt, \quad (46)$$

Integrating

$$F_T(t_a) = 1 + \frac{\delta}{\beta - \delta} e^{-\beta t_a} - \frac{\beta}{\beta - \delta} e^{-\delta t_a} \quad (47)$$

and differentiating, we get the density function,

$$f_T(t_a) = \frac{\beta \delta}{\delta - \beta} [e^{-\beta t_a} - e^{-\delta t_a}] \quad (48)$$

from which

$$m_{t_a} = \left( \frac{\beta + \delta}{\beta \delta} \right) = \omega^{-1} \quad (49)$$

For Boston we have found that

$$\beta/\delta \ll 1$$

thus for large  $t_a$  the distribution of interarrival times approaches the exponential, as is required for a Poisson process. This exponential approximation is given by (10),  $\omega$  being defined by (49).

5. The storm depth  $h$  is related functionally to the storm duration and average intensity by

$$h = it_r \quad (50)$$

and its cdf is thus given by

$$F_H(h) = \int \int_{R(h)} f(i, t_r) di dt_r \quad (51)$$

What is the joint distribution  $f(i, t_r)$ ? Are  $i$  and  $t_r$  independent? Assuming for the moment that they are independent, we can write

$$f(i, t_r) = f(i)f(t_r) = \alpha \delta e^{-\alpha i - \delta t_r} \quad (52)$$

whereupon (51) can be integrated to yield

$$F_H(h) = 1 - 2(\alpha \delta h)^{1/2} K_1[2(\alpha \delta h)^{1/2}] \quad (53)$$

where  $K_1[ ]$  is the modified Bessel function of the first order. Equation (53) is compared with Boston observations in Figure 5. In spite of its consistency with the chosen marginal distribu-

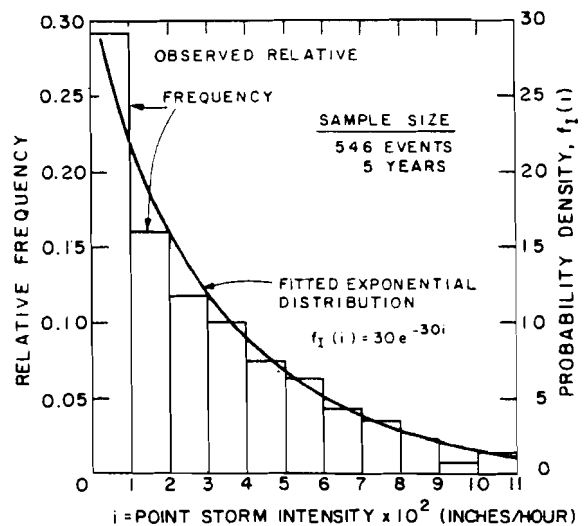


Fig. 4. Distribution of point storm intensities at Boston, Massachusetts.

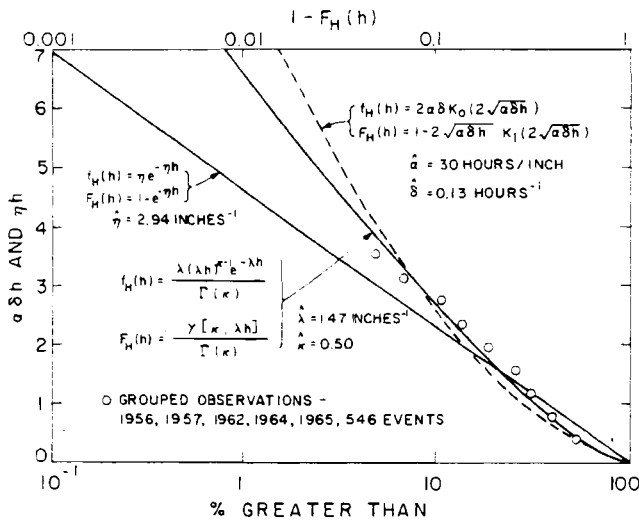


Fig. 5. Point storm precipitation at Boston, Massachusetts.

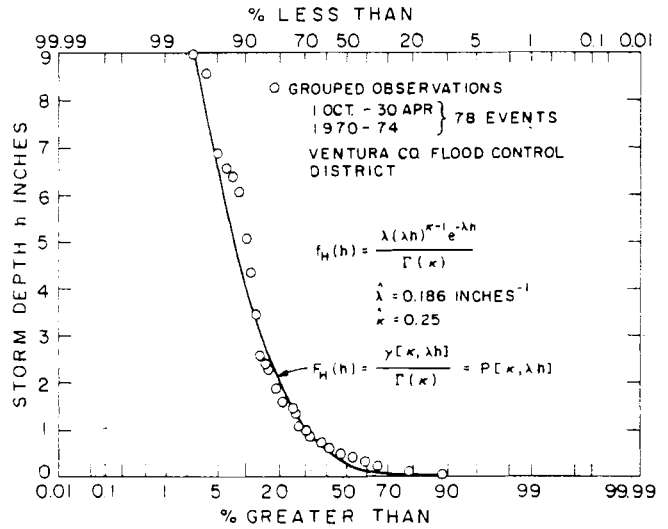


Fig. 7. Point storm precipitation at Santa Paula Canyon, California (at Ferndale Ranch).

tions for intensity and duration, (53) is unsatisfactory for our purposes. Its density function is not self-preserving, as we required in deriving (22). We thus will fit the observed storm depths with the gamma distribution of (15) by the method of moments, acknowledging that this distribution is inconsistent with (37) and (41).

In Figure 5 the grouped observations are also compared with (15) and with the special  $\kappa = 1$  case of (15), which gives the exponential distribution having  $\lambda = \eta$ . Of the three alternatives presented, it seems clear that the gamma function provides the best fit to the observations.

A similar 5-yr analysis of storm data was made for Pasadena, California, and for Santa Paula, California (in Ventura County about 75 mi northwest of Pasadena). In both of these cases, only daily rainfall records were available, and the criterion for independence of successive storm events was taken to be their separation by at least 1 day that has no recorded rainfall. The fitting of (15) to the observed storm depths is shown in Figures 6 and 7. The values found for  $m_H$  and  $\kappa$ , as well as for the other storm properties determined, are given in Table 1.

Notice in Figures 6 and 7 that for large storm depths the chosen gamma distribution does not fit the observations well.

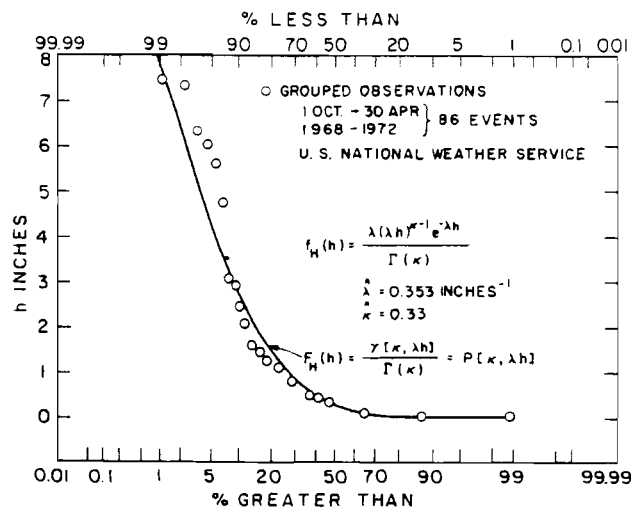


Fig. 6. Point storm precipitation at Pasadena, California.

It is believed that this is due to southern California precipitation being composed of two distinct storm types rather than a single homogeneous population, as was assumed.

OBSERVED NUMBER OF STORMS PER YEAR

The monthly average precipitation for Boston, Massachusetts, and for Santa Paula, California, is shown in Figure 8 for 5 years of observation. From this it is seen that the average length of the Boston rainy season is  $m_r = 365$  d. For Santa Paula it is not so clear, but from these monthly totals we can pick the nominal value  $m_r = 212$  d. Using this as a guide, we can count the number of storms per season and average this number to obtain for Santa Paula,

$$\hat{m}_v = \omega m_r = 15.7$$

If we use  $m_r = 212$  d, the minimal storm arrival rate is defined to be

$$\omega = 15.7/212 = 0.074 \text{ d}^{-1}$$

which will differ somewhat from that calculated by using (49), as is shown in Table 1. This problem can be eliminated by using an actual (i.e., first storm to last storm) season length rather than a nominal (i.e., calendar) one when computing  $\omega$ . It does not arise, of course, when the season is a full year.

TABLE 1. Five-Year Average Properties of Storm Series

Properties	Boston	Santa Paula	Pasadena
$\hat{m}_t, \text{d}$	0.32	1.43	
$\hat{m}_s, \text{d}$	2.98	10.42	
$\hat{m}_H, \text{cm}$	0.86	3.41	2.37
$\hat{\kappa}$	0.50	0.25	0.33
$\hat{m}_r, \text{d}$	365	212	212
$\hat{m}_v$	109	15.7	17.2
$\hat{m}_l, \text{cm/d}$	2.03		
$m_l \approx \hat{m}_H / \hat{m}_t, \text{cm/d}$	2.69	2.38	
$\omega, \text{d}^{-1}$	0.30	0.084	
$\omega = \hat{m}_v / \hat{m}_r, \text{d}^{-1}$	0.30	0.074	0.081
$m_{PA}, \text{cm}^\dagger$	102.6	54.4	51.5
$m_{PA}, \text{cm}$	94.1	53.6	40.8

\*From (49).  
†Full station record.

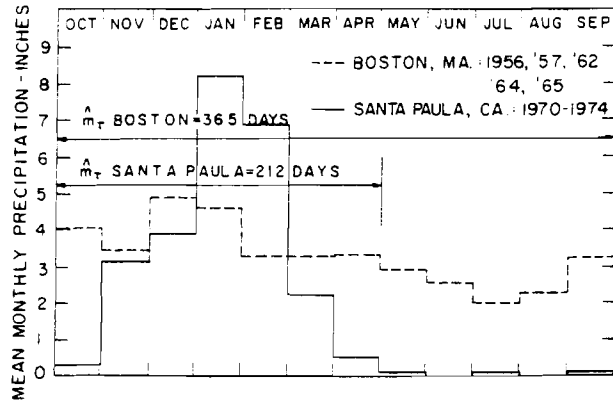


Fig. 8. Seasonality of annual precipitation.

ANNUAL PRECIPITATION

*Humid climates.* In Figure 9 the total annual point precipitation at Boston for each of the 5 years for which the individual storms were studied is presented in the form of the cumulative distribution function. The probability was calculated from an ordered ranking of the five observations according to the Thomas [1948] relation,

$$\text{Prob} \left[ \frac{P_A}{m_{P_A}} < z \right] = \frac{m_z}{N + 1} \quad (54)$$

where  $m_z$  is the rank order of observation of magnitude  $z$  and  $N$  is the number of years of record. The five values are shown by the solid symbols in Figure 9.

In a humid climate such as that of Boston it is generally accepted [Linsley et al., 1975, p. 356] that annual precipitation is distributed normally. It is conventional therefore to fit such an a priori distribution to small data sets. The dashed line of Figure 9 represents the normal distribution fitted by the method of moments to the five annual precipitation totals.

Shown as circles on Figure 9 are the annual precipitation

totals for the period 1886-1974 at Boston. (Earlier observations were discarded when a cumulative precipitation curve showed an apparent abrupt change in slope in about 1886.) We may consider these observations to define the true statistics of  $P_A$ , which we are attempting to estimate from a short 5-yr period of observation.

The solid curve in Figure 9 represents the cdf of  $P_A/m_{P_A}$  as derived by (36) by using values of the parameters  $m_v = \omega m_r$  and  $\kappa$  estimated from the same 5 years of individual storm observations. Notice the close agreement between this prediction and the 'population' cdf (given by the circles). Use of the commonly neglected (although not always available) storm properties appears to provide, through the Poisson model, an estimate of the variance of the annual precipitation, which is considerably more accurate than that obtained through use of the annual totals alone. One might intuitively expect this kind of 'leverage' due to the larger sample size for the parameters to be estimated provided, of course, that the Poisson model is a true representation of the process.

Clinton, Massachusetts, is in the Nashua River basin about 50 mi northwest of Boston. It seems safe to assume that the storm properties will be homogeneous over this short distance. By using the storm parameters  $m_v$  and  $\kappa$  determined from the 5 years of Boston record the cdf of normalized annual point precipitation was calculated for Clinton (since the parameters are identical, the cdf are identical) and is presented as the solid line in Figure 10. Here it is compared with the total available record of annual point totals at Clinton, as is shown by the circles plotted according to (54). Once again the agreement is remarkable.

*Arid climates.* In arid climates it is generally accepted [Linsley et al., 1975, p. 356] that annual precipitation is distributed log normally. In Figures 11 and 12 we present the cdf of annual precipitation for Pasadena and Santa Paula, California. The comparisons are similar to those for Boston and Clinton, Massachusetts, except that log probability paper is used. Here the divergence of the observed and the calculated distributions in both tails may be due to the apparent dual

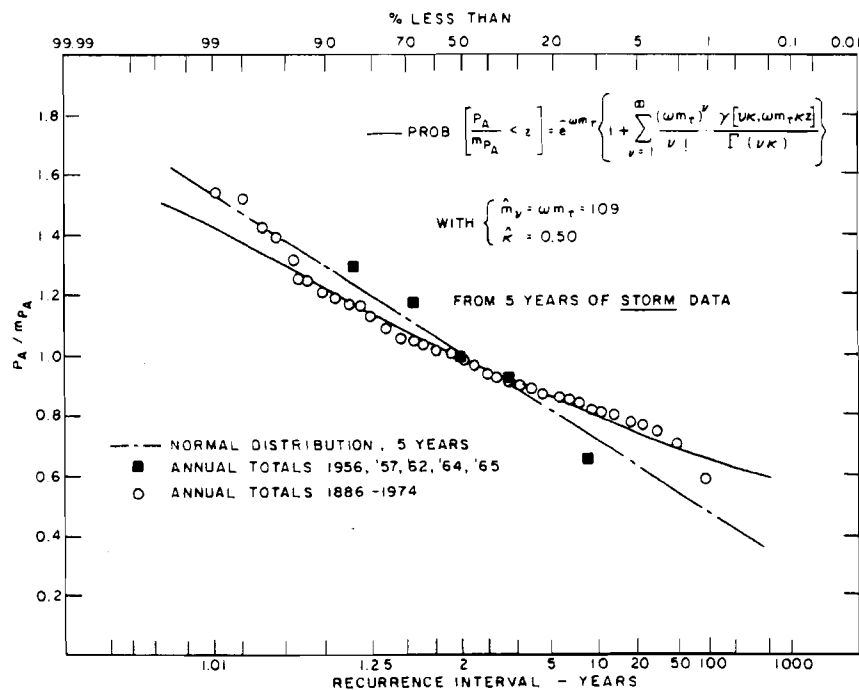


Fig. 9. Annual point precipitation interval at Boston, Massachusetts.

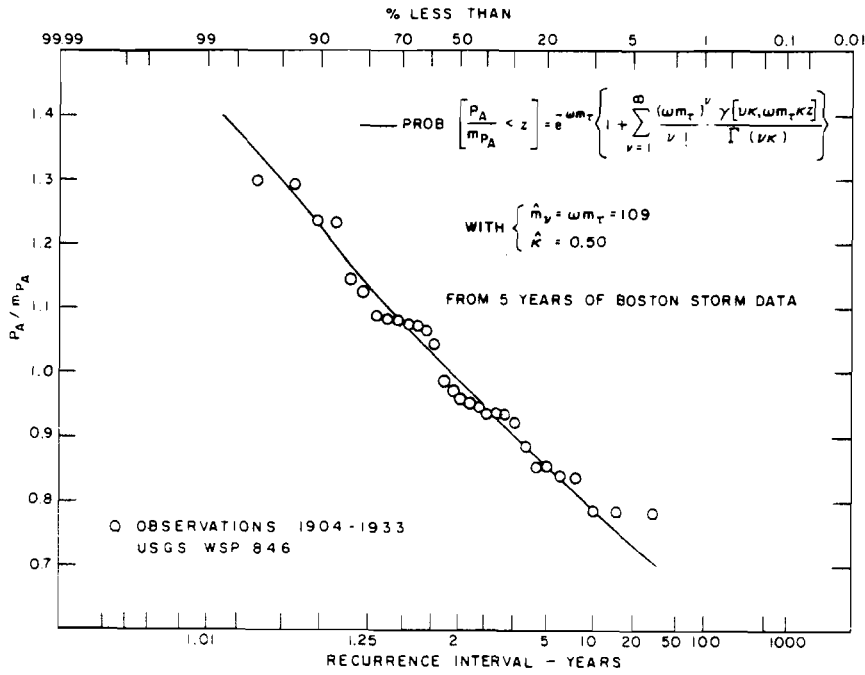


Fig. 10. Annual precipitation in Nashua River basin at Clinton, Massachusetts.

storm population mentioned earlier or to the assumption of constant  $\tau$ .

**SUMMARY**

By using a Poisson distribution of point rainstorm arrivals and a gamma distribution of point rainstorm depths the cumulative distribution function of annual point precipitation is derived.

This distribution allows the generation of the frequency curve of annual precipitation, given short-period observations of storm characteristics, and is shown through comparisons with observations in both arid and subhumid climates to provide a better estimate of the variance of the annual precipitation than do conventional hydrologic techniques which utilize the observed annual totals.

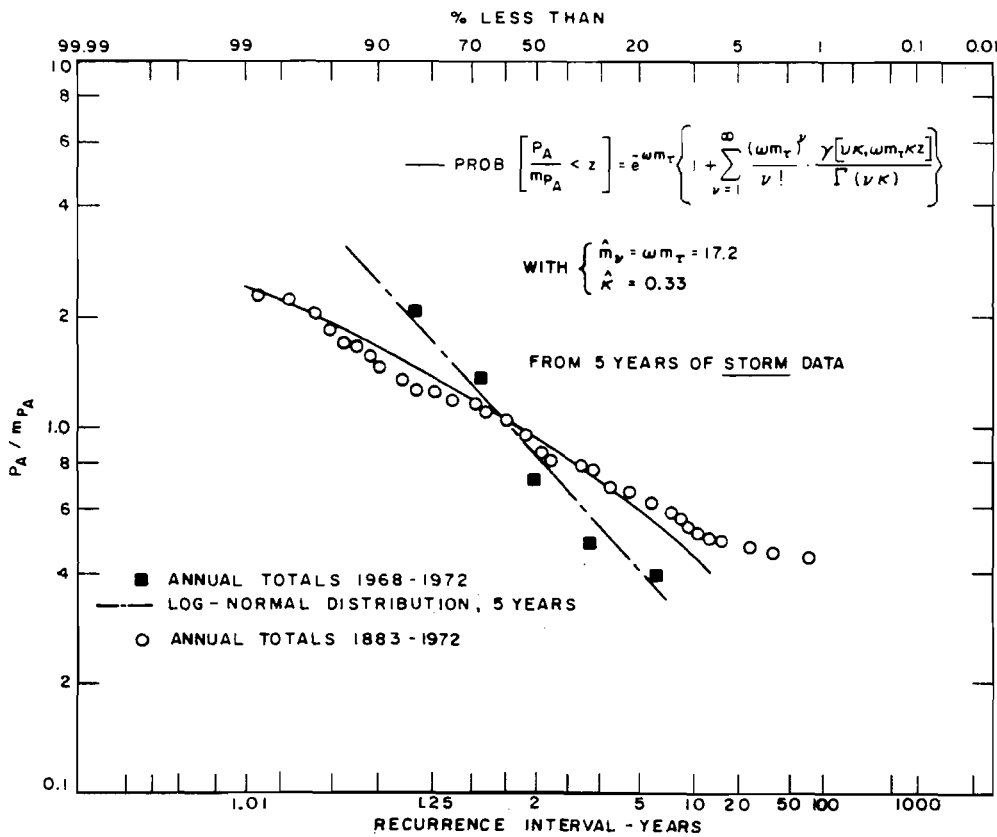


Fig. 11. Seasonal point precipitation at Pasadena, California.

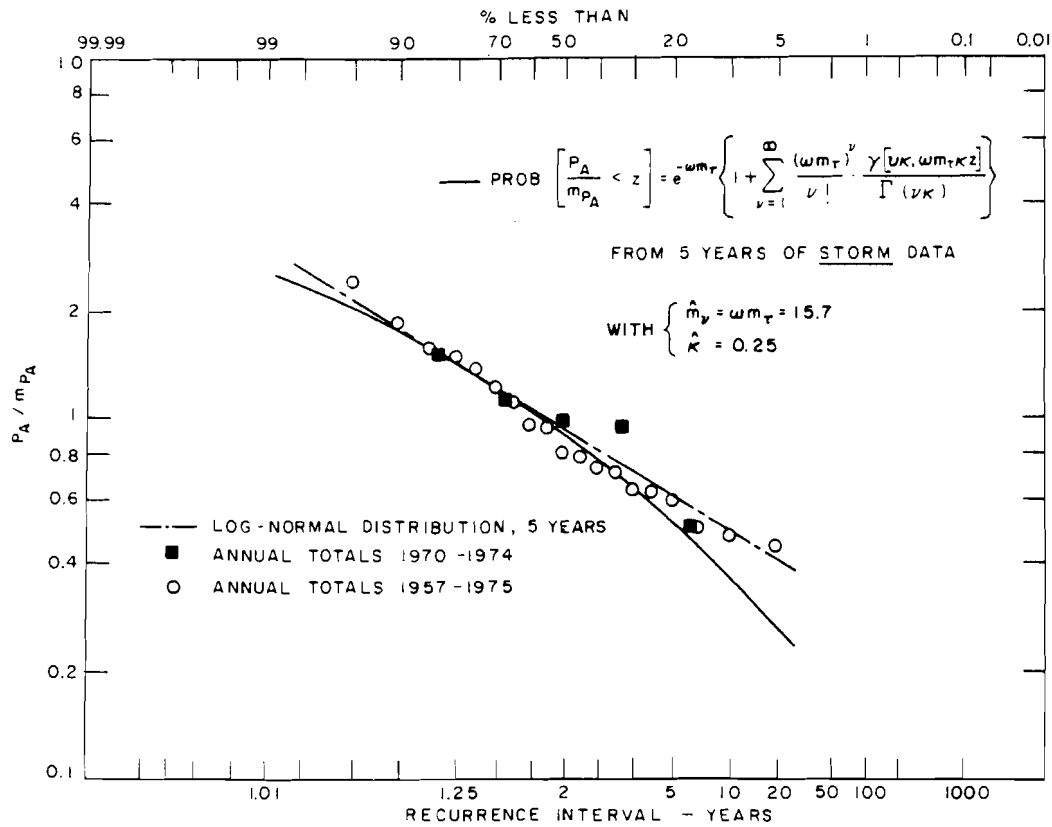


Fig. 12. Seasonal point precipitation at Santa Paula Canyon, California (at Ferndale Ranch).

NOTATION

- $h$  storm depth, centimeters.
- $i$  precipitation rate, centimeters per second.
- $j$  counting variable.
- $m$  number of time intervals.
- $m_H$  mean storm depth, centimeters.
- $m_i$  mean storm intensity, centimeters per second.
- $m_{P_A}$  average annual precipitation, centimeters.
- $m_{t_a}$  mean storm interarrival time, days.
- $m_{t_b}$  mean time between storms, days.
- $m_{t_r}$  mean storm duration, days.
- $m_z$  rank order of observation of magnitude  $z$ .
- $m_v$  mean number of storms per year.
- $m_\tau$  mean length of rainy season, days.
- $N$  number of years of record.
- $P$  precipitation, centimeters.
- $P_A$  annual precipitation, centimeters.
- $p$  probability value.
- $T$  time to arrival of first storm, days.
- $t$  time, seconds.
- $t_a$  storm interarrival time, days.
- $t_b$  time between storms, days.
- $t_r$  storm duration, days.
- $t_0$  storm spacing for independence, days.
- $\alpha$  reciprocal of average rainstorm intensity, equal to  $m_i^{-1}$ , seconds per centimeter.
- $\beta$  reciprocal of average time between storms, equal to  $m_{t_b}^{-1}$ , days<sup>-1</sup>.
- $\Delta t$  arbitrary small time interval, seconds.
- $\delta$  reciprocal of average storm duration, equal to  $m_{t_r}^{-1}$ , days<sup>-1</sup>.

- $\eta$  reciprocal of mean storm depth, equal to  $m_H^{-1}$ , centimeters<sup>-1</sup>.
- $\theta$  number of storms in wet season.
- $\kappa$  parameter of gamma distribution of storm depth.
- $\lambda$  parameter of gamma distribution of storm depths, equal to  $\kappa/m_H$ , centimeters<sup>-1</sup>.
- $\nu$  counting variable for number of storms.
- $\sigma_x$  standard deviation of  $x$ .
- $\sigma_x^2$  variance of  $x$ .
- $\tau$  length of rainy season, days.
- $\omega$  average arrival rate of storms, days<sup>-1</sup>.
- $E[ ]$  expected value of  $[ ]$ .
- $F[ ]$  cumulative distribution function.
- $f( )$  probability density function of  $( )$ .
- $G( , )$  gamma distribution.
- $K_1[ ]$  Bessel function of order 1.
- $P[ , ]$  Pearson's incomplete gamma function.
- $\text{Var}[ ]$  variance of  $[ ]$ .
- $\Gamma( )$  gamma function.
- $\gamma[a, x]$  incomplete gamma function.
- $( \hat{ } )$  estimate of  $( )$ .

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