Derivation of unit hydrograph using a transfer function approach

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Received 2 May 2005; revised 26 October 2005; accepted 16 November 2005; published 31 January 2006.

[1] The unit hydrograph (UH) concept and model have been widely used in the hydrological field over the past decades. However, the estimation of such a model in practice has always been a challenge for researchers and practitioners because such a model is usually ill formed in mathematical terms. The large number of parameters (or the number of ordinates) in a unit hydrograph model are correlated to a certain degree, and this could cause unstable results. So far, the research has been mainly focused on restricting the negative values and smoothing the oscillation by brute force methods, such as linear programming, and has achieved a certain degree of success. However, the number of parameters involved and the lack of stable model response are still a problem. In this study, a new model structure has been proposed that would inherently remove the negative UH ordinates and guarantee a smooth curve. This model is derived by the unit pulse response of a given discrete transfer function in the time domain by restricting its poles along the positive real axis in its Z domain (that is, there are no imaginary components and no negative real values). The model is termed the physically realizable transfer function. The strengths of its structure are that it is numerically stable, physically realizable, parsimonious in parameters, and easy to implement in real time for its state and parameter updating. Its shortcomings are that it has nonlinear pole positions and a more complicated parameter estimation process. A case study with two events in England has been used to demonstrate the application of such a model.

Citation: Yang, Z., and D. Han (2006), Derivation of unit hydrograph using a transfer function approach, *Water Resour. Res.*, *42*, W01501, doi:10.1029/2005WR004227.

1. Introduction

[2] The theory of the unit hydrograph (UH) was first introduced by *Sherman* [1932], and since then it has been acting as an important technique for lumped rainfall-runoff modeling in hydrology. The unit hydrograph of a catchment is defined as a direct runoff hydrograph resulting from a unit of excess rainfall generated uniformly over the drainage area at a constant rate for an effective duration [*Chow et al.*, 1988]. Some computation methods have been developed to estimate the unit hydrograph in the past which include the methods based on the least squares [*Snyder*, 1955] and matrix inversion [*Eagleson et al.*, 1966]. However, the large number of parameters involved in the derivation of the unit hydrograph using these methods can lead to problems of computational instability, which restricts the practical application of these techniques [*Singh*, 1976, 1982; *Rao and Tirtotjondro*, 1995]. For years, hydrologists have attempted to develop a method that can estimate a stable oscillation-free and negative-free unit hydrograph without losing model accuracy. Linear programming techniques [*Deininger*, 1969; *Singh*, 1976; *Mays and Coles*, 1980] have been used to set constraints in negative values, and the Bayesian method [*Rao and Tirtotjondro*, 1995] has been proposed to obtain instantaneous unit hydrographs by limiting some negative values of the model parameters. All these methods suffer from the common problem that there are a large number of correlated parameters in the unit hydrograph model, and mathematically, such an estimation problem is ill formed; therefore these constraints in the estimation methodology could be ineffective in some practical applications. In contrast to these nonparametric approaches mentioned above, parametric methods have also been explored for creating more stable unit hydrographs [*Nash*, 1957; *Koutsoyiannis and Xanthopoulos*, 1989] that fit UH to some prescriptive functional curves (e.g., gamma density function or some other known probability density functions) by moment estimation or by optimization. Although they can be applied to many catchment conditions, there are unsuitable occasions when the catchment responses could deviate from these known simple curves (for example, in an irregularly shaped catchment the outflow could show a second (or maybe a third) smaller peak as a result of a small (remotely located) part of the catchment draining with a large delay, and these parametric methods would fail to cope with such situations). In this paper, a new method based on transfer functions has been developed to obtain a unit hydrograph with no negative and oscillatory ordinates that is also able to cope with various shapes (i.e., not limited to some prescribed density functions). A major advantage of using

0043-1397/06/2005WR004227$09.00

a transfer function to generate UH models is its parsimony in parameterization, so that its real-time updating ability could be improved. The developed model is termed the physically realizable transfer function (PRTF) model, as proposed by *Han* [1991].

2. Foundation of the Physically Realizable Transfer Function Model

[3] The transfer function (TF) model originated from the control theory and has played a very important role in hydrology in past decades [*Amorocho and Hart*, 1964; *Salas et al.*, 1980; *Wood*, 1980; *O'Connell and Clarke*, 1981; *Sorooshian*, 1983; *Gupta*, 1984; *Bras and Rodriguez-Iturbe*, 1985; *Singh*, 1988; *Cluckie*, 1993; *Jakeman et al.*, 1990; *Lees*, 2000; *Beven*, 2001]. Although the TF model appears as a "black box" type, it actually has the ability to reveal the underlying input-output mechanism in hydrological systems. In one important sense, TF models provide a rather natural model form for hydrologists because the impulse response of a continuous-time TF model in hydrology is equivalent to the instantaneous unit hydrograph (IUH), while the discrete pulse response of the discrete time TF model is equivalent to the discrete time UH [*Chow et al.*, 1988]. It should be pointed out that the famous Nash model is actually a special type of TF model [*Nash*, 1957].

[4] In the time domain a rainfall-runoff process modeled by a TF model can be described as [*Ljung and Glad*, 1994]

$$y(t) = \sum_{i=1}^{m} a_i y(t-i) + \sum_{j=0}^{n} b_j r(t-j), \quad t = 1, 2, \ldots, \tag{1}$$

and the corresponding unit pulse response (i.e., unit hydrograph) can be derived as

$$u(0) = b_0, u(k) = b_k - \sum_{i=1}^{k} a_i u(k-i), \quad k = 1, 2, \ldots. \tag{2}$$

The linkage between equations (1) and (2) is

$$y(t) = \sum_{k=0}^{t} r(t-k)u(k) = \sum_{k=0}^{t} r(k)u(t-k), \tag{3}$$

where r is net rainfall, y is river flow, t is the time index, a_i and b_i are parameters, and m and n are the orders of the autoregressive and moving average parts. The identification of a TF model involves the estimation of m, n, a_i, and b_i parameters. The common practice is to try various combinations of m and n and then estimate a_i and b_i with the least squares or instrument variable method [*The MathWorks*, 2004]. The final model can be decided by the Akaike information criterion and root-mean-square error (RMSE) of the model's performance [*Ljung*, 1999]. MATLAB's user guide on the System Identification Toolbox [*The MathWorks*, 2004] provides detailed guidance to facilitate the identification process for general TF models.

[5] Although the unit pulse response of a TF model can be derived by feeding a unit pulse (equivalent to the well-known Dirac delta function in continuous form) into equation (1), it is not easy to reveal the complete working mechanism of a TF model in the time domain. In this study, the derivation of the new UH based on the TF model is carried out in the Z domain using Z transform (a popular tool in the control theory and signal processing field [see *Bellanger*, 1989]). For hydrologists it is instructive to view the Z transform as a transformation that maps a real variable in the time domain into a complex Z domain (similar to the Fourier transform, which maps a real signal in the time domain into the frequency domain). Applying the Z transform to both sides of equation (1), we derive the following formula (detailed derivation is provided in Appendix A):

$$U(z) = \frac{Y(z)}{R(z)} = \frac{B(z)}{A(z)} = \frac{\sum_{i=0}^{n} b_i z^{-i}}{1 - \sum_{i=1}^{m} a_i z^{-i}}. \tag{4}$$

$U(z)$ is the response function for the TF model in the Z domain. To convert it back in the time domain, we apply an inversion transform for $U(z)$,

$$u(k) = Z^{-1}[U(z)] = \sum_{i=1}^{N} \operatorname*{Res}_{z=p_i} [U_0(z)], \tag{5}$$

where N is a positive integer representing the total number of individual poles p_i and $U_0(z) = U(z)z^{k-1}$. Since $U_0(z)$ is a rational function (i.e., a polynomial divided by another polynomial), its residues can be simply derived by a combination of residues from simple poles (order of 1) and poles with higher orders (i.e., repeating poles; for example, if two poles have the same value, they are called repeating poles, and they would be treated as one pole with an order of 2). From polynomial algebra the roots of a polynomial can only have two types: real numbers and conjugate complex numbers. By analyzing the residues for both types we can cover all the possible response curves for all TF models.

[6] Let's start with a single real-pole case (the pole value is a real number p). From equation (5) its unit pulse response function would be derived as

$$u(k) = \operatorname*{Res}_{z=p} [U_0(z)] = b_0 p^k, \tag{6}$$

where $k = 0, 1, 2, 3, \ldots$ and $|p| < 1$ (note that if $|p| > 1$, the TF model is not stable). This indicates that a negative real pole would produce an undesirable unit pulse response curve with alternate positive and negative ordinates.

[7] For a TF model with a pair of conjugate complex poles ($p_1 = x + jy$, $p_2 = x - jy$) we can derive its unit response curve as

$$u(k) = \operatorname*{Res}_{z=x+jy} [U_0(z)] + \operatorname*{Res}_{z=x-jy} [U_0(z)] = \frac{c^k}{\sin \phi} \sin[(k+1)\phi], \tag{7}$$

where $c = \sqrt{x^2 + y^2}$ and $\phi = \tan^{-1}(y/x)$. It can be seen that $u(k)$ represents a damped sinusoidal curve decaying exponentially to zero. To avoid a fluctuating response, ϕ should be zero. This means that there should be no imaginary part in the complex poles for a stable and nonfluctuating TF model. To understand other more complicated forms of TF models (i.e., when $A(z)$ has more than three poles and $B(z)$ has more than one zero), it is necessary to analyze the combination forms in these TF models. Since $U(z)$ can be factored into the realization of two transfer functions $U(z) = B(z)/A(z)$, the unit pulse response function of the TF model can then be derived from these two parts in a convoluted way. As the poles of $B(z)$ are all located at the origin in the Z plane, $B(z)$ is always stable and nonfluctuating; hence we only need to consider the $1/A(z)$ part. If the unit pulse response of $B(z)$ can be viewed as rainfall and $1/A(z)$ can be viewed as a unit hydrograph (just an analogy), the unit response of $U(z)$ can be interpreted as a convoluted river flow series. It is then straightforward to show that to achieve a physically realizable TF model, $1/A(z)$ must be a positive stable TF model without fluctuations in its response curve. Since $U(z)$ is a rational function, it can be separated into different TF models on the basis of their poles,

$$U(z) = \frac{B(z)}{A(z)} = \frac{c(z-o_0)(z-o_1)\ldots(z-o_n)}{(z-p_1)(z-p_2)\ldots(z-p_m)} = \frac{f_1(z)}{z-p_1} + \frac{f_2(z)}{z-p_2} + \ldots \frac{f_m(z)}{z-p_m}. \quad (8)$$

[8] This shows that no matter how complicated a TF model is, it can always be decomposed into a sum of several simpler TF models. A PRTF model can then be derived if all poles are restricted along the real axis. Equation (8) also shows that a TF model is able to simulate a unit hydrograph with two peaks (with two different poles) or more, which is a clear advantage over other parametric methods which can only cope with a single peak. From equation (5) a general PRTF response function in the Z domain is

$$U(z) = \frac{Y(z)}{R(z)} = \frac{B(z)}{A(z)} = \frac{\sum_{i=0}^{n} b_i z^{-i}}{\prod_{i=1}^{N} \left(1 - p_i z^{-1}\right)^{m_i}}, \quad (9)$$

where the poles are restricted in (0, 1)($p_i \in R$ and $0 < p_i < 1$).

3. Modeling Application of PRTF

[9] The first task with a PRTF model in hydrological applications is to decide the poles (orders and values). Although it is desirable to develop a system which has no limit to the number of poles, in practice a single-pole TF model works extremely well (albeit not in all cases). This is because in practice a single-peak unit hydrograph is usually versatile enough for most catchment conditions; hence a TF model with a single pole is further explored in this study for application purposes. From equation (9) a PRTF with a single pole p of order m in the time domain can be expressed as (derivation details can be found in Appendix A)

$$y(t) = \sum_{i=1}^{m} a_i y(t-i) + \sum_{j=0}^{n} b_j r(t-j)$$
$$a_i = -\binom{m}{i}(-p)^i, (p_i \in R, 0 < p_i < 1). \quad (10)$$

It has been found that a PRTF model with a single pole of order 3 worked quite well for most catchments analyzed by the authors in the southwest and northwest of England (a single pole of other orders did not produce better results). In this case we can derive this simplified TF model from equation (10) by substituting $m = 3$ and $i = 1, 2, 3$. The final model is derived as (a time lag is introduced for the catchment with a time delay τ)

$$y(t) = 3py(t-1) - 3p^2y(t-2) + p^3y(t-3) + \sum_{j=0}^{n} b_j r(t-j-\tau). \quad (11)$$

Equation (11) has been adopted for real-time flood forecasting by two regions of the Environment Agency in the United Kingdom [*Cluckie and Han*, 2000]. This model has been favored by the duty hydrologists because of its very simple form. In our experience, two or three b_i parameters produced the best results for many practical cases.

[10] For the development of traditional transfer function models, there are many conventional techniques that can be used for parameter estimation, such as least squares, instrumental variables, etc. [*Box et al.*, 1970; *Young*, 1984; *The MathWorks*, 2004]. They are usually quite efficient for linear transfer function models. However, because of the nonlinear relationship between the model's response and the poles in PRTF models a modified least squares method should be used to convert the nonlinear problem into a linear one. The objective function for PRTF in equation (11) is a function of pole p and b_i parameters based on the RMSE of a one-step prediction of the model as

$$\min S(p,b) = \sum_t e_t^2 = \sum_t [y(t) - \widehat{y}(t)]^2 = \sum_t \left\{ y(t) - \left[3py(t-1) - 3p^2y(t-2) + p^3y(t-3) + \sum_{j=0}^{n} b_j r(t-j-\tau)\right]\right\}^2. \quad (12)$$

Clearly, S and p are in a nonlinear relationship. A new variable is introduced to encapsulate this nonlinear part,

$$w(t) = y(t) - 3py(t-1) + 3p^2y(t-2) - p^3y(t-3).$$

Equation (11) is then simplified as

$$w(t) = \sum_{j=0}^{n} b_j r(t-j). \quad (13)$$

This is a linear function, and b_i parameters can be easily estimated by least squares with a simple matrix inversion (for example, with MATLAB's QR matrix decomposition function "\").

[11] The computation process can be broken down into the following steps. First, it is suggested that several major flow events are concatenated into one series to calibrate the PRTF model; therefore the derived UH from PRTF is an averaged one. As it is known, different events may have different UH shapes, and this could cause some major problems with traditional UH models. In practice, practitioners usually calibrate many UH curves from individual events and then classify them into different

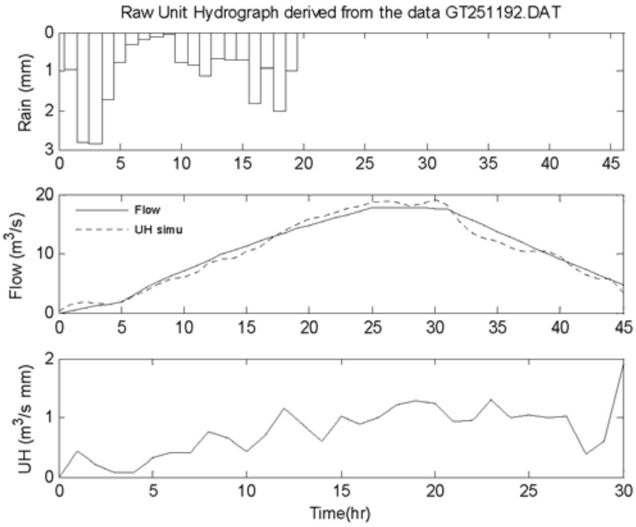

Figure 1. Nonparametric unit hydrograph (UH) derived from the data of event Gt251192.

groups. During the modeling stage, actual storm characteristics (i.e., storm center location, intensity, etc.) are used to identify the suitable UH to be used for individual storms [*Ge*, 2001]. PRTF has a useful potential to overcome this problem (see section 4 for further details). Second, the number of b_i parameters is set from 1 to 6, and for each b_i number (e.g., 4) a nonlinear search is carried out to locate the p value coupled with a least squares estimation of b_i. There are many nonlinear optimization tools available for carrying out this task (in this study, Optimization Toolbox in MATLAB is used). This is iterated for every b_i parameter number (i.e., 1, 2,. . ., 6). The time delay factor τ can then be estimated in a similar style as the b_i parameter number. The final model is chosen on the basis of the comparison of RMSE between the numbers of b_i parameters (the minimum is chosen). Although RMSE of the one-step prediction of the model is used in this study as the objective function, other forms of objective func-

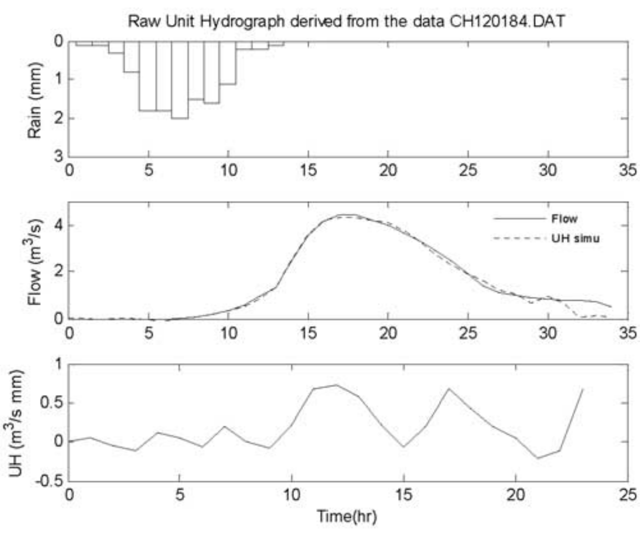

Figure 2. Nonparametric unit hydrograph derived from the data of event Ch120184.

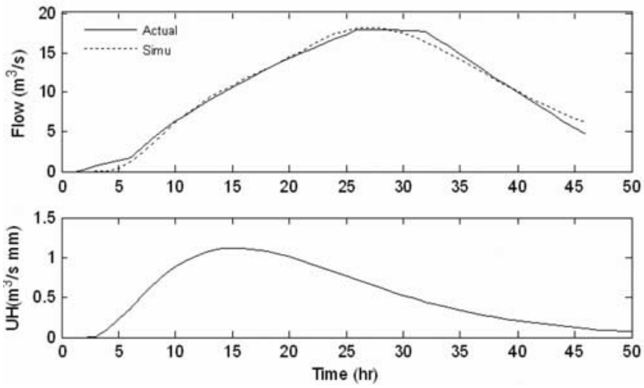

Figure 3. UH derived by physically realizable transfer function (PRTF) model for event Gt251192.

tions are also possible, e.g., RMSE of the simulation of the model or other prediction steps. As different objective functions could yield different models, it may be useful to explore the Pareto front of a combination of these objective functions and find a suitable scheme for multi-objective optimization of PRTF models. Last, it is important to note that unlike other numerical schemes, where a finer time step may be able to produce a higher accuracy, PRTF models have a limited suitable working range. If the time step is too fine (e.g., using a 1 min interval instead of a 1 hour interval), a PRTF model would not work very effectively (a nonparametric UH would suffer even more since a short time interval would cause a huge number of correlated UH ordinates). In our experience, an hourly interval is suitable for many catchments in England. Further research is needed to link the optimal model time interval with the catchment concentration time.

[12] In its application for real-time flood forecasting, PRTF has been proven as a very easy model to use by duty hydrologists. A case study is shown here to illustrate the effectiveness of this model. Two examples are presented on the basis of the events Ch120184 in Chiselborough and Gt251192 in Great Somerford in southwest England. If no restrictions are applied to a least squares method (the matrix inversion is carried out by MATLAB routine inv, which is based on LAPACK), the estimated unit hydrograph models can be clearly seen with oscillating and even with negative values (Figures 1 and 2), although the simulated flows from such models are very close to the calibrated events. These UH models would perform poorly in simulating other unforeseen events (for example, it is very unrealistic for any rainfall with a unit duration to produce these UH runoff curves in real life; they are typical overfitting cases in mathematical model estimations and would have poor generalizations). When PRTF structure based on equation (11) is applied (Figures 3 and 4), the derived UH models show much more realistic responses, and the simulated flows are very close to the measured ones. The comparisons between the computation results are summarized in Table 1 on the basis of R^2 and RMSE (with a further note in Appendix B). It is interesting to note that the restriction of pole positions has little detrimental effect on the simulated closeness to the measured flow. The result is even better than the unrestricted one for event Gt251192. This is very puzzling initially since

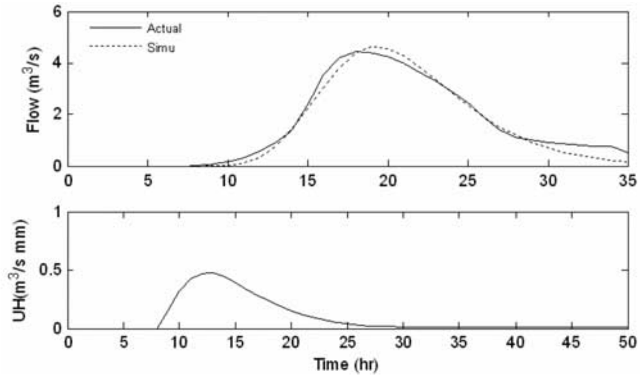

Figure 4. UH derived by PRTF model for event Ch120184.

it is usually assumed that the more parameters in a model, the closer the simulated model would be to the calibrated data points (not necessarily to the test data points). We believe that this phenomenon might be caused by the ill-formed model structure of a large number of correlated ordinates in the nonparametric UH, which prevented an effective matrix inversion to minimize the least squares errors. Therefore, although in theory a model with a large number of parameters should perform better than a much smaller one in calibration cases, numerically, the result may be different because of the limited computer precision and the data noise.

4. Discussion

[13] The application of PRTF so far has been focused on a single pole of order 3 as presented in equation (11). *Yang* [2000] proposed a new term, all-pole PRTF (APPRTF), to emphasize that multiple poles should be used instead of a single pole. For APPRTF models the identification would be more complicated [*Yang*, 2000]. In general, there are three different categories of poles to be considered: (1) the distinct pole condition, (2) the repeating pole condition, and (3) the combination of distinct and repeating poles. Each of these can realize a PRTF model. An APPRTF model has no restriction on how many poles are to be used; therefore the APPRTF model includes or covers the following three types of PRTF models: (1) PRTF models with repeating poles, (2) PRTF models with distinct poles, and (3) PRTF models with both repeating and distinct poles.

[14] The explanation for the number of poles and their values in the hydrological domain can be helped by the linear reservoir models. The concept of linear reservoirs was first introduced by *Zoch* [1934] and further expanded by other researchers [*Nash*, 1957; *Doodge*, 1959; *Chow*, 1964] to explain their hydrological models on the basis of linear reservoirs in series and in parallel. The concept is very useful here to explain the model structure of PRTF. It is clear that the number of poles represents the order of the system in a linear reservoir model, i.e., the number of reservoirs. If the final calibrated model has n poles, it may be explained that the rainfall-runoff process in the corresponding hydrological system can be simulated as a cascade of n linear reservoirs in series. There are three occasions when the cascade system should be used. In the case that all the poles have the same value (repeating poles) it indicates that each reservoir may have similar characteristics (equal linear reservoirs) that produce a similar response to that contained in the Nash model. When all the poles have different values (distinct poles), it implies that reservoirs in the system may be quite inhomogeneous in characteristics and that their responses to unit pulse input would be different. Finally, if the system has some poles of the same value and some with different values, then it means that among those reservoirs, some of them may be similar in characteristics and some may be different. As a result, their responses to unit pulse input are a combination of the above two. *Yang* [2000] has shown that if a PRTF model is built embracing all pole conditions, it is more flexible and versatile for catchments with variable characteristics (e.g., multiple peaks). As a result, the APPRTF model has the potential to produce a better estimation of the UH and a better fit of the direct runoff for some complicated catchments (e.g., several peaks, as mentioned in section 1). However, for the APPRTF model estimation the objective function may have multiple peaks, and the standard nonlinear searching tools may be trapped to a local optimum. Yang proposed to use genetic algorithms (GAs) to solve the optimization problem (the scheme was an advanced version based on the GA first used in hydrology by *Wang* [1991]). However, more research work on APPRTF is still needed before it could be used in operational flood forecasting systems (e.g., how to update APPRTF in real-time operations).

[15] In addition to PRTF's parsimonious parameters and physically realizable characteristics an even more important feature of PRTF is that the shape of its response function can easily be continuously altered by moving its pole location. Because of the change of storm and catchment characteristics (intensity, duration, storm center, seasons, etc.) the catchment response to a storm event would be different, and hence the unit hydrograph also changes [*Ge*, 2001]. Traditionally, practitioners have to calibrate many UH models for different situations and retrieve those models on the basis of the operational environment. This is very tedious, and it is not always easy to classify and retrieve the calibrated UH models [*Ge*, 2001]. In a related

Table 1. Modeling Statistics[a]

Model	Statistics	Gt251192	Ch120184
Nonparametric UH	R^2	0.9703	0.9836
Nonparametric UH	RMSE	0.9789	0.1957
PRTF UH	R^2	0.9886	0.9776
PRTF UH	RMSE	0.6072	0.2282
Model parameters		b = [0.00721, 0.07202], lag = 2, p = 0.8562	b = [0.157], lag = 8, p = 0.6693

[a]UH, unit hydrograph; RMSE, root-mean-square error; PRTF, physically realizable transfer function.

field, although the TF model has been used in real-time flood forecasting for many years, it has been a major problem in making the model adaptive to different catchment states and storm characteristics. In this field the most famous advancement has been in using the Kalman filter to update the TF model's state and parameters [*Todini*, 1978; *Cluckie and Harpin*, 1980; *Bras and Rodriguez-Iturbe*, 1985; *Ge*, 2001]. Despite the reported success in literature none of the techniques can work well all the time in actual operational systems. For example, the well-known mutually interactive state and parameter (MISP) and the constrained linear systems (CLS) model by *Todini* [1978] has been applied to many countries. However, in operational situations at some major catchments in China it has been found that the updated model could occasionally produce unstable responses, and despite the relatively rare occurrence of these unstable cases this potential uncertainty in model divergence is a serious shortcoming in operational flood forecasting systems and undermines the confidence of such an adaptive scheme by practitioners [*Ge*, 2001]. Interestingly, this is similar to the main reason why artificial neural networks (ANN) models have not been successful in penetrating into the operational flood forecasting systems so far despite the large number of academic papers published in this field, as demonstrated by *Han et al.* [2005] in their bootstrap statistical analysis of ANN model uncertainties. We believe that the problem with traditional TF models is caused by updating a_i parameters since the poles of the TF model are very sensitive to tiny alterations of a_i parameters (see Appendix B). When any of the poles is updated outside of the unit circle, its TF response would immediately become unstable, and the model would produce very unrealistic outcomes. The physically realizable TF model reported in this paper is able to overcome this shortcoming since only the poles are updated along the positive real axis within the unit circle, so the model is always stable under any circumstances. An adaptive form of PRTF has been developed on the basis of the principles described above, which have three parameters for adjusting the volume, shape, and time delay of the catchment response. Each of the parameters is independent of the others. Instead of calibrating, classifying, and retrieving a large number of UHs as used to be the conventional way to use the UH, hydrologists now only need to calibrate one PRTF model on the basis of an averaged catchment and storm environment (based on the concatenation of many major flow events) and then link the adaptive PRTF model's adjustment factors to different storm and catchment states. In comparison to other model forms the implementation of this adaptive PRTF in an operational system is quite convenient because of its stability and small number of parameters. So far, such an adaptive model has been successfully integrated into a real-time expert system by hydrologists in southwest England and has been used extensively in flood forecasting operations in the last 5 years. The derivation of the adaptive form of PRTF and its working experience will be reported in the near future.

5. Conclusions

[16] The PRTF structure provides a new means for estimating stable and realistic unit hydrograph models. Because of its much smaller number of parameters and inherent stability, PRTF is able to overcome the ill-formed structure of a large number of correlated ordinations in traditional unit hydrograph models. The following advantages are associated with this new structure: It is parsimonious, physically realizable (no oscillation and no negative values), numerically stable, and easy to be implemented in real time. It should be noted that an important feature of this new model is its response stability during real-time updating; this is crucial for operational flood forecasting systems. However, although an expert system has been implemented to use PRTF's adaptivity by the Environment Agency, the effort in building, updating, and maintaining an operational expert system is huge, and we believe that a combination of PRTF with a Kalman filter could be a very useful alternative approach. It should be pointed out that the PRTF model also has many drawbacks. (1) Its calibration is mathematically more complicated than a simple matrix inversion. (2) Its a_i parameters have a nonlinear relationship with its pole positions. (3) The model is still relatively new, and there are limited application results published in the literature. However, for those who are familiar with traditional linear transfer function models, PRTF model structure is a simple extension in concept and should impose very little difficulty in its implementation in practical problems. Especially, for a single pole of order 3 TF model its structure is surprisingly simple, and the parameter calibration can be carried out very efficiently without using the more complicated GA technique. There are still many unsolved problems with this new model (e.g., the optimal model time interval, application in ungauged catchments (linking the pole locations with catchment physical characteristics), coupling with a Kalman filter, etc.).

Appendix A: Mathematical Derivation of PRTF Concept and Formulas

[17] Knowledge of a residue theorem in complex variable analysis is needed for the following derivations, and unfamiliar readers are encouraged to refer to *Zill and Cullen* [1992] or to other similar books on this topic.

[18] Rearranging equation (1) with y terms to the left-hand side gives

$$y(t) - \sum_{i=1}^{m} a_i y(t-i) = \sum_{j=0}^{n} b_j r(t-j). \tag{A1}$$

Employing the Z transform to both sides of equation (A1) gives

$$Z\left[y(t) - \sum_{i=1}^{m} a_i y(t-i)\right] = Z\left[\sum_{j=0}^{n} b_j r(t-j)\right].$$

Let

$$A(z) = 1 - a_1 z^{-1} - a_2 z^{-2} - \ldots - a_m z^{-m}$$

$$B(z) = b_0 + b_1 z^{-1} + b_1 z^{-2} + \ldots + b_n z^{-n},$$

so we have $A(z)Y(z) = B(z)R(z)$; then we get equation (4).To convert $U(z)$ back in the time domain, we apply an inversion integral for $U(z)$. It takes the form of a contour integral (the contour S includes the poles of $U(z)$ within it):

$$u(k) = Z^{-1}[U(z)] = \frac{1}{2\pi j} \oint_S U(z) z^{k-1} dz, \tag{A2}$$

where j is the imaginary unit (in this paper, j is used instead of i since i has been used for index).

[19] Because of the properties of complex functions the above integral can be rewritten as a sum of residues (from Cauchy's residue theorem):

$$u(k) = Z^{-1}[U(z)] = \sum_{i=1}^{N} \operatorname*{Res}_{z=p_i} \left[U(z) z^{k-1}\right] = \sum_{i=1}^{N} \operatorname*{Res}_{z=p_i} [U_0(z)], \tag{A3}$$

where N is a positive integer representing the total number of individual poles p_i and $U_0(z) = U(z)z^{k-1}$ (note that from complex variable theory, if a function has a pole (i.e., singularity) at some point, it means that its value "explodes" at that point without a finite value).

[20] For a simple pole with an order of 1 in $U_0(z)$ the residue can be derived as

$$\operatorname*{Res}_{z=p_i} [U_0(z)] = \lim_{z \to p_i} [(z - p_i) U_0(z)] = (z - p_i) U_0(z)\big|_{z=p_i}. \tag{A4}$$

For a vth-order pole (v = 2, 3, ...) the residue in $U_0(z)$ can be derived as

$$\operatorname*{Res}_{z=p_i} [U_0(z)] = \frac{1}{(v-1)!} \lim_{z \to p_i} \frac{d^{v-1}}{dz^{v-1}} [(z-p_i)^v U_0(z)] = \frac{1}{(v-1)!} \frac{d^{v-1}}{dz^{v-1}} [(z-p_i)^v U_0(z)]\big|_{z=p_i}. \tag{A5}$$

From polynomial algebra the roots of a polynomial can only have two types: real numbers and conjugate complex numbers. By analyzing the residues for both types we can cover all the possible response curves regardless of complicated combinations in any TF models. We will start with a single real-pole case,

$$U_o(z) = U(z) z^{k-1} = \frac{b_0}{1 - pz^{-1}} z^{k-1} = \frac{zb_0}{z-p} z^{k-1} = \frac{b_0 z^k}{z-p},$$

where p is the pole vale of $U_0(z)$. This is equivalent to a TF model in the time domain as $y(t) = py(t-1) + b_0 r(t)$. From equations (A4) and (A5) its unit pulse response function would be

$$u(k) = \operatorname*{Res}_{z=p} [U_0(z)] = \operatorname*{Res}_{z=p} \left[\frac{b_0 z^k}{z-p}\right] = [z-p] \frac{b_0 z^k}{z-p}\big|_{z=p} = b_0 p^k, \tag{A6}$$

where k = 0, 1, 2, 3, ... and $|p| < 1$ (note that if $|p| > 1$, the TF model is not stable).

[21] It can be seen that for $-1 < p < 0$ we have $u(k) > 0$ when $k = 2i$ and i = 0, 1, 2, 3,..., and we have $u(k) < 0$ when $k = 2i + 1$ and i = 0, 1, 2, 3, ... This indicates that a negative real pole would produce an undesirable unit pulse response curve with alternate positive and negative ordinates.

[22] For a TF model with a pair of conjugate complex poles we can derive $U_0(z)$ as

$$U_o(z) = U(z) z^{k-1} = \frac{b_0 z^{k-1}}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})} = \frac{b_0 z^{k+1}}{(z-p_1)(z-p_2)}.$$

This is equivalent to a TF model in the time domain as

$$y(t) = -(p_1 + p_2) y(t-1) + p_1 p_2 y(t-2) + b_0 r(t).$$

Since the poles are a pair of conjugate complex numbers, $p_1 = x + jy$ and $p_2 = x - jy$. The unit response curve is derived as

$$u(k) = \operatorname*{Res}_{z=x+jy} [U_0(z)] + \operatorname*{Res}_{z=x-jy} [U_0(z)] = \frac{z^{k+1}}{z - x + jy}\big|_{z=x+jy} + \frac{z^{k+1}}{z - x - jy}\big|_{z=x-jy} = \frac{1}{j2y}\left[(x+jy)^{k+1} - (x-jy)^{k+1}\right].$$

Since

$$x \pm jy = ce^{\pm j\phi},$$

where $c = \sqrt{x^2 + y^2}$ and $\phi = \tan^{-1}(y/x)$, one obtains

$$u(k) = \frac{c^k}{j2y}\left[ce^{j(k+1)\phi} - ce^{-j(k+1)\phi}\right].$$

Finally, since $y = c \sin \phi$, the unit pulse response curve can be derived as

$$u(k) = \frac{c^k}{\sin \phi} \sin[(k+1)\phi], \qquad k = 0, 1, 2, 3, \ldots. \tag{A7}$$

It can be seen that $u(k)$ represents a damped sinusoidal curve decaying exponentially to zero. Since $U(z)$ is a rational function, it can be separated into different TF models on the basis of their poles, shown in equation (8). From equation (4) a general PRTF response function in the Z domain is

$$U(z) = \frac{Y(z)}{R(z)} = \frac{B(z)}{A(z)} = \frac{\sum_{i=0}^{n} b_i z^{-i}}{1 - \sum_{i=1}^{m} a_i z^{-i}} = \frac{\sum_{i=0}^{n} b_i z^{-i}}{z^{-m}\left(z^m - \sum_{i=1}^{m} a_i z^{m-i}\right)} = \frac{\sum_{i=0}^{n} b_i z^{-i}}{z^{-m} \prod_{i=1}^{N} (z - p_i)^{m_i}} = \frac{\sum_{i=0}^{n} b_i z^{-i}}{\prod_{i=1}^{N} \left(1 - p_i z^{-1}\right)^{m_i}}, \tag{A8}$$

where the poles are restricted in (0, 1)($p_i \in R$ and $0 < p_i < 1$).

[23] Here a TF model with a single pole is further explored for application purposes. From equation (A8), $U(z)$ can be simplified as (with one pole p of order m)

$$U(z) = \frac{\sum_{i=0}^{n} b_i z^{-i}}{(1-pz^{-1})^m} = \frac{\sum_{i=0}^{n} b_i z^{-i}}{\sum_{i=0}^{m} \binom{m}{i} (-pz^{-1})^i} = \frac{\sum_{i=0}^{n} b_i z^{-i}}{1 - \sum_{i=1}^{m} \left[-\binom{m}{i} (-p)^i z^{-i} \right]}. \quad \text{(A9)}$$

(Note that equation (A11) uses the binomial theorem $(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} = \sum_{i=0}^{n} \binom{n}{i} y^i x^{n-i}$ and $\binom{x}{y} = x!/y!(x-y)!$ Since $1 - \sum_{i=1}^{m} a_i z^{-i} = 1 - \sum_{i=1}^{m} [-\binom{m}{i}(-p)^i z^{-i}]$ (from equations (A8) and (A9)),

$$a_i = -\binom{m}{i}(-p)^i, i = 1, 2, \ldots, \text{m}.$$

Therefore PRTF with a single pole p of order m in the time domain can be expressed as equation (10). Its unit pulse response function can then be derived from the equation (A5) as,

$$\begin{aligned} u(k) &= \operatorname*{Res}_{z=p} [U_0(z)] = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-p)^m U_0(z)]\big|_{z=p} \\ &= \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[z^m z^{k-1} \sum_{i=0}^{n} b_i z^{-i} \right]\big|_{z=p} \\ &= \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[\sum_{i=0}^{n} b_i z^{m+k-i-1} \right]\big|_{z=p} \\ &= \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[\sum_{i=0}^{n} b_i z^{m+k-i-1} \right]\big|_{z=p} \\ &= \frac{1}{(m-1)!} \left[\sum_{i=0}^{n} b_i \prod_{h=0}^{m-2} (m+k-i-1-h) z^{k-i} \right]\big|_{z=p} \\ &= \frac{1}{(m-1)!} \left[\sum_{i=0}^{n} b_i \prod_{h=0}^{m-2} (m+k-i-1-h) p^{k-i} \right]. \quad \text{(A10)} \end{aligned}$$

In practice (from the computational point of view) it is more convenient to get the unit pulse response function from equation (2) than from equation (A10).

Appendix B: Discussion Note for Table 1

[24] It is important to use the pole value directly in equation (11) because a tiny error in a_i can have a large influence on the pole positions. Take the example at Gt251192: For a root of $p = 0.8562$ we have this polynomial (with MATLAB code: ≫r = [0.8562, 0.8562, 0.8562]; A = poly(r)),

$$[1.00000000000000 \quad -2.56860000000000 \; 2.19923532000000 \; -0.62766176032800].$$

The roots of this polynomial are (with MATLAB ≫ roots(A)),

$$\begin{aligned} p = [&0.85620332041198 + 0.00000575124534i, \\ &0.85620332041198 - 0.00000575124534i, \\ &0.85619335917604]. \end{aligned}$$

[25] There are visible deviations (albeit very small) from these numerical calculations (see the imaginary parts) in spite of the fact that MATLAB uses double precision in its computation. In this case the three poles are close to the original pole value of 0.8562.

[26] If four significant digits in a_i parameters are used (with MATLAB code: ≫A = 1.000, −2.569, 2.199, −0.6277]; p = roots(A))

$$\begin{aligned} p = [&0.94119713970686, 0.81390143014657 \\ &+ 0.06694076404419i, 0.81390143014657 \\ &- 0.06694076404419i]. \end{aligned}$$

It can be seen that the model's poles have moved away from the actual pole (0.8562) by a large discrepancy (0.9412, 0.8139 + 0.067i, 0.8139 − 0.067i). This has a serious implication in using transfer function models in hydrological modeling since a hydrologist would naturally think there is no harm in rounding the model parameter 2.5686 to −2.569 (a mere 0.016% change) and such a tiny rounding would significantly relocate the model's poles. This phenomenon is much more serious if a modeler would try to update TF's a_i parameters directly in real time, and as a result one of the poles could be pushed outside of the unit circle. This may explain the reason why the real-time updating of TF parameters in hydrology so far has not been successful in operational usages, even with the powerful updating tools such as the Kalman filter.

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