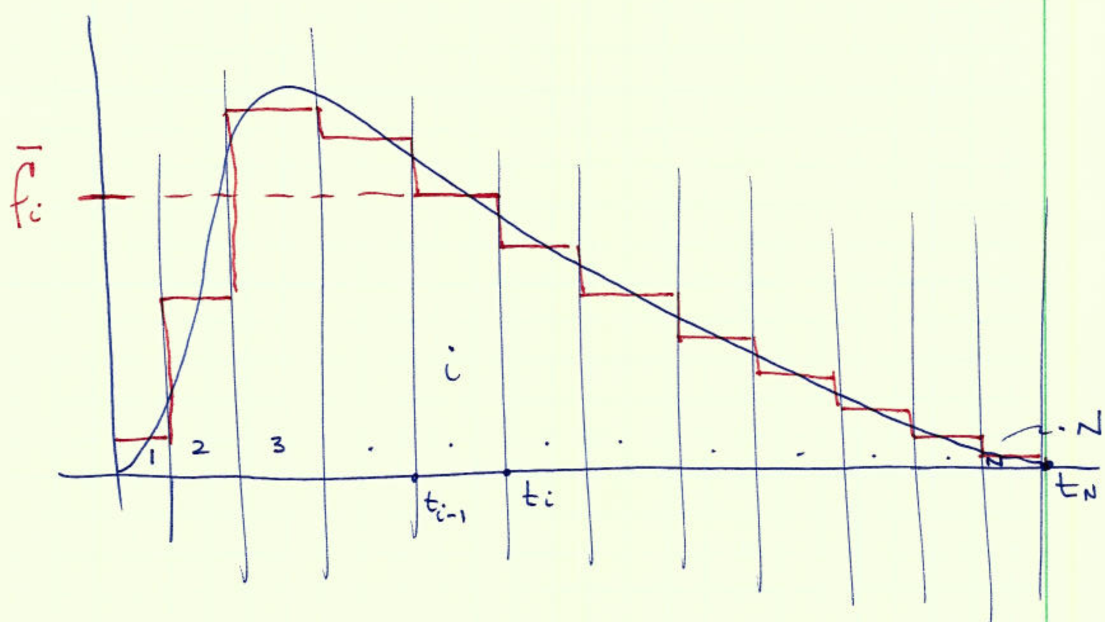
Empirical Moments

$$M_2(f'(t)) = \frac{\int_0^{\infty} t^2 f'(t)\,dt}{\int_0^{\infty} f'(t)\,dt} = \frac{I_{n_{2}}}{I_d}$$

$$I_{n_{2}} = \int_0^{\infty} t^2 f'(t)\,dt = \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} t^2 f'(t)\,dt = \sum_{i=1}^{N} \bar{f}_i \int_{t_{i-1}}^{t_i} t^2\,dt$$



$$I_{n_{2}} = \sum_{i=1}^{N} \bar{f}_i \left.\frac{t_3}{3}\right|_{t_{i-1}}^{t_i} = \sum_{i=1}^{N} \bar{f}_i \left[\frac{t_i^3}{3} - \frac{t_{i-1}^3}{3}\right]$$

But $t_i = t_{i-1} + \Delta t \longrightarrow t_i^3 = t_{i-1}^3 + 3t_{i-1}^2\,\Delta t + 3t_{i-1}(\Delta t)^2 + (\Delta t)^3$

and

$$I_{n_{2}} = \sum_{i=1}^{N} \bar{f}_i \left[\frac{t_{i-1}^3}{3} + t_{i-1}^2\,\Delta t + t_{i-1}(\Delta t)^2 + \frac{(\Delta t)^3}{3} - \frac{t_{i-1}^3}{3}\right]$$

$$I_{n_{②}} = \Delta t \sum_{i=1}^{N} \bar{f}_i \left[t_{i-1}^2 + t_{i-1}(\Delta t) + \frac{(\Delta t)^2}{3} \right]$$

Adding & subtracting $\frac{(\Delta t)^2}{4}$ to the sum in brackets in order to complete the square of $\left(t_{i-1} + \frac{\Delta t}{2}\right)$:

$$I_{n_{②}} = \Delta t \sum_{i=1}^{N} \bar{f}_i \left[\left(t_{i-1} + \frac{\Delta t}{2}\right)^2 + \frac{(\Delta t)^2}{12} \right]$$

As you can see $t_{i-1} + \frac{\Delta t}{2}$ is the mid-point of the i^{th} interval. Let $\bar{t}_i = t_{i-1} + \frac{\Delta t}{2}$; then

$$I_{n_{②}} = \Delta t \sum_{i=1}^{N} \bar{f}_i \left[\bar{t}_i^{\,2} + \frac{(\Delta t)^2}{12} \right]$$

Proceeding similarly for I_d:

$$I_d = \int_0^\infty f(t)\,dt = \Delta t \sum_{i=1}^{N} \bar{f}_i$$

For the case of $M_1(f(t)) = \frac{\int_0^\infty t f(t)\,dt}{\int_0^\infty f(t)\,dt} = \frac{I_{n_①}}{I_d}$

Proceeding similarly one obtains:

$$I_{n_①} = \Delta t \sum_{i=1}^{n} \bar{f}_i \, \bar{t}_i$$

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