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A SIMPLE PRECIPITATION MODEL[1]

The precipitation model presented in class is based on the so-called adiabatic parcel dynamics. Moist air parcels ascend following an adiabatic process to the level of condensation, and a pseudo-adiabatic process above the level of condensation. The products of the condensation process, liquid water droplets, contribute to the formation of the cloud system. Eventually, if these droplets grow sufficiently so that their terminal velocities exceed the updraft velocity of the air, they will fall out of the cloud and potentially make it to the surface as precipitation.

The mass of liquid water content of the cloud system evolves in time according to the following continuity equation,

$$\frac{dX(t)}{dt} = I(t) - O_t(t) - O_b(t) \tag{1}$$

where $X(t)$ is the mass of liquid water in the unit cloud column, $I(t)$ is the input flux of liquid water as a result of condensation following an ascent as described earlier, $O_t(t)$ and $O_b(t)$ are the output fluxes of liquid water droplets through the cloud top and cloud bottom respectively.

The continuity equation above can be solved in finite difference form as,

$$X(t+\Delta t) = X(t) + [I(t) - O_t(t) - O_b(t)]\Delta t \tag{2}$$

Expressions to compute all of the terms in this equation are given below. The problem statement provides all of the necessary initial values so that this equation can be propagated in time.

For the first time step, compute $X(t=0)$, $I(t=0)$, $O_t(t=0)$, $O_b(t=0)$ using equation 3, 4, 20, and 21 below and using the initial values given in the problem statement. For the second and later time steps, obtain the value of the inverse mean diameter at the corresponding time step by using equation 18 below, and then proceed to evaluate the net fluxes as indicated.

MASS OF LIQUID WATER IN THE CLOUD SYSTEM

[1] For further details or a more complex version of this model, the reader is referred to Georgakakos and Bras, 1984a, 1984b; and Bras, 1990.

The mass of liquid water per unit area in a cloud column can be computed using the following equation,

$$X(t) = (\pi\rho_w N_o Z_c \delta / c_l{}^4)$$
$$\delta = (1/\gamma + 1/\gamma^2 + 1/\gamma^3)/3 \qquad (3)$$

where N_o is the parameter of the Marshall-Palmer distribution of droplet sizes and c_l is the inverse of the mean of the diameters of the droplets at the cloud bottom; Z_c is the cloud depth; and γ is the ratio of the mean cloud diameter at the cloud bottom to that at the top (typically > 1). The Marshall-Palmer distribution of cloud droplet sizes is

$$\eta(D) = N_o e^{-cD}$$

where $\eta(D)d$D is the number of cloud droplets with diameters in the range *[D, D + dD]* per unit volume of cloud.

INPUT MASS FLUX:

In order to evaluate the input of liquid water mass to the cloud system, we need to evaluate the following equation.

$$I = -\rho_m v \Delta w = \rho_m v [w_o - w_s(T_t, p_t)] \qquad (4)$$

For this, we need to obtain the moist air density. In addition, we need to compute the net change in water vapor mixing ratio for a parcel undergoing an adiabatic-pseudo-adiabatic ascent from the surface to the cloud top. The steps necessary to do so are outlined below.

BELOW LFC:

For moist air ascending adiabatically between the surface and the lifting condensation level (LCL), the potential temperature, θ, and water vapor missing ratio, w_o, are conserved quantities. Thus, at the LCL:

$$\theta = T_s\left(\frac{100000}{p_s}\right)^{R/C_p} = \text{constant} \qquad (5)$$

$$w_o = \frac{\varepsilon e_s(T_s)}{p_s} = \text{constant} \qquad (6)$$

$$\theta_e = \theta \exp\left[\frac{L_v w_s(T_s, p_s)}{C_p T_s}\right] \qquad (7)$$

In addition, for an adiabatic ascent, the temperature, T_s, and pressure, p_s, of the air parcel at the LCL are related through the following equation:

$$p_s = 100000\left(\frac{T_s}{\theta}\right)^{C_p/R} \tag{8}$$

Putting equations 5-8 together leads to,

$$\frac{C_p T_s}{L_v}\ln\left(\frac{\theta_e}{\theta}\right) = \frac{\varepsilon\theta^{C_p/R} e_s(T_s)}{100000 T_s^{C_p/R}} \tag{9}$$

where θ and θ_e are known values (given in the problem statement) and constant below LCL. Equation 9 must be solved iteratively for the parcel temperature at LCL, T_s. After obtaining T_s, the pressure at the LCL, p_s, can be obtained using equation 8; the initial mixing ratio, w_o, can be obtained from equation 7. Finally, the altitude of the cloud base can be obtained as,

$$Z_b = \frac{T_o - (T_{amb})_s}{\Gamma_{amb}} \tag{10}$$

where $(T_{amb})_s$ is the ambient prewssure at the LCL, which can be computed evaluating equation 16 below at the LCL pressure.

ABOVE LCL:

For a pseudo-adiabatic ascent above the LCL, the equivalent potential temperature is conserved, θ_e, whereas the potential temperature, θ, is not. Thus, at the cloud top,

$$\theta_e = \theta\exp\left[\frac{L_v w_s(T_t, p_t)}{C_p T_t}\right] = T_t\left(\frac{100000}{p_t}\right)^{R/C_p}\exp\left[\frac{L_v w_s(T_t, p_t)}{C_p T_t}\right] \tag{11}$$

This equation is a function of the temperature, T_t, and the pressure, p_t, of the parcel at the cloud top. However, the pressure of the cloud top is given in the problem statement. Thus, equation 11 is a function of only one unknown, namely, T_t, which can be obtained using an iterative approach.

Once T_t is obtained, the final water vapor mixing ratio at the cloud top can be obtained from,

$$w_s(T_t, p_t) = \frac{\varepsilon e_s(T_t)}{p_t} \tag{12}$$

Finally,

$$\Delta w = -[\, w_o - w_s(T_t, p_t)\,] \tag{13}$$

For purposes of this homework, use the density of the moist air at the surface and the updraft velocity given in the problem statement in the equation for *I* above.

$$\rho_m = \frac{p}{R_d T}(1 - 0.378\frac{e}{p}) \tag{14}$$

The ambient air temperature at the cloud top, $(T_{amb})_t$, can be obtained using the result obtained in Homework No.1 for the distribution of pressure in an atmosphere in which the temperature decreases linearly with elevation,

$$p(z) = p(z = z_o)\left(\frac{T(z)}{T(z_o)}\right)^{g/R_d \Gamma_{amb}} \tag{15}$$

Evaluating equation 15 at the cloud top for which $p(z) = p_t$ and $p(z_o) = p_o$ and solving for the ambient air temperature at the cloud top, $T(z) = (T_{amb})_t$,

$$(T_{amb})_t = T_o\left(\frac{p_t}{p_o}\right)^{R_d \Gamma_{amb}/g} \tag{16}$$

which can then be used to obtain the altitude of the cloud top,

$$Z_t = \frac{T_o - (T_{amb})_t}{\Gamma_{amb}} \tag{17}$$

OUTPUT MASS FLUX:

The output mass flux is composed of two contributions, the output of cloud droplets from the top of the cloud, O_t, and the output of cloud droplets from the bottom of the cloud, O_b. Both of these fluxes are a function of the inverse of the mean of the diameters of the population of cloud droplets at the cloud top and cloud bottom, respectively. The inverse of the mean diameter can be obtained from the following equation:

$$c_l = \left[\frac{\pi \rho_w N_o Z_c \delta}{X}\right]^{1/4} \tag{18}$$

where X is the total mass of liquid water contained in the cloud column and Z_c is the cloud depth,

$$Z_c = Z_t - Z_b \tag{19}$$

OUTPUT FROM CLOUD TOP:

Compute the output from the cloud top using the following equation,

$$O_t = \frac{\pi \rho_w N_o \alpha}{(c_l \gamma)^5} \left(\frac{1}{6} [\Gamma(5, \gamma \frac{c_l v}{\alpha}) - \gamma \frac{c_l v}{\alpha} \Gamma(4, \gamma \frac{c_l v}{\alpha})] + \gamma \frac{c_l v}{\alpha} - 4 \right) \tag{20}$$

OUTPUT FROM CLOUD BOTTOM:

Compute the output from the cloud bottom using the following equation,

$$O_b = \frac{\pi \rho_w N_o}{6 c_l^{\,4}} [\frac{\alpha}{c_l} \Gamma(5, \frac{c_l v}{\alpha}) - v \Gamma(4, \frac{c_l v}{\alpha})] \tag{21}$$

where α is the parameter of the terminal velocity function, and v is the updraft velocity. In these equations, everything is known at the initial time as given in the problem statement. For other time steps, the inverse of the cloud diameter changes as a result of changes in the mass of liquid water contained in the cloud column. Thus, the inverse of the mean diameter should be computed at the beginning of each time step using equation 18 above, in which X is evaluated at the beginning of the interval.

In equations 20 and 21, $\Gamma(,)$ stands for the Complementary Gamma function, as defined in the appendix.

PRECIPITATION RATE:

Cloud droplets falling out from the cloud bottom may eventually reach the ground and contribute to the precipitation rate. However, for unsaturated sub-cloud layer conditions, the mass of the droplet is reduced by evaporation as it falls to the ground. Taking into account this process, the precipitation rate at the surface can be computed using the following equation,

$$P = \frac{\pi \rho_w N_o}{6 c_l^{\,4}} [\frac{\alpha}{c_l} \Gamma(5, c_l D_l) - v \Gamma(4, c_l D_l) - D_c^3 \alpha c_l^{\,2} \Gamma(2, c_l D_l) + D_c^3 v c_l^{\,3} e^{-c_l D_l}] \tag{22}$$

where D_l is a threshold diameter defining the distribution of droplets at the cloud base that actually contribute to the precipitation rate at the surface. Droplets with smaller diameters will either not have terminal velocities that exceed the updraft velocity at the cloud base, or evaporate completely before reaching the surface.

$$D_l = \max(D_c, v / \alpha) \tag{23}$$

In the above expression, D_c represents the critical diameter at cloud base. Cloud droplets of this diameter will completely evaporate exactly upon reaching the ground surface. Thus, only droplets whose diameter at cloud base is larger than D_c will contribute to the precipitation rate. For steady state Fickian diffusion, the critical diameter can be obtained as,

$$D_c = [\frac{4D^*}{c^* R_v} Z_b (\frac{e_s(T_w)}{T_w} - \frac{e_s(T_d)}{T_o})]^{1/3} \tag{24}$$

where c^* is a constant with units of $kg/m^3/s$ which takes on the following values,

$$c^* = \left\langle \begin{array}{l} 7.\,\mathrm{x}\,10^5 \ \ \text{rain} \\ 1.4\,\mathrm{x}\,10^5 \ \ \text{snow} \end{array} \right. \tag{25}$$

and D^* is the diffusivity of water vapor in air in m^2/s, which is a function of temperature and pressure and can be computed as,

$$D^* = 2.11 * 10^{-5} (\frac{T_o}{273.15})^{1.94} (\frac{101325}{p_o}) \tag{26}$$

In equation 24, the absolute humidity at the surface of the droplet is that corresponding to the wet-bulb temperature, T_w, at the surface conditions, and the ambient absolute humidity is that corresponding to the dew-point temperature, T_d, at the surface conditions.

ENERGY GENERATED PER UNIT MASS

The energy generated per unit mass as a result of the work done by the positive buoyancy force acting on the parcel between the level of free convection (LFC) and the cloud top can be obtained using the following numerical approximation (trapezoidal rule) to the energy integral,

$$Energy / Mass = \int_{LFC}^{CloudTop} g(\frac{T_p(z)}{T_a(z)} - 1) dz \cong g \sum_{i=1}^{n} \Delta z_i \overline{\left(\frac{T_p}{T_a} - 1 \right)}_i \tag{27}$$

where,

$$\overline{\left(\frac{T_p}{T_a} - 1 \right)}_i = \frac{1}{2} (\frac{T_p(z_{i+1}) - T_a(z_{i+1})}{T_a(z_{i+1})} + \frac{T_p(z_i) - T_a(z_i)}{T_a(z_i)})$$

and,

$$\Delta z_i = z_{i+1} - z_i$$

APPENDIX

The incomplete gamma function required in the above computations is defined as,

$$\Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t} dt$$

For integer values of *a* (which is the case for this homework), the incomplete gamma function can be evaluated using the following expression,

$$\Gamma(a,x) = (a-1)! e^{-x} \sum_{k=1}^{a} \frac{x^{k-1}}{(k-1)!} = (a-1)! e^{-x} (1 + x + \frac{x^2}{2!} + ... + \frac{x^{a-1}}{(a-1)!})$$

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