Concentration of very fine silts in a steady vortex

Concentration de limons très fins dans un vortex en régime permanent

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SUMMARY
The presence of fine sediment particles in a steady horizontal vortex is examined theoretically and experimentally. Starting from an homogeneous mixture, particles denser than the surrounding fluid are moved outside of the vortex core by centrifugal action. As a sediment concentration gradient gradually builds up across the vortex, a diffusive flux opposite to the centrifugal flux is induced. Equilibrium is reached when the two fluxes are equal. Theoretical relationships describing the suspended sediment concentration profiles in a steady Rankine vortex are verified with experimental data.

RÉSUMÉ
La présence de fines particules solides dans un vortex plan de Rankine est étudiée analytiquement et expérimentalement. Partant d'un mélange homogène, les particules plus denses que le fluide sont projetées à l'extérieur du noyau central par centrifugation. Ceci crée un gradient de concentration dans le vortex et induit un flux diffusif opposé au flux centrifuge. L'équilibre est obtenu lorsque les flux centrifuges et diffusifs sont égaux et opposés. Les équations analytiques décrivant les profils d'équilibre de concentration de sédiments en suspension sont conformes aux mesures expérimentales.

1 Introduction

A Rankine combined vortex is a simple two-dimensional model describing vortices in homogeneous fluids. It combines a rotational vortex core surrounded by an irrotational vortex. The fundamental flow properties in a Rankine combined vortex have been discussed in great detail in the literature: Lamb (1932), Prandtl and Tietjens (1934), Rouse and Hsu (1952), Sedov (1959), Rouse (1966), Batchelor (1967), Fortier (1967), Happel and Brenner (1973) and Daily and Harleman (1973).

As opposed to flow in homogeneous fluids, the flow characteristics of a water-sediment mixture are obscured by the presence of solid particles. It is intuitively recognized that particles in a vortex tend to separate from the fluid by centrifugal action. Full understanding of this problem is as yet incomplete considering the interaction of viscous, gravitational and inertia forces and the unsteady nature of eddies in turbulent flows.

In this paper a steady two-dimensional horizontal vortex is considered. The fluid mixture under investigation is composed of very fine cohesionless sediment particles in distilled water. With very small particles, the analysis is simplified since the viscous forces exerted on the particles are dominant compared to their weight and inertia.

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The forces exerted on the particles in circular motion are first examined in order to define the acceleration components and the limit velocity of small particles in the vortex. Diffusion concepts are then introduced to determine the steady-state sediment concentration profiles.

2 Velocity and pressure distribution in a Rankine combined vortex

The mathematical concept of vorticity is known as a vector quantity ω having three orthogonal components, each of which is expressible in terms of transverse gradients of the velocity vector. The vorticity ω is equal to twice the angular velocity Ω of a fluid element and the product of vorticity by the cross-sectional area defines the circulation Γ which is constant along a vortex filament. As schematized in Fig. 1 the Rankine combined vortex is composed of a forced vortex core of finite radius r_0 and constant vorticity ω surrounded by a free vortex of constant circulation $\Gamma = 2\pi\Omega r_0^2$. In the following analysis, the two-dimensional vortex rotates around a vertical axis for which the fluid motion in a horizontal plane is considered gravity-free.

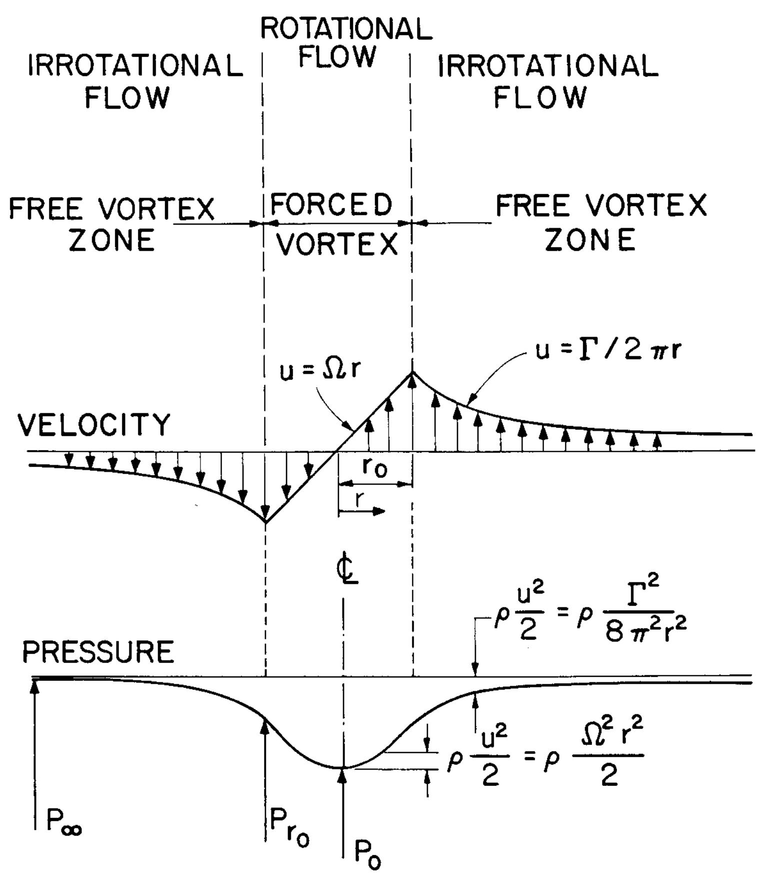

Fig. 1. Velocity and pressure distribution in a two-dimensional Rankine combined vortex.
Répartition de la vitesse et de la pression dans un vortex de Rankine.

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2.1 *Free vortex zone*

In the free vortex zone the flow is irrotational with constant circulation. The velocity varies inversely with the radius to satisfy the requirement of constant circulation and the boundary condition of no motion at infinity:

$$u=\frac{\Gamma}{2\pi r} \tag{1}$$

in which u is the tangential velocity at a distance r from the center of the vortex. The Bernoulli theorem for a steady irrotational flow of an inviscid incompressible fluid is:

$$p+\varrho\frac{u^2}{2}=\text{constant} \tag{2}$$

in which ϱ is the mass density of fluid and p is the pressure. The pressure distribution in a free vortex is obtained as follows from equations (1) and (2) after taking the constant equal to the pressure p_∞ at $r=\infty$:

$$p_\infty-p=\varrho\frac{u^2}{2}=\varrho\frac{\Gamma^2}{8\pi^2r^2} \tag{3}$$

The Bernoulli sum along different stream lines remains constant.

2.2 *Forced vortex core*

In a homogeneous fluid, the forced vortex flow is rotational without any viscous energy dissipation though the Bernoulli sum between different streamlines is not constant. The velocity varies linearly with the radius ($u=\Omega r$) and the pressure gradient is balanced by the centrifugal acceleration:

$$\frac{1}{\varrho}\frac{\partial p}{\partial r}=\frac{u^2}{r} \tag{4}$$

The pressure distribution is:

$$p-p_0=\frac{\varrho u^2}{2}=\frac{\varrho\Omega^2r^2}{2} \tag{5}$$

in which p_0 is the pressure at the center of the vortex.

3 Particle motion in a vortex

The following analysis (Julien, 1985b) describes the motion of a small spherical particle of diameter d_s, specific mass ϱ_s and mass $\pi\varrho_s d_s^3/6$ located at distance r from the center of the vortex. Small particles refer to cohesionless particle sizes much smaller than the vortex core radius r_0. The symbol v denotes the velocity of the particle while u designates the fluid velocity. The forces exerted on sediment particles in a vortex can be defined from the velocity and pressure distribution described in Section 2. The relative magnitude of pressure and centrifugal forces defines the acceleration of solid particles in a forced and a free vortex. Viscous forces are exerted on particles in motion relative to the fluid and equilibrium conditions are reached when the sum of all forces reduces to zero.

The pressure force F_p exerted on a small particle toward the center of the vortex is:

$$F_p = \frac{\pi}{6} \varrho d_s^3 \frac{u^2}{r} \quad (6)$$

The centrifugal force F_c depends on the tangential velocity of a sediment particle v_t and the radius of curvature. Assuming that the radius of curvature is equal to the distance r from the center of the vortex, the centrifugal force outward from the center is equal to:

$$F_c = \frac{\pi}{6} \varrho_s d_s^3 \frac{v_t^2}{r} \quad (7)$$

It is considered that for small particles, the velocity of the particle is very close to the velocity of the fluid. The Coriolis acceleration is thus considered negligible compared to the centrifugal acceleration. A radial acceleration a_r is imparted to a sediment particle provided the centrifugal and pressure forces are not equal. In this case, the motion of a sediment particle in a vortex is given when the viscous forces are included in the analysis. The velocity components of a particle in the radial v_r and tangential v_t directions are assumed to differ from the fluid velocity u as sketched in Fig. 2. The velocity of the particle relative to the fluid v' is equal to:

$$v' = \sqrt{v_r^2 + (v_t - u)^2} \quad (8)$$

and the angle $\psi = \tan^{-1} v_r/(v_t - u)$ between the two relative velocity components is indicated in Fig. 2.

The friction force exerted on the particle in the direction opposite to v' is function of the relative velocity v', the Reynolds number of the particle $Re' = v' d_s / \upsilon$, the surface area and the drag coefficient C_D:

$$F_V = C_D \frac{\pi d_s^2 \varrho v'^2}{8} \quad (9)$$

It is assumed that the drag coefficient C_D in the curvilinear vortex flow field is not significantly different from the C_D value in a parallel flow field. The friction forces exerted on small particles are predominantly viscous and the drag coefficient is inversely proportional to the Reynolds number:

$$C_D = \frac{24}{Re'} = \frac{24 \upsilon}{v' d_s} \quad (10)$$

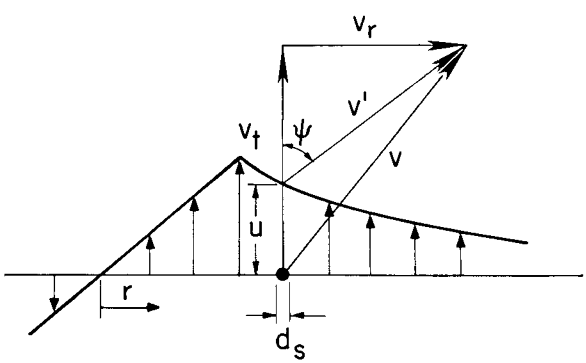


Fig. 2. Relative velocity of a sediment particle in a vortex.
Vitesse relative d'une particule solide dans un vortex.

The resulting radial acceleration of the particle is obtained after considering the equilibrium of pressure, viscous ($F_v \sin \psi$) and centrifugal forces divided by the mass of the particle:

$$a_r = \frac{v_t^2}{r} - \frac{\varrho}{\varrho_s}\frac{u^2}{r} - 18\,\frac{\varrho \upsilon}{\varrho_s}\frac{v_r}{d_s^2} \tag{11}$$

The tangential acceleration a_t equals the tangential viscous force, $F_v \cos \psi$, per unit mass of the particle.

$$a_t = -18\,\frac{\varrho \upsilon}{\varrho_s}\frac{(v_t - u)}{d_s^2} \tag{12}$$

Equation (12) indicates that as the ratio υ/d_s^2 becomes very large the tangential acceleration term prevails until the tangential velocity of the particle v_t reaches the velocity of the fluid u. Therefore, the tangential velocity of small particles should always remain close to the fluid velocity. Equilibrium condition in the radial direction for $v_t = u$ occurs when the acceleration component a_r from equation (11) vanishes. The corresponding limit velocity v_e' of the particles in the radial direction is:

$$v_e' = \frac{1}{18}\left(\frac{\varrho_s}{\varrho} - 1\right)\frac{u^2}{r\upsilon}\,d_s^2 \tag{13}$$

Which can be rewritten in a dimensionless form as a function of the Reynolds number, $Re = ud_s/\upsilon$, as:

$$\frac{v_e'}{u} = \left[\frac{1}{18}\left(\frac{\varrho_s}{\varrho} - 1\right)\frac{d_s}{r}\right] Re \tag{14}$$

From these relationships, the limit radial velocity of the particle is proportional to ϱ_s/ϱ, d_s and u and decreases as the viscosity υ and the radius r increase. These equations also verify the previous assumptions regarding the circular motion of small sediment particles.

4 Sediment concentration profile in a vortex

In a viscous fluid, small sediment particles reach the fluid velocity very rapidly and particles heavier than the fluid are moved outside of the vortex core. Therefore the concentration of particles decreases toward the center, thus creating a concentration gradient across the vortex. A diffusion flux proportional to this gradient induces the transport of solid particles toward the regions of lower concentration. Equilibrium conditions are reached when the flux of sediment particles due to centrifugal force is balanced by the diffusion flux in the opposite direction. This equilibrium condition implies that the radius of curvature of the particles is r as assumed previously in the derivation of equation (7). Equilibrium condition can be mathematically described as follows:

$$\frac{\mathrm{d}C}{C} = \frac{v_e'}{\varepsilon}\,\mathrm{d}r \tag{15}$$

in which C is the sediment concentration by volume and ε is the diffusion coefficient. It is assumed that the diffusion coefficient ε remains independent of r regardless of whether laminar or turbulent diffusion is involved. After substituting equation (13) into equation (15), the concentration profile is described by the following integral:

$$\int \frac{dC}{C} = \frac{1}{18} \int \left(\frac{\varrho_s}{\varrho} - 1 \right) \frac{u^2}{r} \frac{d_s^2}{\upsilon \varepsilon} dr \tag{16}$$

With the boundary condition at infinity $C = C_\infty$, the integration of equation (16) yields two expressions for the sediment concentration profile since two velocity relationships are applicable to free and forced vortex.

These two concentration profiles are respectively:

$$\frac{C}{C_\infty} = e^{\alpha \left(\frac{r^2}{r_0^2} - 2 \right)} \quad \text{when} \quad r \leq r_0 \tag{17}$$

and

$$\frac{C}{C_\infty} = e^{-\alpha \frac{r_0^2}{r^2}} \quad \text{when} \quad r > r_0 \tag{18}$$

A dimensionless parameter α describing the distribution of sediments is defined as follows:

$$\alpha = \left(\frac{\varrho_s}{\varrho} - 1 \right) \frac{\Gamma^2}{144\pi^2 \varepsilon \upsilon} \left(\frac{d_s}{r_0} \right)^2 \tag{19}$$

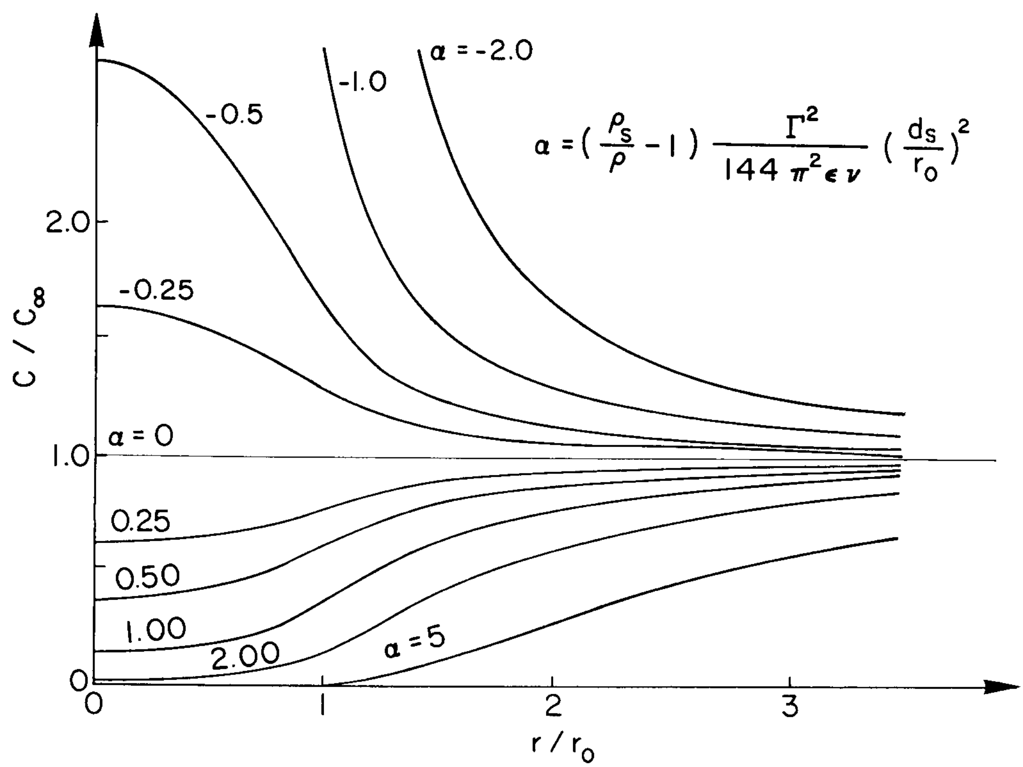


Fig. 3. Steady-state sediment concentration profile in a Rankine combined vortex.
Profil de concentration en régime permanent dans un vortex de Rankine.

The theoretical concentration profiles in a Rankine combined vortex with various α values are shown in Fig. 3 as a function of r/r_0. It is concluded from this analysis that the concentration is constant when $\alpha = 0$ which corresponds to infinitely small particles or neutrally buoyant particles $\varrho_s = \varrho$. The curves for $\alpha > 0$ indicate a decrease in concentration toward the center of the vortex while the concentration increases toward the center when $\varrho_s < \varrho$. The boundary condition at $r = r_0$ and at the center of the vortex are respectively:

$$C_{r_0} = C_\infty e^{-\alpha} \tag{20}$$

$$C_0 = C_\infty e^{-2\alpha} \tag{21}$$

Interestingly the ratio C_{r_0}/C_∞ equals the square root of C_0/C_∞.

5 Experimental investigation

Laboratory experiments were conducted at the Engineering Research Center to determine the sediment concentration profile in a steady vortex for validation of the theoretical profiles when $\alpha > 0$. A water sediment mixture composed of very fine silts (0.0053 mm $< d_s <$ 0.0074 mm) at a concentration of 50 g/l was used for the experiments. A steady vortex was induced in three liters of the mixture by rotating a 76 mm (3 in.) magnetic stirring bar at the bottom of a fixed cylindrical flask. The nearly horizontal motion of sediment particles, as observed through the glassed walls indicates that no significant secondary circulation was present in the vortex. Water surface profiles were measured using a point gage and several 30 ml samples were pipetted, dried and weighed (± 0.1 mg) to measure the sediment concentration profiles across the steady vortex. The data and their analysis are detailed in Julien (1985a). The circulation Γ, the angular velocity Ω and the velocity u_0 at the radius of the core r_0 are obtained from the surface profile data. The parameter u_0 is calculated from $\varrho u_0^2 = p_\infty - p_0$ and r_0 from $\Gamma = 2\pi r_0 u_0$. The coefficients α and C_∞

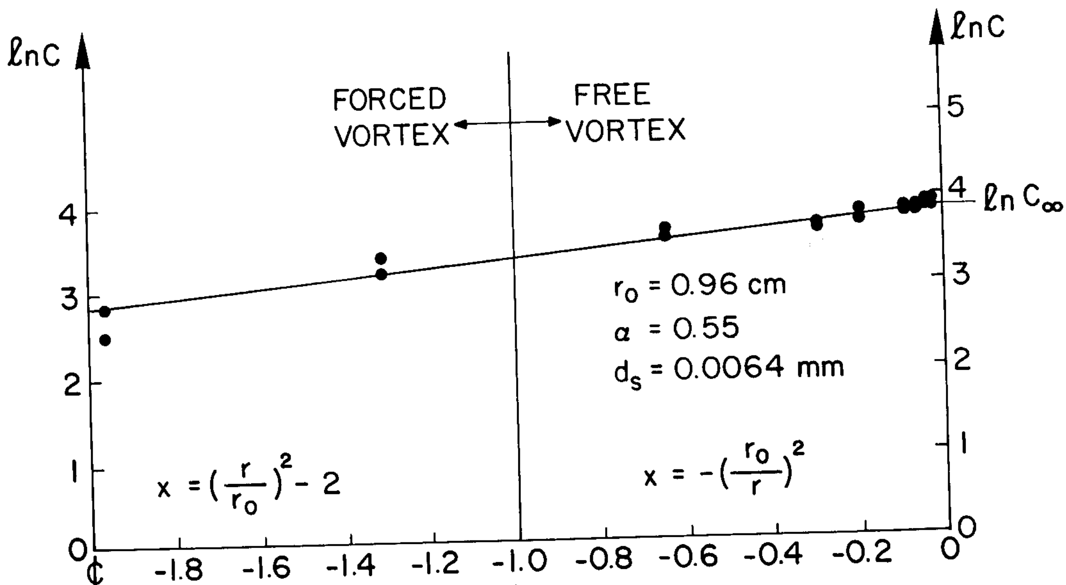


Fig. 4. Linearized sediment concentration profile.
Profil linéarisé de concentration en sédiments.

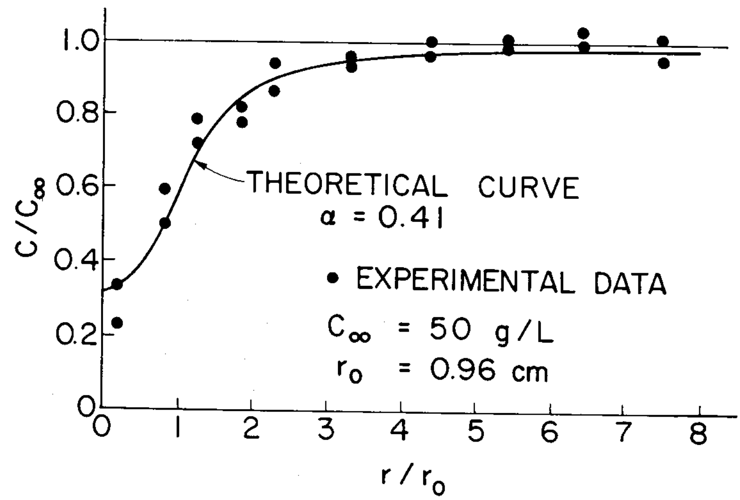
are then evaluated from the sediment concentration data after equations (17) and (18) are linearized as follows:

$$\ln C = \ln C_\infty + \alpha \left[\left(\frac{r}{r_0} \right)^2 - 2 \right] \quad \text{when} \quad r \leq r_0 \tag{22}$$

and

$$\ln C = \ln C_\infty - \alpha \left(\frac{r_0}{r} \right)^2 \quad \text{when} \quad r \geq r_0 \tag{23}$$

A linearized sediment concentration profile is presented in Fig. 4 for the evaluation of C_∞ and α. In Fig. 5, two sediment concentration profiles measured in this preliminary experimental study are compared with the theoretically derived relationships. In both cases, the agreement is excellent and it is concluded that when $\varrho_s > \varrho$, the sediment concentration decreases toward the



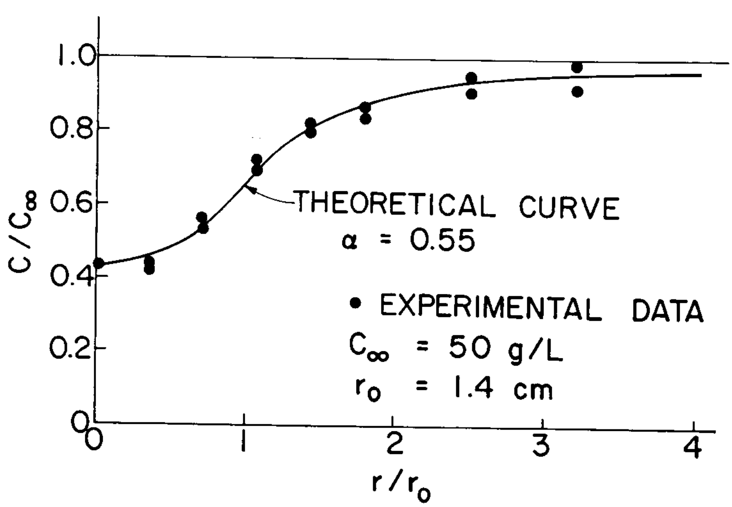


Fig. 5. Sediment concentration profile.
Profil de concentration en sédiments.

center of a vortex. The sediment concentration profile depends on three major factors: diffusion, friction and centrifugal force exerted on small particles. The diffusion coefficients for the two experiments shown in Fig. 5 are respectively 1.67 cm^2/s and 9.4 cm^2/s which corresponds very likely to turbulent diffusion. It is recognized that further experiments are required to better define the diffusion coefficient. At this point, however, it seems relevant to emphasize that the experimental data are in good agreement with the theoretical results for which the diffusion coefficient is assumed constant throughout the vortex.

6 Summary and conclusions

The presence of small sediment particles in a steady horizontal Rankine vortex is examined. For small particles, the viscous force is dominant compared to inertia and gravity forces. The similarities of motion in both the forced vortex and the free vortex are apparent. A particle at rest is accelerated toward the center. The tangential velocity of small particles reaches rapidly the velocity of the fluid and particles denser than the fluid move toward the outside of the vortex. The centrifugal flux created by the outward motion of denser particles induces a diffusion flux in the opposite direction. Equilibrium conditions are obtained when the two fluxes are equal and opposite. Under steady state conditions, the sediment concentration profiles of fine particles in suspension are theoretically derived and plotted using dimensionless scales. Preliminary experiments verified the theoretical profiles for $\alpha > 0$. Experimental evidence supports the assumption that the diffusion coefficient remains constant throughout the vortex.

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Notations

a_r	radial acceleration of a sediment particle
a_t	tangential acceleration of a sediment particle
a_{vr}	radial acceleration due to viscous forces
a_{vt}	tangential acceleration component due to viscous forces
C	sediment concentration
C_0	sediment concentration at $r = 0$
C_{r_0}	sediment concentration at $r = r_0$
C_∞	sediment concentration at $r = \infty$
C_D	drag coefficient
d_s	diameter of a sediment particle
F_c	centrifugal force on a sediment particle
F_p	pressure force on a sediment particle
F_v	viscous force on a sediment particle
p	pressure

p_0 pressure at the center of the vortex
p_{r_0} pressure at $r=r_0$
p_∞ pressure at $r=\infty$
r radial distance from the center of the vortex
r_0 radius of the vortex core
Re' Reynolds number using the relative velocity of the particle
Re Reynolds number of the particle using the fluid velocity
u tangential velocity of the fluid
v total velocity of a sediment particle
v_r radial velocity of a sediment particle
v_t tangential velocity of a sediment particle
v' velocity of the particle relative to the fluid velocity
v'_e limit radial velocity of the particle
α parameter governing the sediment concentration profile
Γ circulation
ε diffusion coefficient
υ kinematic viscosity of the fluid
ϱ mass density of the fluid
ϱ_s mass density of the sediment particles
ψ angle between the radial and tangential components of a particle velocity relative to the fluid
ω vorticity
Ω angular velocity of the forced vortex

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