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Stochastic Hydrology and Hydraulics

On predicting upland erosion losses from rainfall depth

Part 1: Probabilistic approach

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Abstract: Point rainfall triggers the complex processes of overland flow and surface erosion. The probability density functions of rainfall duration and intensity are coupled with a physically based dynamic formulation of rainfall-runoff-sediment transport relationships for upland areas. When considering a single storm, rainfall depth alone is a poor predictor of sediment transport because of the dispersion introduced by the effect of rainfall intensity. On a long terms basis, however, the total amount of rainfall can be used to predict total erosion losses.

Key words: Upland erosion, annual erosion losses

1 Introduction

It has long been recognized that rainfall induced runoff has the ability to detach and transport considerable amounts of sediments from upland areas. Raindrop impact on the earth surface loosens soil particles for downslope transport into the fluvial system. Soil erosion deteriorates soil structure and increase nutrient loss, which decreases the potential productivity of agricultural land. The amount of material transported is a function of the soil erodibility and the sediment transport capacity of surface runoff. Upland erosion losses depend on the size of sediment particles, infiltration rate, surface sealing, vegetation cover, resistance to flow and soil conservation practices.

Earlier investigations have been undertaken to quantify upland erosion losses. Musgrave's (1947) contribution has been cast into the well-known Universal Soil-Loss Equation (Wischmeier and Smith 1978). Mathematical models have been developed with increasing interest since the early sixties with contributions of Meyer and Wischmeier (1969), Foster and Meyer (1972), Simons et al. (1975), and Knisel (1980). In these computer models, the complex physical processes of precipitation over soil surface for a single storm are replaced by idealized deterministic configurations of physical elements that preserve the essential characteristics of a drainage area. The relationships derived from the realm of physics are used to simulate the response of watershed systems to the input variables.

When considering the variability of rainfall-runoff relationships, Eagleson (1978) showed that the probability distributions of input variables such as rainfall could be transformed into probability distributions of the output variables using deterministic physical processes. This approach has been extended to the analysis of soil erosion from upland areas by Julien and Frenette (1985). The probability density functions (pdf) of both rainfall duration and intensity serve as inputs to calculate the expected value of rainfall erosion losses from upland areas. This method has been applied to several large watersheds in Canada (Julien and Frenette 1985). This approach, however, requires detailed knowledge of both rainfall duration and intensity which limits its applicability to locations where sufficient meteorological data are available. The purpose of the present investigation is to combine the effects of both duration and intensity into a single variable, namely rainfall depth, and then examine the relationship between erosion losses and rainfall depth. Previous studies of the probability distribution of rainfall duration and intensity serve as a basis for this theoretical investigation. The relationship between erosion losses and rainfall depth is first examined for a single storm and then for a large number oi storms.

2 Problem Formulation

2.1 Rainfall precipitation

Point rainfall precipitation has been treated successfully as a random process by Todorovic (1968) and by Eagleson (1978). A dimensionless rainfall depth parameter ϕ_D has been defined after dividing the rainfall depth of a single storm by the average rainfall depth of a large number of storms. The exponential probability density function (pdf) of the dimensionless rainfall depth can be written:

$$f(\phi_D) = e^{-\phi_D}. \tag{1}$$

Similarly, Grayman and Eagleson (1969), Eagleson (1978) and Julien (1982) showed that both the rainfall duration and intensity can be represented by exponential pdf's that are independent of each other. After normalizing rainfall duration and intensity of a single event with average values, the pdf of the dimensionless rainfall duration ϕ_t and that of intensity ϕ_i can be written:

$$f(\phi_t) = e^{-\phi_t}, \quad \text{and} \quad f(\phi_i) = e^{-\phi_i}. \tag{2a,b}$$

The rainfall depth being equal to the product of storm duration and intensity, the following relation also holds true in dimensionless form:

$$\phi_D = \phi_t \phi_i. \tag{3}$$

As the rainfall intensity exceeds the rate of infiltration in the soil, the excess rainfall intensity generates surface runoff.

2.2 Soil erosion losses

The principal variables involved in the unit sediment transport q_s are the surface slope S, the rainfall intensity i, the runoff length L, the mass density ρ, and the kinematic viscosity ν of the fluid. The method of dimensional analysis reduces the number of parameters. The dimensionless sediment transport rate per unit width ϕ_{q_s} can be written (Julien 1982) as:

$$\phi_{q_s} = \bar{\alpha}\left(\frac{L\bar{i}}{\nu}\right)^{\gamma} S^{\beta} \phi_i^{\gamma} \tag{4}$$

in which $\phi_{q_s} = q_s/\nu$; $\bar{i}$ is the long term average rainfall intensity; and the empirical parameters $\bar{\alpha}$, β, and γ have been determined by Julien and Simons (1985) for: (1) several empirical equations derived from field data; (2) theoretical relationships; and (3) several bed-load formulas describing sediment transport in laboratory flumes and streams. The analysis showed that the exponent β varied between 1.2

and 1.9 whereas the exponent of the sediment rating curve, γ, typically ranged from 1.4 to 2.4 for upland erosion losses.

A single storm of given duration and uniform rainfall intensity, over an impervious rectangular plane of width W, and runoff length L yields the following dimensionless soil loss, ϕ_s, after using Eqs. (3) and (4):

$$\phi_s = \phi_W \phi_{q_s} \phi_t = \pi \phi_D \phi_i^{\gamma - 1} \tag{5}$$

where $\pi = \bar{\alpha} S^{\beta} (\frac{L\bar{i}}{\nu})^{\gamma} \frac{W}{L}$ and $\phi_W = \frac{W}{L}$.

It is noted that Eq. (5) neglects the effects of the rising and falling limbs of the runoff hydrographs. Julien and Frenette (1985) demonstrated that this is a sufficiently good first-order approximation for field applications.

For a given field site the dimensionless group of constants π remains invariant, thus the dimensionless soil loss ϕ_s is proportional to the dimensionless amount of rainfall ϕ_D and the dimensionless rainfall intensity ϕ_i whenever $\gamma > 1$. The issue to be clarified relates to the influence of ϕ_i on ϕ_s in Eq. (5). If ϕ_s could be demonstrated to be insensitive to changes in ϕ_i, then Eq. (5) would indicate a linear relationship between ϕ_s and ϕ_D.

3 Soil erosion losses during single storms given the rainfall depth

The influence of rainfall intersity ϕ_i on soil loss ϕ_s given a constant rainfall depth ϕ_D is examined through the concept of conditional probability schematized in Fig. 1. The expected value, $E(\phi_s \mid \phi_D)$, and the variability of ϕ_s around the expected value are calculated after the joint probability density function is defined. Considering the independence of rainfall duration and rainfall intensity, the joint probability density function equals the product of the individual exponential distributions from Eqs. (2a) and (2b). The conditional probability density function (cpdf) of rainfall intensity ϕ_i for a given rainfall depth (ϕ_D constant) is obtained from $\phi_t = \phi_D / \phi_i$. Thus the cpdf is:

$$f(\phi_i \mid \phi_D) = \frac{e^{-\phi_i} e^{-\phi_D/\phi_i}}{\int_0^{\infty} e^{-\phi_i} e^{-\phi_D/\phi_i} \, d\phi_i} . \tag{6}$$

The denominator in Eq. (6) is a constant normalizing the distribution. The result of integrating Eq. (6) is the conditional cumulative distribution function (ccdf) denoted by $F(\phi_i \mid \phi_D)$ and written as:

$$F(\phi_i \mid \phi_D) = \int_0^{\phi_i} f(\phi_i \mid \phi_D) \, d\phi_i . \tag{7}$$

Interestingly, the function $F(\phi_i \mid \phi_D)$ also serves to describe the ccdf of ϕ_s because from Eq. (5), with π and ϕ_D constant, ϕ_i is directly related to ϕ_s as

$$\phi_i = [\frac{\phi_s}{\pi \phi_D}]^{\frac{1}{\gamma - 1}} . \tag{8}$$

Figure 2 shows the ccdf of soil erosion for different values of ϕ_D ranging from 0.1 to 10 as calculated from numerical integration of Eq. (7). It is seen from this figure that for a given ϕ_i, $F(\phi_i \mid \phi_D)$ decreases with increasing ϕ_D.

The first two moments of the ccdf are calculated to yield insight on the variability of ϕ_s around the expected value of ϕ_s as a function of γ.

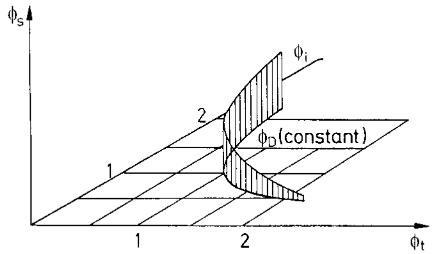

Figure 1

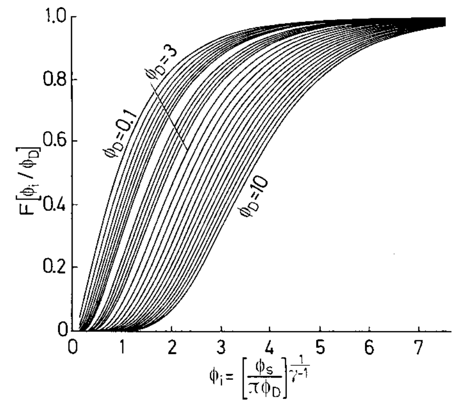

Figure 2

Figure 1. Soil loss ϕ_s given constant values of ϕ_D

Figure 2. Conditional cumulative distribution function of ϕ_i given ϕ_D constant

3.1 Expected value of soil erosion

The first moment of the distribution corresponds to the expected value $E(\phi_s \mid \phi_D)$ of the soil loss for one rainstorm event, given the rainfall depth ϕ_D. This is expressed mathematically by the following integral:

$$E(\phi_s \mid \phi_D) = \int_0^\infty \phi_s f(\phi_i \mid \phi_D)\, d\phi_i \,. \tag{9}$$

From Eq. (5), ϕ_s can be replaced by a function of ϕ_i, thus:

$$E(\phi_s \mid \phi_D) = \frac{\pi\phi_D \int_0^\infty \phi_i{}^{\gamma-1} e^{-\phi_i} e^{-\phi_D/\phi_i}\, d\phi_i}{\int_0^\infty e^{-\phi_i} e^{-\phi_D/\phi_i}\, d\phi_i} \tag{10}$$

in which the denominator and ϕ_D are constant. Both integrals in Eq. (10) can be expressed in terms of modified Bessel functions, $K_\gamma(2\sqrt{\phi_D})$, of order γ defined in Gradshteyn and Ryzhik (1965):

$$K_\gamma(2\sqrt{\phi_D}) = \frac{1}{2(\phi_D)^{\gamma/2}} \int_0^\infty \phi_i{}^{\gamma-1} e^{-\phi_i} e^{-\phi_D/\phi_i}\, d\phi_i \,. \tag{11}$$

After substituting Bessel functions of order γ (numerator) and of order 1 (denominator), the expected value of soil erosion losses from Eq. (10) can be written as:

$$\frac{E(\phi_s \mid \phi_D)}{\pi\phi_D} = \phi_D{}^{\frac{(\gamma-1)}{2}} \frac{K_\gamma(2\sqrt{\phi_D})}{K_1(2\sqrt{\phi_D})} \,. \tag{12}$$

Solution of the Modified Bessel functions is possible with the aid of mathematical tables (Abramowitz and Stegun 1972) for integer values of γ. Asymptotic series expansions or numerical integrations of Eq. (12) can also be used for non-integer values of γ. Dawod (1986) verified numerical solutions with mathematical tables for integer values of γ. The numerical method was then used for non-integer values of γ. The results for the expected value of soil losses conditional to a given ϕ_D are shown in Fig. 3 as a function of γ ($1.5<\gamma<3.5$) for ϕ_D held fixed between 0.1 and 10. The expected value $E(\phi_s \mid \phi_D)$ is found to increase both with ϕ_D and γ.

3.2 Second moment and coefficient of variation

The second central moment M_2 or variance has been determined from:

$$M_2(\phi_D) = \int_0^{\infty} [\phi_s - E(\phi_s \mid \phi_D)]^2 f(\phi_i \mid \phi_D)\, d\phi_i \tag{13}$$

with ϕ_s given from Eq. (5). It is noted that $M_2(\phi_D)$ is a random variable. This integral has been solved numerically (Dawod 1986) and is shown to increase with γ and ϕ_D. The results are presented in the form of a coefficient of variation.

$$\eta(\phi_D) = \sqrt{M_2(\phi_D)}/E(\phi_s \mid \phi_D)\,. \tag{14}$$

The coefficient of variation is shown in Fig. 4 to decrease with increasing ϕ_D. For example, the standard deviation reaches twice the expected value of soil losses when $\gamma = 3$ and $\phi_D = 0.1$. This indicates that for a single storm, if calculations of soil erosion are to be based on rainfall depth only, the influence of rainfall intensity cannot be neglected because the variability introduced by the rainfall intensity is excessive. The results shown in Fig. 4 reveal that the influence of rainfall intensity decreases as ϕ_D increases and as γ decreases.

4 Expected soil erosion losses from single storms of unknown rainfall depth

When the amount of rainfall from a given storm is unknown, the expected value of soil erosion $E(\phi_s)$ can still be determined after considering the pdf of rainfall depth (Eq. (1)), thus:

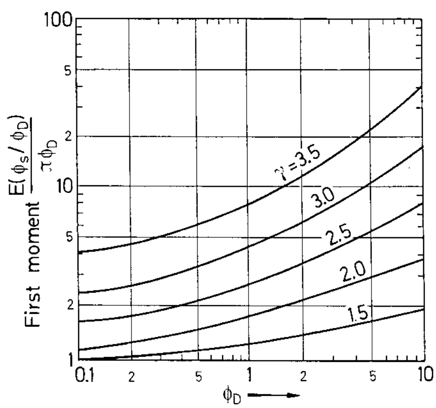

Figure 3

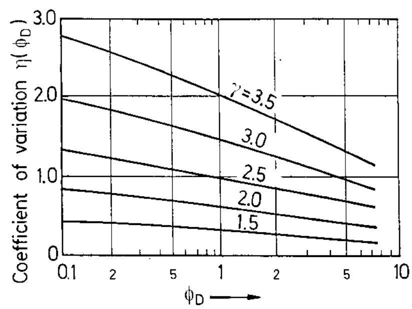

Figure 4

Figure 3. Expected value of soil loss ϕ_s as a function of ϕ_D and γ

Figure 4. Coefficient of variation $\eta(\phi_D)$ as a function of ϕ_D and γ

Table 1. Expected values of first moment $E(\phi_s)$, second moment $E\{M_2(\phi_D)\}$, and coefficient of variation $\eta\{M_2(\phi_D)\}$, for different γ values

γ	$E(\phi_s)$	$E\{M_2(\phi_D)\}$	$\eta\{M_2(\phi_D)\}$
1.5	1.14	0.14	0.33
2.0	1.61	1.19	0.68
2.5	2.50	7.18	1.07
3.0	4.21	42.78	1.55
3.5	7.64	272.96	2.16

$$E(\phi_s) = \int_0^{\infty} E(\phi_s \mid \phi_D) e^{-\phi_D} \, d\phi_D \, . \tag{15}$$

Simirlarly, the expected values of the variance, $E\{M_2(\phi_D)\}$, and coefficient of variation $\eta\{M_2(\phi_D)\}$ when the rainfall volume is unknown are:

$$E\{M_2(\phi_D)\} = \int_0^{\infty} M_2(\phi_D) e^{-\phi_D} d\phi_D \, , \text{ and } \eta\{M_2(\phi_D)\} = \frac{\sqrt{E\{M_2(\phi_D)\}}}{E(\phi_s)} . \tag{16a,b}$$

The results of the evaluation of Eqs. (15), (16a), and (16b) are summarized in Table 1. As γ increases beyond unity, $E(\phi_s)$ increases while the second moment increases rapidly when $\gamma > 2.5$. The coefficient of variation $\eta\{M_2(\phi_D)\}$, however, increases steadily from 0 (when $\gamma = 1$) to about 2.2 (when $\gamma = 3.5$). This demonstrates the greater sensitivity of soil erosion calculations based on the rainfall intensity as γ becomes larger. It can be concluded that soil erosion losses from individual storms cannot be estimated accurately from rainfall depth only, except when the exponent γ approaches unity.

5 Soil erosion losses from a large number of storms

A different perspective arises when the sum of individual erosion losses from a large number of storms is considered. The central limit theorem states that the distribution of sums of random variables tends toward a normal distribution when the number of storms becomes large.

The expected value of the sum of n erosion losses resulting from the occurrence of n storms, $E(\sum^n \phi_s)$, equals the sum of expected values from individual storms, $\sum^n E(\phi_s)$. The coefficient of variation of this sum, however, decreases with the number of storms as follows:

$$[1 - \frac{\eta\{M_2(\phi_D)\}}{\sqrt{n}} t_{1-\alpha;n-1}] < \frac{E(\sum^n \phi_s)}{\sum^n E(\phi_s)} < [1 + \frac{\eta\{M_2(\phi_D)\}}{\sqrt{n}} t_{1-\alpha;n-1}] \tag{17}$$

where n = number of storms, t = student's t-distribution, α = specified probability level, and $n-1$ = degrees of freedom.

Hence, the expected value of the sum of n storms remains bounded at a probability level α. For example, the expected value of the sum of 100 storms at a 95% probability level is calculated from Eq. (19) with the aid of Fig. 5 ($t_{0.05;99} = 1.66$) given $\eta\{M_2(\phi_D)\} = 0.68$ when $\gamma = 2$ as obtained from Table 1:

$$0.89 < [\frac{E(\sum^n \phi_s)}{\sum^n E(\phi_s)}] < 1.11 \tag{18}$$

Consequently, as opposed to the previous conclusion regarding single storm events,

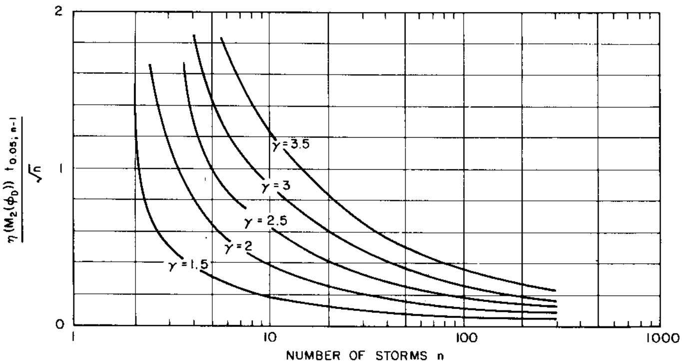

Figure 5. Coefficient of variation of the expected value of erosion losses for n storms

long term erosion losses can be related to the total amount of rainfall provided the number of storms is sufficiently large.

6 Conclusion

On a single storm basis, soil erosion losses cannot be predicted accurately from the rainfall depth only because of the dispersion induced by rainfall intensity. The coefficient of variation increases rapidly with the exponent γ as shown in Table 1.

Total erosion losses from a large number of storms can be calculated from the sum of individual erosion losses. The variance of the expected value of the total erosion losses is shown to decrease as the number of rainstorms increases.

The importance of this conclusion can be seen as it enables the prediction of long term upland erosion losses from long term rainfall amounts. The question as to whether the number of storms is sufficiently large for monthly or annual predictions can be answered from Eq. (17) with the aid of the results shown in Table 1, and Fig. 5.

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Appendix A Notations

$\bar{i}$	average rainfall intensity of a large number of storms
K_γ	Modified Bessel function of order γ
L	length of overland flow
$M_2(\phi_D)$	second central moment conditional on given ϕ_D
n	number of rainstorm events
q_s	unit sediment discharge
S	surface slope
$t_{1-\alpha;n-1}$	student's t-distribution at a probability level α and $n-1$ degrees of freedom
W	width of a rectangular plane
α	probability level
$\bar{\alpha}, \beta, \gamma$	empirical coefficients of the sediment transport relationship
π	group of variables which remain constant at a given field state
ν	kinematic viscosity of the fluid
$\eta(\phi_D)$	coefficient of variation conditional on given ϕ_D
ϕ_D	dimensionless rainfall depth during one storm
ϕ_i	dimensionless average rainfall intensity during one storm
ϕ_{q_s}	dimensionless sediment transport rate per unit width
ϕ_s	dimensionless soil loss
ϕ_t	dimensionless rainfall duration of one storm
ϕ_W	ratio of plane width W to runoff length L

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