Journal of Hydraulic Research Vol. 41, No. 5 (2003), pp. 493–501

Modified log-wake law for turbulent flow in smooth pipes

Loi log-trainée modifiée pour écoulement turbulent en conduite à paroi lisse

JUNKE GUO, *Assistant Professor, Department of Civil Engineering, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260. Email: cveguoj@nus.edu.sg*

PIERRE Y. JULIEN, *Professor, Engineering Research Center, Department of Civil Engineering, Colorado State University, Fort Collins, CO 80523, USA. Email: pierre@engr.colostate.edu*

ABSTRACT
A modified log-wake law for turbulent flow in smooth pipes is developed and tested with laboratory data. The law consists of three terms: a log term, a sine-square term and a cubic term. The log term reflects the restriction of the wall, the sine-square term expresses the contribution of the pressure gradient, and the cubic term makes the standard log-wake law satisfy the axial symmetrical condition. The last two terms define the modified wake law. The proposed velocity profile model not only improves the standard log-wake law near the pipe axis but also provides a better eddy viscosity model for turbulent mixing studies. An explicit friction factor is also presented for practical applications. The velocity profile model and the friction factor equation agree very well with Nikuradse' and other recent data. The eddy viscosity model is consistent with Laufer's, Nunner's and Reichardt's experimental data. Finally, an equivalent polynomial version of the modified log-wake law is presented.

RÉSUMÉ
Une loi log-traînée modifiée est développée pour décrire les écoulements turbulents dans les conduites à parois lisses. La loi modifiée consiste en trois termes: un terme logarithmique, un terme de traînée, et un terme cubique de condition limite axiale. Le terme logarithmique représente les conditions de la paroi, le terme de traînée définit le gradient de pression de l'écoulement, et le dernier terme satisfait la condition limite au centre de la conduite. Le terme de traînée décrit physiquement les conditions de mélange turbulent causées par le gradient de pression. Le coefficient de frottement et le coefficient de viscosité tourbillonnaire sont décrits de façon analytique. Les équations de profils de vitesse et de coefficient de frottement sont en accord avec les données expérimentales de Nikuradse et autres données récentes. Le modèle de viscosité tourbillonnaire est consistent avec les données expérimentales de Laufer, Nunner et Reichardt. Finalement, une version polynomiale de la loi log-traînée modifiée est également présentée.

Keywords: Pipe flow; turbulence; logarithmic matching; log law; log-wake law; velocity profile; velocity distribution; eddy viscosity; friction factor.

1 Introduction

Fully developed turbulent flow in circular pipes has been investigated extensively not only because of its practical importance, but also for the extension of the results to open-channel flows and boundary layer flows. The first systematic study of the velocity profile in turbulent pipe flows may be credited to Darcy in 1855 (Schlichting, 1979, p. 608) who deduced a 3/2-nd-power velocity defect law from his careful laboratory measurements, i.e.

$$\frac{u_{\max} - u}{u_*} = 5.08\left(1 - \frac{y}{R}\right)^{3/2} \tag{1}$$

in which $u_{\max}$ = the maximum velocity at the pipe axis, u = the time-averaged velocity at a distance y from the pipe wall, u_* = the shear velocity, and R = the pipe radius. Equation (1) is seldom used because it is invalid near the pipe wall with $\xi = y/R < 0.25$. Near the wall or in the inner region, Spalding's law (White, 1991, p. 415) or O'Connor's law (1995) may be applied.

This study emphasizes the velocity profile away from the pipe wall where $yu_*/\nu > 30$ given ν = fluid kinematic viscosity. The law of the wall or the logarithmic law proposed by von Karman and Prandtl (Schlichting, 1979, p. 603) is widely used in this region. For smooth pipe flow, it is written as

$$\frac{u}{u_*} = \frac{1}{\kappa}\ln\frac{yu_*}{\nu} + A \tag{2a}$$

where ν = the von Karman constant. Nikuradse (1932) has verified (2a) with his classical experiments. When (2a) was first proposed, the value of κ was suggested to be 0.4. Most researchers subsequently preferred the value of 0.41 (Nezu and Nakagawa, 1993, p. 51). Recently, Barenblatt (1996, p. 278) showed that for very large Reynolds number, the value of von Karman constant tends to

$$\kappa = \frac{2}{\sqrt{3e}} \approx 0.42 \tag{2b}$$

Revision received February 17, 2003. Open for discussion till February 29, 2004.

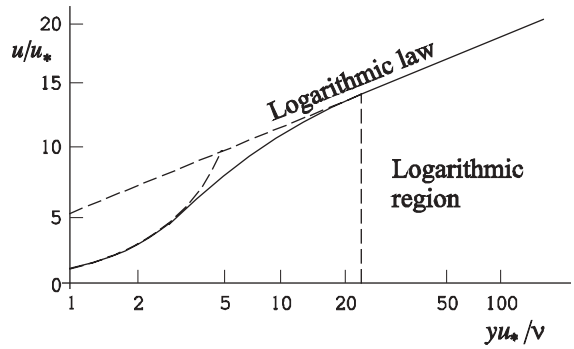


Figure 1 The law of the wall or the logarithmic law.

The value of A varies in literature; it is about 5.29 ± 0.47 according to Nezu and Nakagawa (1993, p. 51). Equation (2b) will be examined later in this paper.

Equation (2a) is applicable away from the wall where $yu_*/\nu > 30$, as shown in Fig. 1. It is not only valid for steady flow, but is also frequently used as a reference condition in unsteady flow simulations (Ferziger and Peric, 1997, p. 277). This is because even in unsteady flows, the wall shear stress predominates in the near-wall flow, and the influence of inertial forces and pressure gradient are vanishingly small.

Laufer (1954) found that the logarithmic law (2a) actually deviates from experimental data when $\xi = y/R > 0.1 \sim 0.2$. Coles (1956) further confirmed this finding and claimed that the deviation has a wake-like shape when viewed from the freestream. Thus, he called the deviation the law of the wake. Based on Coles' digital data, Hinze (1975, p. 98) proposed the following expression for the wake function, i.e.

$$W(\xi) = \frac{2\Pi}{\kappa} \sin^2 \frac{\pi \xi}{2} \tag{3}$$

in which Π is Coles wake strength. Finally, one can modify the logarithmic law by adding the wake function, i.e.

$$\frac{u}{u_*} = \left(\frac{1}{\kappa} \ln \frac{yu_*}{\nu} + A \right) + \frac{2\Pi}{\kappa} \sin^2 \frac{\pi \xi}{2} \tag{4}$$

This is called the log-wake law. When applied to pipe flows, the reader can easily show that the log-wake law (4) does not satisfy the axial symmetrical condition, i.e., the velocity gradient is nonzero at the pipe axis. Besides, the physical interpretation of the wake function is not clear in pipe flows.

Similar to flows in narrow open-channels (Guo and Julien, 2001), to correct the velocity gradient at the pipe axis, Guo (1998) proposed the following velocity profile model for pipe flows,

$$\frac{u}{u_*} = \underbrace{\frac{1}{\kappa} \ln \frac{yu_*}{\nu} + A}_{\text{The law of the wall}} + \underbrace{\frac{2\Pi}{\kappa} \sin^2 \frac{\pi \xi}{2} - \frac{\xi}{\kappa}}_{\text{Law of the wake}} \tag{5a}$$

or

$$\frac{u_{\max} - u}{u_*} = -\frac{1}{\kappa}(\ln \xi + 1 - \xi) + \frac{2\Pi}{\kappa} \cos^2 \frac{\pi \xi}{2} \tag{5b}$$

in which the Coles wake strength Π is due to the pressure-gradient. The log term in (5a) expresses the restriction of the wall, the sine-square term indicates the effect of the pressure-gradient, and the last term reflects the axial boundary correction. According to Coles, the last two terms in (5a), which are the deviation from the log law, define a law of the wake. It is noted that although (5a) or (5b) satisfies the axial symmetrical condition, the constant derivative of the linear correction term brings an additional shear stress in the near wall region, which slightly perturbs the law of the wall. It is therefore concluded that the linear function is not the best axial boundary correction.

The purpose of this paper is to develop a physically-based velocity profile model for turbulent pipe flows called the modified log-wake law. It is proposed to improve upon Eq. (5b) in the light of a better physical interpretation of the wake law. The proposed modified log-wake law is also tested with experimental velocity measurements in pipes. Analytical relationships for the turbulent eddy viscosity and for the friction factor are then compared with experimental data.

2 Development of the modified log-wake law

This section formulates the modified log-wake law in smooth pipes. First a theoretical analysis is considered. Section 2.2 proposes the basic structure of the modified log-wake law. Section 2.3 defines the axial boundary correction function. The final formulation of the modified log-wake law is written in velocity-defect form in Section 2.4.

2.1 Theoretical analysis

Consider fully developed turbulent flow with homogeneous density through a pipe of radius R. For convenience, cylindrical coordinates are used with the x-axis coinciding with the axis of the pipe, as shown in Fig. 2(a). One can show that the continuity equation is automatically satisfied and none of the flow variables depend on θ. The momentum equation in the r-direction gives that the pressure p is a function of x alone, i.e.

$$\frac{\partial p}{\partial r} = 0 \tag{6}$$

in which p = dynamic pressure. The only nonzero component of velocity is the axial velocity $u(r)$ or $u(y)$ where y = distance from the wall, i.e. $y = R - r$. With reference to Fig. 2(b), for steady and incompressible flow, the force balance in the x-direction gives

$$-\tau \cdot 2\pi(R-y)dx + (\tau + d\tau) \cdot 2\pi(R-y-dy)dx$$
$$-dp \cdot \pi[(R-y)^2 - (R-y-dy)^2] = 0$$

in which τ = local shear stress. Neglecting the 2nd order terms, the above equation reduces to

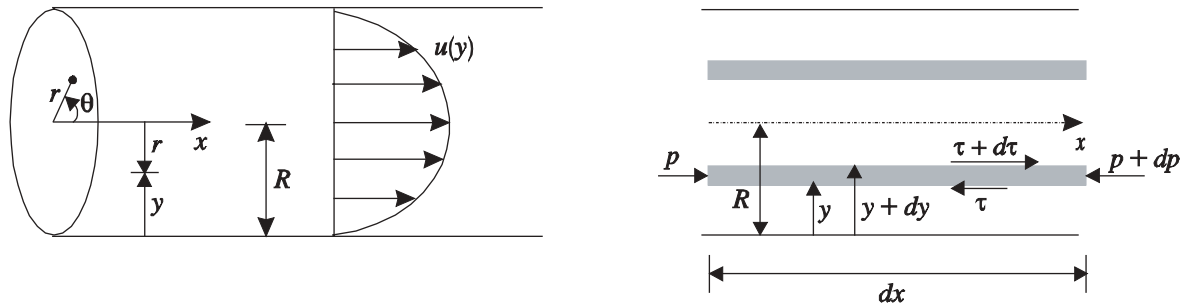
$$-\tau \cdot 2\pi dxdy + d\tau \cdot 2\pi(R-y)dx - dp \cdot 2\pi(R-y)dy = 0$$

Dividing by $2\pi dxdy$, the above becomes

$$-\tau + \frac{d\tau}{dy}(R-y) - \frac{dp}{dx}(R-y) = 0$$

This can be rearranged as

$$-\frac{dp}{dx}(R-y) + \frac{d[(R-y)\tau]}{dy} = 0 \tag{7}$$



(a) Coordinates of the system (b) Analysis of the momentum equation in the flow direction

Figure 2 Scheme of a developed turbulent pipe flow.

This is the momentum equation in the x-direction. Integrating the above with respect to y and applying the wall shear stress τ_w at $y = 0$ gives

$$\tau = \frac{R\tau_w}{R-y} + \frac{R}{2}\frac{dp}{dx}\frac{(2R-y)y}{(R-y)R} \tag{8}$$

Although the relation

$$\tau_w = -\frac{R}{2}\frac{dp}{dx} \tag{9}$$

which can be found from $\tau(\xi = 1) = 0$, can further reduce (8) to a linear function, the current form in (8) has a clear physical interpretation. Its first term expresses the effect of the wall shear stress, and the second term reflects the effect of the pressure-gradient. Near the wall, the effect of the pressure gradient can be neglected, and the fluid shear stress is balanced by the wall shear stress.

Applying the eddy viscosity model,

$$\tau_t = \rho\nu_t\frac{du}{dy} \tag{10}$$

in which τ_t = turbulent shear stress, ρ = fluid density, and ν_t = eddy viscosity, to (8) and neglecting the viscous shear stress gives

$$\rho\nu_t\frac{du}{dy} = \frac{R\tau_w}{R-y} + \frac{R}{2}\frac{dp}{dx}\frac{(2R-y)y}{(R-y)R} \tag{11}$$

According to previous experience (Hinze, 1975, p. 730), one can assume an eddy viscosity as

$$\nu_t = R\sqrt{\frac{\tau_w}{\rho}}f\left(\frac{y}{R}\right) \tag{12}$$

in which f is an unknown function. Applying the definition of the shear velocity $u_* = \sqrt{\tau_w/\rho}$ and the normalized distance $\xi = y/R$ into (11) and (12) leads to

$$\frac{1}{u_*}\frac{du}{d\xi} = \frac{1}{(1-\xi)f(\xi)} + \frac{R}{2}\frac{dp}{dx}\frac{1}{\rho u_*^2}\frac{\xi(2-\xi)}{(1-\xi)f(\xi)} \tag{13}$$

This shows that the pipe velocity profile is a result of the effects of the wall shear stress (the first term) and the pressure-gradient (the second term). After making clear the physical meaning of the second term, one can apply (9) to (13) and eliminate the pressure-gradient. Integrating (13) gives

$$\frac{u}{u_*} = \int\frac{d\xi}{(1-\xi)f(\xi)} + \int\frac{\xi(\xi-2)d\xi}{(1-\xi)f(\xi)} \tag{14}$$

Clearly, the solution of the above equation requires the knowledge of $f(\xi)$ that is complicated and unknown. Therefore, this paper tries to construct an approximate velocity profile model, based on a physical and mathematical reasoning.

2.2 *Approximation of the velocity profile model*

The effect of the wall shear stress is often expressed by the law of the wall (2a). This implies that the first integral of (14) must reduce to the logarithmic law (2a) near the wall. It is then assumed that the first integral can be approximated by

$$\int\frac{d\xi}{(1-\xi)f(\xi)} = \left(\frac{1}{\kappa}\ln\frac{yu_*}{\nu} + A\right) + F_1(\xi) \tag{15}$$

in which F_1 is a correction function of the effect of the wall on the core flow region. Obviously, F_1 must satisfy

$$F_1(\xi = 0) = 0 \tag{16a}$$

and

$$F_1'(\xi = 0) = 0 \tag{16b}$$

Condition (16a) keeps F_1 negligible near the wall, and condition (16b) guarantees F_1 does not bring a shear stress to the near wall region.

For simplicity, one can define the second integral of (14) as

$$F_2(\xi) = \int\frac{\xi(\xi-2)d\xi}{(1-\xi)f(\xi)} \tag{17}$$

Like F_1, F_2 must be negligible and not bring a shear stress near the wall, i.e.

$$F_2(\xi = 0) = 0 \tag{18a}$$

and

$$F_2'(\xi = 0) = 0 \tag{18b}$$

This is equivalent to removing the wall restriction in the flow direction. Therefore, the effect of the pressure gradient can be considered a wall-free shear, like a jet, except that the pressure gradient is the driving force in pipe flows while the inertia is

the driving force in a developed jet. Furthermore, the pipe flow can be considered a superposition of a wall-bounded shear and a wall-free shear. Because of the symmetrical condition, F_2 must reach its maximum value at the pipe axis and satisfy

$$F_2'(\xi = 1) = 0 \tag{18c}$$

According to (18b) and (18c), one can approximate the derivative of F_2 as

$$F_2'(\xi) \propto \sin \pi\xi \tag{19}$$

The integration of (19) with the boundary condition (18a) gives

$$F_2(\xi) = \frac{2\Pi}{\kappa} \sin^2 \frac{\pi\xi}{2} \tag{20a}$$

or

$$\int \frac{\xi(\xi - 2)d\xi}{(1-\xi)f(\xi)} = \frac{2\Pi}{\kappa} \sin^2 \frac{\pi\xi}{2} \tag{20b}$$

in which the constant $2/\pi$ gets buried in Π in the integration and Π is introduced as per Coles wake function. Clearly, the sine-square function in pipe flows is due to the effect of pressure-gradient. The value of Π might vary with a Reynolds number slightly, but a universal constant might be good enough for large Reynolds number flows.

Substituting (15) and (20b) into (14) gives

$$\frac{u}{u_*} = \left(\frac{1}{\kappa} \ln \frac{yu_*}{\nu} + A\right) + \frac{2\Pi}{\kappa} \sin^2 \frac{\pi\xi}{2} + F_1(\xi) \tag{21}$$

The last two terms disappear near the wall, thus the values of κ and A should be the same as those in the law of the wall. It is then suggested $\kappa = 2/(\sqrt{3}e) \approx 0.42$. The value of A will be replaced with the maximum velocity $u_{\max}$ by using the velocity defect formulation in Section 2.4. Besides, it is shown later that the value of $\Pi = \kappa$ fits the modified log-wake law well with experimental data. Therefore, this paper assumes

$$\kappa = \Pi = \frac{2}{\sqrt{3}e} \approx 0.42 \tag{22}$$

2.3 *The axial boundary correction*

Because of the axial symmetry, the velocity gradient must be zero at $\xi = 1$. From (21), one has

$$\frac{1}{u_*} \frac{du}{d\xi}\bigg|_{\xi=1} = \frac{1}{\kappa} + F_1'(1) = 0$$

which gives

$$F_1'(1) = -\frac{1}{\kappa} \tag{23}$$

From (16b) and (23), one can assume

$$F_1'(\xi) = -\frac{\xi^{n-1}}{\kappa} \tag{24}$$

in which $n > 1$. Integrating the above equation and applying (16a) gives

$$F_1(\xi) = -\frac{\xi^n}{n\kappa} \tag{25}$$

Since pipe flows are completely symmetrical about the axis, mathematically, the pipe velocity profile is an even function about the axis $\xi = 1$. In terms of Taylor series, all odd derivatives at $\xi = 1$ must be zero, i.e.

$$u = u_{\max} + \frac{1}{2!} \frac{d^2u}{d\xi^2}\bigg|_{\xi=1} (1-\xi)^2 + \frac{1}{4!} \frac{d^4u}{d\xi^4}\bigg|_{\xi=1} (1-\xi)^4 + \cdots$$

To correct the modified log-wake law to the third order term, letting

$$\frac{d^3u}{d\xi^3}\bigg|_{\xi=1} = 0$$

in (21) where (25) gets applied, one can show that

$$n = 3 \tag{26}$$

2.4 *The modified log-wake law and its defect form*

Combining (21), (22), (25) and (26) leads to the following modified log-wake law:

$$\frac{u}{u_*} = \underbrace{\frac{\sqrt{3}e}{2} \ln \frac{yu_*}{\nu} + A}_{\text{the law of the wall}} + \underbrace{2\sin^2 \frac{\pi\xi}{2} - \frac{\sqrt{3}e}{2}\frac{\xi^3}{3}}_{\text{the modified law of the wake}} \tag{27}$$

Since the last two terms in the above equation express the deviation from the law of the wall, following Coles (1956), they define the law of the wake in this paper, i.e.

$$W(\xi) = 2\sin^2 \frac{\pi\xi}{2} - \frac{\sqrt{3}e}{2}\frac{\xi^3}{3} \tag{28}$$

To distinguish it from the standard sine-square wake law, this paper calls (28) the modified wake law. Furthermore, (27) is called the modified log-wake law that is graphically represented by Fig. 3. Equation (27) has at least two advantages over the standard log-wake law. It shows that the law of the wake in pipes results from the pressure-gradient and the axial symmetrical condition. It meets the symmetrical condition at the pipe axis where the standard log-wake law fails.

To eliminate A from the modified log-wake law (27), one can introduce the maximum velocity $u_{\max}$ at the axis $\xi = 1$ to the modified log-wake law. From (27), one obtains

$$\frac{u_{\max}}{u_*} = \frac{\sqrt{3}e}{2}\left(\ln \frac{Ru_*}{\nu} - \frac{1}{3}\right) + A + 2 \tag{29}$$

Furthermore, eliminating A from (27) and (29) gives the velocity defect form of the modified log-wake law

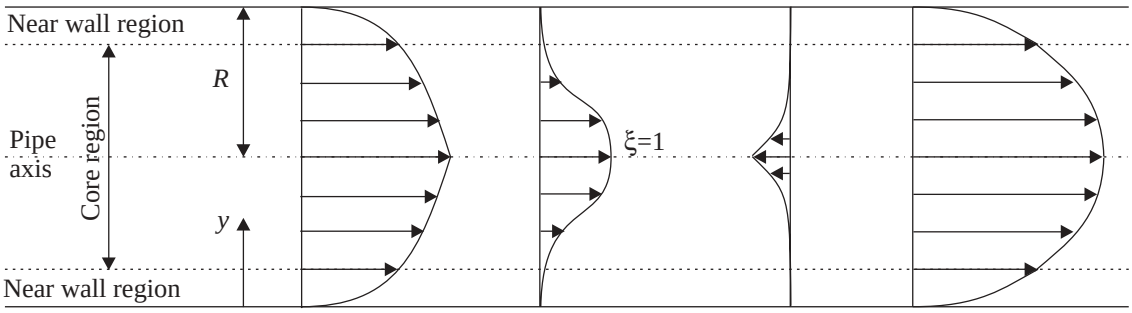
$$\frac{u_{\max} - u}{u_*} = -\frac{\sqrt{3}e}{2}\left(\ln \xi + \frac{1-\xi^3}{3}\right) + 2\cos^2 \frac{\pi\xi}{2} \tag{30}$$

This is the most important result of this study.

3 Determination of the maximum velocity and comparison with recent experiments

3.1 *Determination of the maximum velocity*

The maximum velocity in (30) plays the role of wall friction that will be discussed in Section 5. This section directly correlates the maximum velocity $u_{\max}$ with Reynolds number Ru_*/ν. A plot of



(a) The effect of the wall (b) The effect of the pressure-gradient (c) The boundary correction (d) The composite velocity profile

Figure 3 Components of the modified log-wake law.

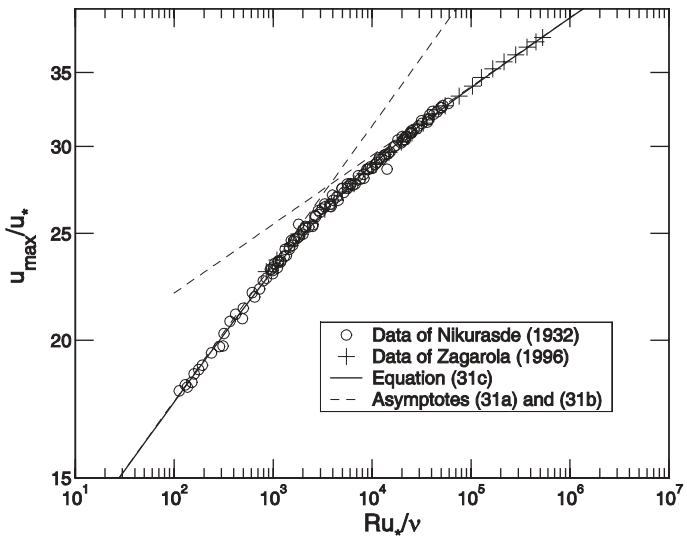


Figure 4 The relation of $u_{\max}/u_*$ versus Ru_*/ν in loglog coordinates.

Nikurasde's (1932) classical data and the recent Princeton University experiments (Zagarola, 1996) is shown in Fig. 4. One can see that for $Ru_*/\nu < 2000$, the data follows the following power law,

$$\frac{u_{\max}}{u_*} = 9.9\left(\frac{Ru_*}{\nu}\right)^{1/8} \tag{31a}$$

and for $Ru_*/\nu > 2 \times 10^4$, the data fits the following power law well,

$$\frac{u_{\max}}{u_*} = 16.55\left(\frac{Ru_*}{\nu}\right)^{1/16} \tag{31b}$$

According to Guo's (2002) logarithmic matching, an accurate curve-fitting equation is then obtained for any Reynolds number,

$$\frac{u_{\max}}{u_*} = 9.9\left(\frac{Ru_*}{\nu}\right)^{1/8}\left(1+\frac{1}{3720}\frac{Ru_*}{\nu}\right)^{-1/16} \tag{31c}$$

where the shape transition parameter $\beta = 1$ in Guo's (2002) method provides excellent agreement with experimental data, as shown in Fig. 4.

3.2 Comparison with recent experiments

This section first examines the applicability of (30) to describe individual velocity profiles. The universality of the parameters is then tested by plotting all data points according to the defect form (30). Zagarola (1996) at Princeton University measured 26 mean velocity profiles with different Reynolds numbers between 3.1×10^4 and 3.5×10^7. The test pipe was smooth and had a nominal diameter of 129 mm. The complete descriptions of the experimental apparatus and experimental data can be found on the web site http://www.princeton.edu/∼gasdyn/ or in Zagarola (1996).

To illustrate the procedures of analysis, take Run 16 for an example where $R = 6.47$ cm, $u_* = 0.7$ m/s and $\nu = 1.07 \times 10^{-6}$ m^2/s. One can calculate that

$$\frac{Ru_*}{\nu} = 45290$$

and from (31c) one gets

$$\frac{u_{\max}}{u_*} = 31.2$$

The velocity profile, according to (30), is then

$$\frac{u}{u_*} = 31.2 + \frac{\sqrt{3}e}{2}\left(\ln\xi + \frac{1-\xi^2}{3}\right) - 2\cos^2\frac{\pi\xi}{2} \tag{32}$$

Using the above procedures, all 26 profiles are obtained and plotted in Fig. 5. One can conclude that: (i) the basic structure of the modified log-wake law is correct; (ii) Eq. (30) can replicate the experimental data very well; (iii) the empirical Eq. (31c) for the maximum velocity works very well; (iv) the modified log-wake law tends to a straight line in the semilog plot near the wall and then coincides with the log law there; and (v) the zero velocity gradient at the pipe axis can be clearly seen from all the experimental profiles which show that the axial boundary correction is necessary.

Besides, according to the defect form, all 26 profiles including 1040 data points are also plotted in Fig. 6 where all data points fall in a narrow band. This shows that the model parameters κ, Π and n are universal constants.

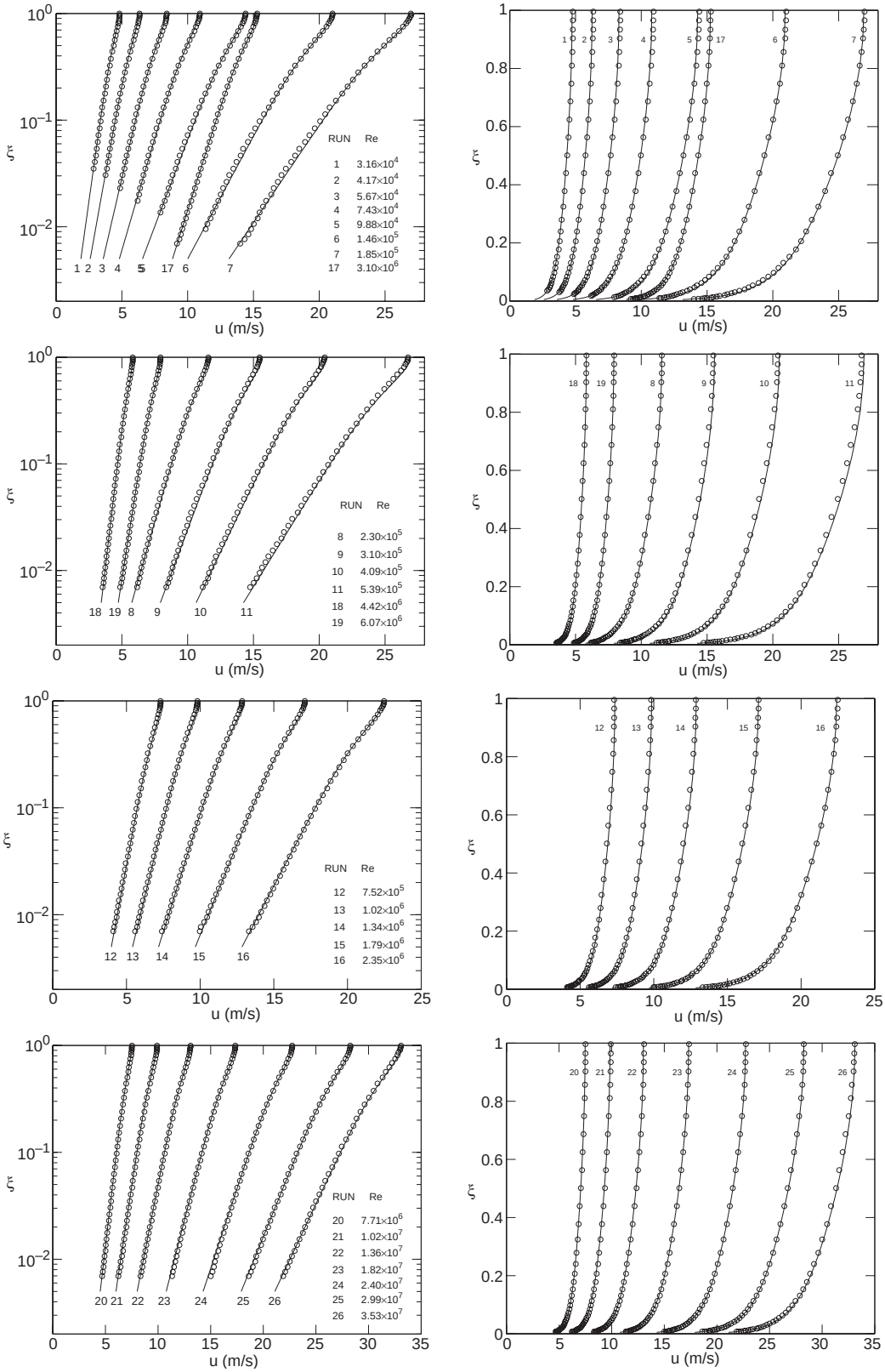


Figure 5 Comparison of the modified log-wake law (30) with Zagarola's (1996) experimental data through individual profiles.

4 Implication for eddy viscosity

Eddy viscosity is important when studying turbulent mixing. With the modified log-wake law, the eddy viscosity can now be determined. First, applying (9) to (8) gives the shear stress distribution:

$$\tau = \tau_w(1 - \xi) \tag{33}$$

Neglecting the viscous shear stress, from (10) and (33), one can show that the eddy viscosity can be expressed by

$$\frac{\nu_t}{Ru_*} = \frac{1 - \xi}{(1/u_*)(du/d\xi)} \tag{34}$$

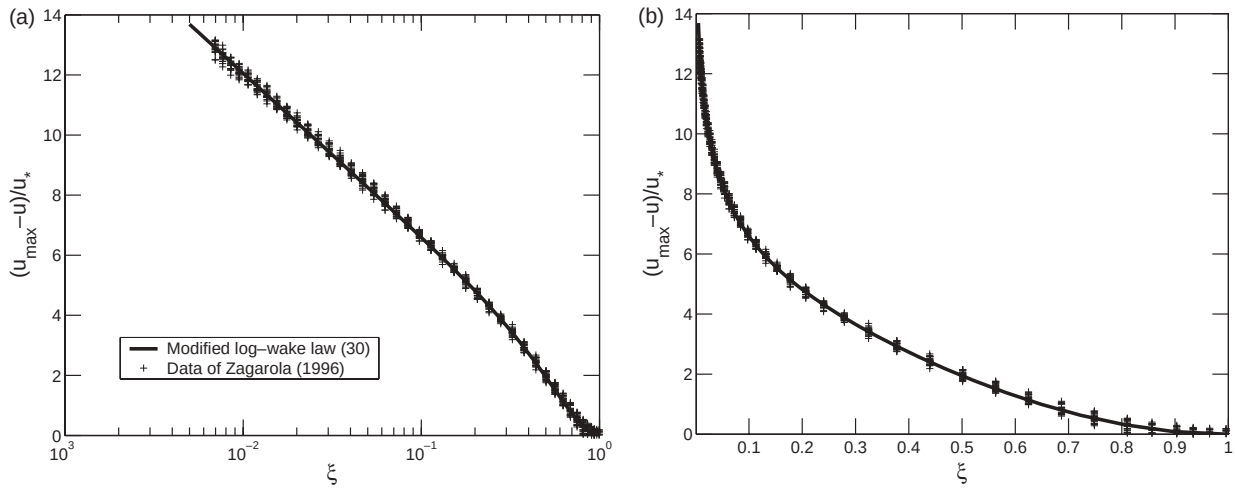


Figure 6 Verification of the model universal constants with the recent Princeton University experiments.

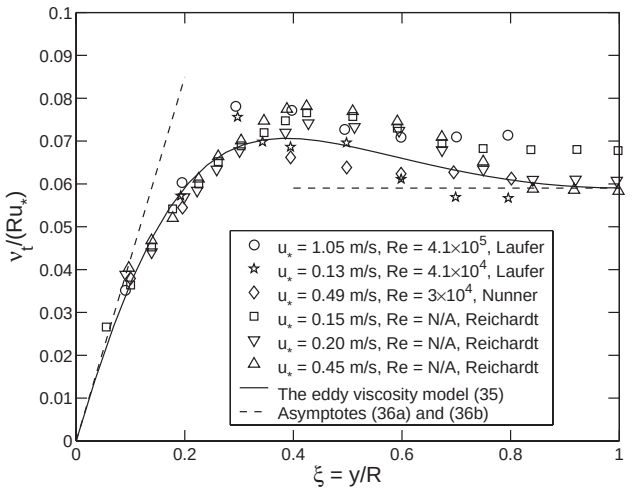


Figure 7 Comparison of the eddy viscosity model with experimental data (Data source: Ohmi and Usui, 1976).

The velocity gradient from the modified log-wake law (30) is

$$\frac{1}{u_*}\frac{du}{d\xi} = \frac{\sqrt{3}e}{2}\left(\frac{1}{\xi} - \xi^2\right) + \pi \sin \pi\xi$$

Substitution into (34) gives

$$\frac{\nu_t}{Ru_*} = \frac{1-\xi}{(\sqrt{3}e/2)(1/\xi - \xi^2) + \pi \sin \pi\xi} = \left[\frac{\sqrt{3}e}{2}\left(\frac{1}{\xi} + 1 + \xi\right) + \frac{\pi \sin \pi\xi}{1-\xi}\right]^{-1} \tag{35}$$

which is the eddy viscosity model corresponding to the modified log-wake law. Figure 7 shows an excellent agreement between (35) and several measured data sets (Ohmi and Usui, 1976). This comparison not only indirectly shows that the modified log-wake law (30) correctly describes the velocity gradients, but also it can be used to study turbulent mixing in pipe flows.

Near the pipe wall, $\xi \to 0$, (35) reduces to

$$\frac{\nu_t}{Ru_*} \to \frac{2\xi}{\sqrt{3}e} \approx 0.42\xi \tag{36a}$$

which is consistent with the mixing length model. Near the pipe axis, $\xi \to 1$, one has

$$\frac{\sin \pi\xi}{1-\xi} = \frac{\sin \pi(1-\xi)}{1-\xi} \to \pi$$

Substituting it into (35) yields

$$\frac{\nu_t}{Ru_*} \to \frac{1}{(3\sqrt{3}e/2) + \pi^2} = 0.059 \tag{36b}$$

The constant eddy viscosity near the axis corresponds to an asymptote of a parabolic law (Hinze, 1975, p. 732).

Equation (35) may be the best result so far for the eddy viscosity model in pipe flows. It may also be used to study some complicated turbulent flows such as a wave turbulent boundary layer flow. All previous velocity profile models, including the log law, the log-wake law and the power law, cannot produce the maximum eddy viscosity at $\xi \approx 0.3$ and the constant eddy viscosity near the axis.

5 Friction factor

The friction factor is an essential parameter in pipe designs and numerical simulations. It is defined as

$$\tau_w = \frac{f}{8}\rho V^2 \tag{37a}$$

in which V = cross-sectional average velocity. Applying the definition $\tau_w = \rho u_*^2$ and rearranging the above gives

$$f = 8\left(\frac{u_*}{V}\right)^2 \tag{37b}$$

Upon integrating (30) over the cross-sectional area, one obtains

$$\frac{V}{u_*} = \frac{u_{\max}}{u_*} - 3.53$$

Substituting (31c) into the above equation and then into (37b) gives

$$f = 8\left[9\mathrm{Re}^{1/8}\left(1 + \frac{\mathrm{Re}}{7440}\right)^{-1/16} - 3.53\right]^{-2} \tag{37c}$$

in which $R = d/2$ and $\mathrm{Re} = Vd/\nu$ have been used and d = pipe diameter. The constants in the above equation must be adjusted

since the viscous sublayer was neglected in the derivation. For simplicity, as per (37c), one can assume the friction factor follows

$$f = \frac{a}{\mathrm{Re}^{1/4}}\left(1 + \frac{\mathrm{Re}}{b}\right)^{1/8} \tag{38a}$$

For low Reynolds number say $\mathrm{Re} < 10^5$, the above equation should reduce to Blasius formula (Schlichting, 1979, p. 597), i.e.

$$f = \frac{0.3164}{\mathrm{Re}^{1/4}} \tag{38b}$$

which gives

$$a = 0.3164 \tag{38c}$$

A curve fitting process for high Reynolds number say $\mathrm{Re} > 2 \times 10^6$ gives

$$b = 4.31 \times 10^5 \tag{38d}$$

Finally the friction factor (38a) is written as

$$f = \frac{0.3164}{\mathrm{Re}^{1/4}}\left(1 + \frac{\mathrm{Re}}{4.31 \times 10^5}\right)^{1/8} \tag{38e}$$

This equation is compared with Nikuradse's (1932) classical data and the recent Princeton University data (Zagarola, 1996) in Fig. 8.

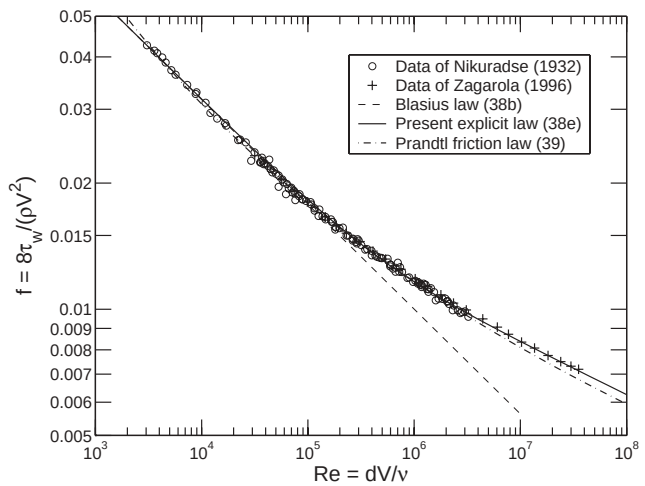


Figure 8 Friction factor for smooth pipe flows.

For comparison, Prandtl's universal friction law (Schlichting, 1979, p. 611) is also plotted in Fig. 8 (dashdot line), i.e.

$$\frac{1}{\sqrt{f}} = 2\log(\mathrm{Re}\sqrt{f}) - 0.8 \tag{39}$$

One can see from Fig. 8 that: (i) for $\mathrm{Re} < 3 \times 10^6$, the present formula (38e) is equivalent to Prandtl's equation (39); (ii) for $\mathrm{Re} > 3 \times 10^6$, the present formula looks better than Prandtl's equation; and (iii) the present formula (38e) is explicit while Prandtl's formula (39) is implicit. Therefore, (38e) is convenient in applications. In particular, it can save much computing time by avoiding iterations in numerical simulations.

6 Polynomial version of the modified log-wake law

At the end of this paper, it is noteworthy that an equivalent polynomial log-wake law exists if one replaces (19) by

$$F_2'(\xi) \propto \xi(1-\xi) \tag{40}$$

For simplicity, directly assuming that (White, 1991, p. 417)

$$\sin^2\frac{\pi\xi}{2} \approx 3\xi^2 - 2\xi^3 \tag{41a}$$

and

$$\cos^2\frac{\pi\xi}{2} \approx 1 - 3\xi^2 + 2\xi^3 \tag{41b}$$

(30) can be rewritten as

$$\frac{u_{\max} - u}{u_*} = -\frac{\sqrt{3}e}{2}\ln\xi + \left(2 - \frac{\sqrt{3}e}{6}\right) - 6\xi^2 + \left(4 + \frac{\sqrt{3}e}{6}\right)\xi^3 \tag{42}$$

Comparison of (42) with (30) is shown in Fig. 9 which shows that the polynomial version (42) is indeed equivalent to the cosine-square version (30). In practice, one may choose the version either (30) or (42) based on convenience according to the problem.

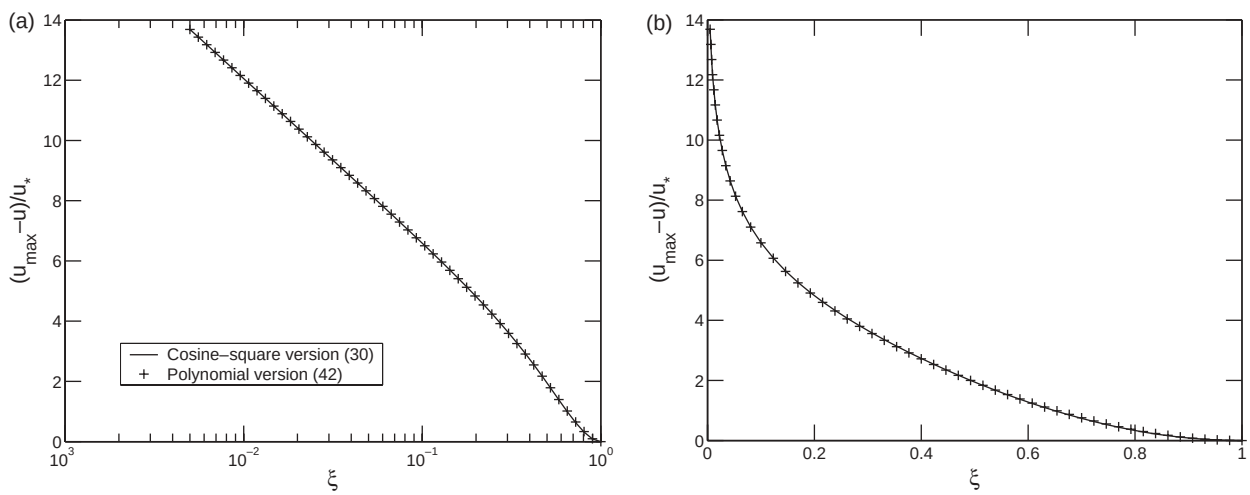


Figure 9 Comparison of the cosine-square version with the polynomial version of the modified log-wake law.

7 Conclusions

Using a theoretical analysis and a physical and mathematical reasoning, smooth pipe turbulent flow is assumed to be a superposition of two flow fields: a wall-bounded shear (the effect of the wall) and a wall-free shear (the effect of the pressure-gradient). The wall-bounded shear results in the logarithmic law and the wall-free shear yields the sine-square function. To correct the logarithmic law at the pipe axis, a cubic function is introduced which is called the axial boundary correction. Finally, the velocity profile in smooth pipe flows is assumed to be a superposition of three terms: a log term, a sine-square term, and a cubic correction term, shown in (27).

The modified log-wake law includes three universal constants $\kappa = \Pi \approx 0.42$ and $n = 3$. The maximum velocity can be accurately determined by an empirical equation (31c). Comparison with recent experiments shows that the modified log-wake law fits the experimental velocity profiles extremely well as shown in Figs. 5 and 6.

The eddy viscosity model (35) deduced from the modified log-wake law not only agrees with the experimental data in the literature very well, as shown in Fig. 7, but also reproduces a maximum eddy viscosity at $\xi \approx 0.3$ and a constant eddy viscosity near the axis.

An explicit friction factor (38e) is also proposed for practical applications. This empirical formula agrees with Nikuradse's (1932) and Zagarola's (1996) data very well and is better than Prandtl's classical equation. It may save much computing time by avoiding iterations in numerical simulations.

Finally, it is pointed out that a polynomial version of the modified log-wake law (42) can be derived. The polynomial version is practically equivalent to the cosine-square version, shown in Fig. 9, and either (30) or (42) can be used in practice.

Notations

A = integral constant in the law of the wall
a, b = fitting constants in (38b)
d = pipe diameter
F_1, F_2 = functional symbols
f = functional symbol or friction factor
n = boundary correction power
p = dynamic pressure
R = pipe radius
Re = global Reynolds number
r = distance from the pipe axis
u = time-averaged velocity at distance y from the wall
$u_{\max}$ = maximum velocity
u_* = shear velocity
V = cross-averaged velocity
W = wake function
x = axial direction in the pipe flow
y = distance from the wall
β = shape transition parameter in Guo's logarithmic matching
θ = angular coordinate in Fig. 2
κ = von Karman constant
ν = kinematic viscosity
ν_t = eddy viscosity
ξ = normalized distance from the wall, y/R
Π = Coles wake strength or the pressure-gradient factor
ρ = fluid density
τ = local shear stress
τ_t = turbulent shear stress
τ_w = wall shear stress

References

1. BARENBLATT, G.I. (1996). "Scaling, Self-Similarity, and Intermediate Asymptotics," Cambridge University Press.
2. COLES, D.E. (1956). "The Law of the Wake in the Turbulent Boundary Layer," *J. Fluid Mech.*, 1, 191–226.
3. FERZIGER, J.H. and PERIC, M. (1997). Computational Methods for Fluid Dynamics. Springer.
4. GUO, J. (1998). Turbulent Velocity Profiles in Clear Water and Sediment-Laden Flows. PhD Dissertation, Colorado State University, Fort Collins, CO.
5. GUO, J. (2002). "Logarithmic Matching and Its Applications in Computational Hydraulics and Sediment Transport," *J. Hydr. Res.* IAHR, 40(5), 555–565.
6. GUO, J. and JULIEN, P.Y. (2001). "Turbulent Velocity Profiles in Sediment-Laden Flows," *J. Hydr. Res.* IAHR, 39(1), 11–23.
7. HINZE, J.O. (1975). Turbulence. 2nd Ed., McGraw-Hill.
8. LAUFER, J. (1954). The Structure of Turbulence in Fully Developed Pipe Flow. Report 1174, National Advisory Committee of Aeronautics, Washington, D. C.
9. NEZU, I. and NAKAGAWA, H. (1993). Turbulent Open-Channel Flows. A. A. Balkema, Rotterdam, Netherlands.
10. NIKURADSE, J. (1932). Gesetzmässigkeiten Der Turbulenten Strömung in Glatten Rohren. Forschg. Arb. Ing.-Wes. No. 356.
11. O'CONNOR, D.J. (1995). "Inner Region of Smooth Pipes and Open-Channels," *J. Hydr. Engrg.* ASCE, 121(7), 555-560.
12. OHMI, M. and USUI, T. (1976). "Pressure and Velocity Distribution in Pulsating Turbulent Pipe Flow, Part 1, Theoretical Treatments," *Bulletin of the JSME*, 19(129), 303–313.
13. SCHLICHTING, H. (1979). Boundary-Layer Theory. 7th edition, McGraw-Hill.
14. WHITE, F.M. (1991). Viscous Fluid Flow. 2nd edition, McGraw-Hill.
15. ZAGAROLA, M.V. (1996). Mean-Flow Scaling of Turbulent Pipe Flow. PhD Dissertation, Princeton University, Princeton, NJ.