Modified log–wake law for zero-pressure-gradient turbulent boundary layers

Loi log-trainée modifiée pour couche limite turbulente sans gradient de pression

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ABSTRACT
This paper shows that the turbulent velocity profile for zero-pressure-gradient boundary layers is affected by the wall shear stress and convective inertia. The effect of the wall shear stress is dominant in the so-called overlap region and can be described by a logarithmic law in which the von Karman constant is about 0.4 while the additive constant depends on a Reynolds number. The effect of the convective inertia can be described by the Coles wake law with a constant wake strength about 0.76. A cubic correction term is introduced to satisfy the zero velocity gradient requirement at the boundary layer edge. Combining the logarithmic law, the wake law and the cubic correction produces a modified log–wake law, which is in excellent agreement with experimental profiles. The proposed velocity profile law is independent of Reynolds number in terms of its defect form, while it is Reynolds number dependent in terms of the inner variables. The modified log–wake law can also provide an accurate equation for skin friction in terms of the momentum thickness. Finally, by replacing the logarithmic law with van Driest's mixing-length model in which the damping factor varies with Reynolds number, the modified log–wake law can be extended to the entire boundary layer flow.

RÉSUMÉ
Le profil de vitesse pour une couche limite turbulente sans gradient de pression dépend de la contrainte de cisaillement à la paroi et de l'inertie convective. L'effet de cisaillement est dominant dans la zone de transition décrite par la loi logarithmique avec constante de von Karman d'environ 0.4. L'effet d'inertie convective est décrit par la loi de trainée de Coles avec un coeffcient de 0.76. Un terme de correction cubique est introduit pour satisfaire la condition limite supérieure sans gradient de vitesse. La loi logarithmique-trainée modifiée qui en résulte se compare très bien avec les profils de vitesse expérimentaux. Sous forme de déviation de vitesse, le profil de vitesse proposé devient indépendant du nombre de Reynolds. La loi proposée produit des équations exactes du coefficient de frottement et d'épaisseur du film de quantité de mouvement. Finalement, en remplaçant la loi logarithmique par la longueur de mélange de van Driest avec coefficient d'amortissement fonction du nombre de Reynolds, la loi log-trainée modifiée devient applicable à la couche limite toute entière.

Keywords: Zero-pressure-gradient boundary layers, turbulence, logarithmic law, wake law, velocity profile, velocity distribution, skin friction.

1 Introduction

Turbulent flows in pipes, zero-pressure-gradient (ZPG) flat plate boundary layers and open-channels are not only three fundamental boundary shear flows but also important in mechanical, aeronautic and hydraulic engineering. These three types of flows have similarities. The flow near the wall can be described by the law of the wall. The flow near the pipe axis, the boundary layer edge and the free surface can be described by the velocity defect law.

In terms of the momentum equation in the primary flow direction, steady pipe flows are the simplest for which differential shear stress is balanced by a constant pressure gradient; ZPG boundary layer flows are more complicated since differential shear stress is balanced by nonlinear convective inertia; and open-channel flows are the most complex because both constant pressure gradient (gravity term) and nonlinear convective inertia (secondary currents) exist. A new turbulent velocity profile model is usually first applied to pipes, then ZPG boundary layers and finally open-channels. For example, the classic logarithmic law proposed by Prandtl and von Karman was first developed for pipes and ZPG boundary layers in the early 1930s (Schlichting 1979, p. 578–665). It was then applied to open-channels by Keulegan (1938). Laufer (1954) first pointed out that experimental data deviate from the logarithmic law away from the pipe wall. Subsequently Coles (1956) confirmed this behavior for boundary layers and suggested the law of the wake. During the 1980s, the

law of the wake was compared with open-channel profiles by Coleman (1981, 1986), Graf (1984) and Nezu and Rodi (1986). This paper is a continuation of Guo and Julien (2003), where a modified log–wake law was proposed for smooth pipe turbulence, by considering the value of the same concepts for ZPG boundary layers. The application of the modified log–wake law to open-channel turbulence will be considered separately because of the complexity of secondary currents and free surface.

The following background is closely related to the development of this study. Combining the logarithmic law and the wake law produces the log–wake law (Coles 1956),

$$\frac{u}{u_*} = \left(\frac{1}{\kappa}\ln\frac{yu_*}{\nu} + B\right) + W(\xi) \tag{1}$$

in which the terms in the parentheses are the logarithmic law while the last term is called the law of the wake, which defines the deviation from the logarithmic law away from the wall, in which u = time-averaged velocity along the wall, u_* = shear velocity, κ = von Karman constant, y = distance from the wall, ν = fluid kinematic viscosity, B = additive constant, $\xi = y/\delta$ the relative distance from the wall, and δ = the boundary layer thickness, which is defined by $u(y = \delta) = 0.999U$ in this paper. Coles (1956) described the wake function $W(\xi)$ in an empirical table. Hinze (1975, p. 698) fit the Coles data analytically,

$$W(\xi) = \frac{2\Pi}{\kappa}\sin^2\frac{\pi\xi}{2} \tag{2}$$

in which Π = Coles' wake strength, which accounts for the effects of Reynolds number in ZPG boundary layers. Including Hinze's equation (2) the log–wake law (1) is often written as

$$\frac{u}{u_*} = \left(\frac{1}{\kappa}\ln\frac{yu_*}{\nu} + B\right) + \frac{2\Pi}{\kappa}\sin^2\frac{\pi\xi}{2} \tag{3}$$

Unfortunately comparison of Eq. (3) with experimental data (Coles, 1969; Hinze, 1975, p. 699) showed that (3) is invalid near the boundary layer edge where the zero velocity gradient requirement is not satisfied.

Except for the above classic results, universal relations for near-wall flows have been extensively debated since the early 1990s (Afzal, 1997, 2001a,b; Barenblatt, 1993; Barenblatt *et al.*, 2000a,b; Buschmann, 2001; Buschmann and Gad-el-Hak, 2003a,b; George and Castillo, 1997; Osterlund *et al.*, 2000; Wosnik *et al.*, 2000; Zagarola, 1996; Zagarola and Smits, 1998; Zagarola *et al.*, 1997; and others). Most of them concluded that the velocity profile for near-wall flows is Reynolds number dependent. In particular, the velocity in the overlap layer follows a power law where the parameters vary with Reynolds number.

This paper tries to incorporate the above state-of-the-art knowledge into the modified log–wake law for ZPG turbulent boundary layers. It will: (a) examine the boundary layer equations and formulate a hypothesis of the modified log–wake law; (b) validate the modified log–wake law by comparing with recent experimental velocity profiles in ZPG boundary layers; (c) derive a skin friction formula in terms of the momentum thickness Reynolds number; and (d) extend the modified log–wake law to the entire boundary layer by applying van Driest's mixing-length model where the damping factor varies with Reynolds number.

2 Hypothesis of the modified log–wake law

This section examines the shear stress distribution in ZPG boundary layers and formulates a hypothesis of the modified log–wake law.

2.1 Shear stress distribution

Consider a steady two-dimensional incompressible viscous flow over a ZPG flat plate where the x direction is along the wall and y normal to the wall. Neglecting gravity, Prandtl's boundary layer equations are (Schlichting, 1979, p. 563)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial \tau}{\partial y} \tag{5}$$

where u = time-averaged velocity in the x direction, v = time-averaged velocity in the y direction, ρ = fluid density, and τ = local shear stress that includes viscous and turbulent shear stresses. Equation (4) is the continuity equation, and (5) is the momentum equation along the wall.

To obtain an expression for the shear stress distribution one may combine the continuity equation and momentum equation and integrate across the boundary layer to develop

$$\tau = \tau_\mathrm{w} + \rho\int_0^y\left(-u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}\right)\mathrm{d}y \tag{6}$$

where $\tau = \tau_\mathrm{w}$ at the wall $y = 0$. One can realize that the shear stress in ZPG boundary layers includes the contributions of the wall shear stress and convective inertia.

2.2 Dimensional analysis of velocity distribution

In the outer region including the overlap layer, the viscous shear stress can be neglected. Applying the eddy viscosity model,

$$\tau_\mathrm{t} = \rho\nu_\mathrm{t}\frac{\mathrm{d}u}{\mathrm{d}y} \tag{7}$$

in which τ_t = turbulent shear stress and ν_t = eddy viscosity, to (6) gives

$$\rho\nu_\mathrm{t}\frac{\partial u}{\partial y} = \tau_\mathrm{w} + \rho\int_0^y\left(-u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}\right)\mathrm{d}y \tag{8}$$

Considering that the eddy viscosity can be expressed by (Hinze, 1975, p. 645)

$$\nu_\mathrm{t} = \delta u_* f\left(\frac{y}{\delta}\right) \tag{9}$$

in which f is an unknown function, and applying the definitions $\xi = y/\delta$ and $\tau_\mathrm{w} = \rho u_*^2$ to (8), one obtains

$$f(\xi)\frac{\partial}{\partial\xi}\left(\frac{u}{u_*}\right) = 1 + \int_0^\xi\left[-\frac{u}{u_*}\frac{\partial}{\partial\xi}\left(\frac{v}{u_*}\right) + \frac{v}{u_*}\frac{\partial}{\partial\xi}\left(\frac{u}{u_*}\right)\right]\mathrm{d}\xi \tag{10}$$

Except for the complicated integrodifferential form in the above, the eddy viscosity function $f(\xi)$ is not really specified. Thus, it is impossible to get an analytical solution for u. However,

the preceding equation suggests the following dimensionless solution form:

$$\frac{u}{u_*} = F\left(\xi, \frac{v}{u_*}\right) \quad (11)$$

in which

$$\frac{v}{u_*} = \eta(\xi) \quad (12)$$

and F and η are velocity distribution functions in the x and y directions. Substituting (12) into (11) gives

$$\frac{u}{u_*} = F(\xi, \eta) \quad (13)$$

Since the transverse velocity v or η is very small compared with the primary velocity u or F in the outer region, one can approximate the primary velocity u/u_* by expanding (13) at $\eta = 0$,

$$\frac{u}{u_*} = F(\xi, 0) + \eta \frac{\partial F(\xi, 0)}{\partial \eta} + \frac{\eta^2}{2!} \frac{\partial^2 F(\xi, 0)}{\partial \eta^2} + \cdots \quad (14)$$

Taking the first two-term approximation, one has

$$\frac{u}{u_*} = F(\xi, 0) + \eta \frac{\partial F(\xi, 0)}{\partial \eta} \quad (15)$$

Note that the above analysis is equivalent to a small perturbation introduced by the transverse velocity function $\eta(\xi)$. In fact, the classical Blasius solution (Schlichting, 1979, p. 136; Kundu, 1990, p. 310) can also be expressed in the above dimensionless form when one expresses the velocity distribution u in terms of the transverse velocity v. In the next subsection, the primary function $F(\xi, 0)$, the transverse velocity distribution function $\eta(\xi)$ and the derivative function $\partial F(\xi, 0)/\partial \eta$ will be specified.

2.3 Approximation of the velocity distribution

2.3.1 The primary function $F(\xi, 0)$

The functions $F(\xi, 0)$, $\eta(\xi)$ and $\partial F(\xi, 0)/\partial \eta$ are approximated asymptotically and empirically. First, consider the overlap layer where the effect of the transverse velocity v or η can be neglected and $\partial F(\xi, 0)/\partial \eta$ is finite. One can conclude that the primary function $F(\xi, 0)$ is the law of the wall, which is often described by the classical logarithmic law or a power law. Recently based on experimental velocity profiles and asymptotic analysis, many researchers (Afzal, 1997; Barenblatt *et al.*, 2000a,b; Buschmann and Gad-el-Hak, 2003a,b; George and Castillo, 1997; Osterlund *et al.*, 2000; Wosnik *et al.*, 2000; Zagarola, 1996; Zagarola and Smits, 1998; Zagarola *et al.*, 1997) showed that a Reynolds number dependent power law can even better represent the velocity profile in the overlap layer. Thus, this paper assumes the following law of the wall:

$$\frac{u}{u_*} = (a \ln \mathrm{Re}_\delta + b)\left(\frac{yu_*}{\nu}\right)^{c/\ln \mathrm{Re}_\delta} \quad (16)$$

in which a, b and c are positive constants and the Reynolds number Re_δ is defined as

$$\mathrm{Re}_\delta = \frac{\delta u_*}{\nu} \quad (17)$$

Since the power exponent $c/\ln \mathrm{Re}_\delta$ in (16) is usually very small say 0.1–0.15 (Barenblatt *et al.*, 2000a,b), one can rewrite (16) as

$$\frac{u}{u_*} = (a \ln \mathrm{Re}_\delta + b) \exp\left\{\frac{c}{\ln \mathrm{Re}_\delta} \ln\left(\frac{yu_*}{\nu}\right)\right\}$$

$$= (a \ln \mathrm{Re}_\delta + b)\left\{1 + \frac{c}{\ln \mathrm{Re}_\delta} \ln\left(\frac{yu_*}{\nu}\right) + \cdots\right\}$$

$$= (a \ln \mathrm{Re}_\delta + b) + \left(ac + \frac{bc}{\ln \mathrm{Re}_\delta}\right) \ln\left(\frac{yu_*}{\nu}\right) + \cdots \quad (18)$$

In the overlap layer, one has $y \ll \delta$ or $\ln(yu_*/\nu) \ll \ln \mathrm{Re}_\delta$, the above equation can then be approximated by

$$\frac{u}{u_*} = (a \ln \mathrm{Re}_\delta + b) + \left(ac + \frac{bc}{\ln \mathrm{Re}_\delta}\right) \ln\left(\frac{yu_*}{\nu}\right) \quad (19)$$

Comparing it with the classical logarithmic law, one has

$$\frac{1}{\kappa} = ac + \frac{bc}{\ln \mathrm{Re}_\delta} \rightarrow ac \quad (20)$$

for large Reynolds number, and

$$B = a \ln \mathrm{Re}_\delta + b \quad (21)$$

where $B =$ additive constant. Note that Eqs (20) and (21) show that: (a) the von Karman constant κ increases with Reynolds number; (b) a universal von Karman constant κ may exist only for large Reynolds number; and (c) the additive constant B increases with Reynolds number even for large Reynolds number. The dependence of Reynolds number accounts for the effect of the "viscous superlayer" (Hinze, 1975, pp. 567, 571, 628), which is near the boundary layer edge where Kolmogoroff length scale energy dissipation exists. In fact, Hinze (1975, p. 628) has noticed that the von Karman constant κ varies slightly about 0.4 whereas the additive constant B corresponds with much greater variations, say, from 4 to 12 (George and Castillo, 1997), which may be explained by (20) and (21). For simplicity, this paper concentrates on large Reynolds number and assumes

$$\kappa = \frac{1}{ac} = 0.4 \quad (22)$$

Furthermore, the primary function $F(\xi, 0)$ can be approximated by (19), i.e.,

$$F(\xi, 0) = \frac{1}{\kappa} \ln \frac{yu_*}{\nu} + B \quad (23)$$

in which $\kappa = 0.4$ and B is estimated by (21) where the constants a and b will be specified in Section 4.

2.3.2 The transverse velocity distribution function $\eta(\xi)$

It is assumed that the shape of the function $\eta(\xi)$ is similar to its counterpart in laminar flows. Inspired by the Blasius classical

solution (Schlichting, 1979, p. 137; Kundu, 1990, p. 311) and the conventional sine-square wake function, one may assume

$$v = V \sin^2 \frac{\pi \xi}{2} \tag{24}$$

in which V is the transverse velocity at the boundary layer edge. Comparing (24) with (12), one must have

$$V = \chi u_* \tag{25}$$

in which χ is a proportional constant. With (25) Eq. (24) can be rewritten as

$$\frac{v}{u_*} = \eta(\xi) = \chi \sin^2 \frac{\pi \xi}{2} \tag{26}$$

The constant χ will be considered together with the derivative function $\partial F(\xi, 0)/\partial \eta$.

2.3.3 The derivative function $\partial F(\xi, 0)/\partial \eta$

With (26) one can write the second term in (15) as

$$\chi \frac{\partial F(\xi, 0)}{\partial \eta} \sin^2 \frac{\pi \xi}{2} \tag{27}$$

which is the same as Hinze's version of the law of the wake assuming

$$\chi \frac{\partial F(\xi, 0)}{\partial \eta} = \frac{2\Pi}{\kappa} \tag{28}$$

and $\partial F(\xi, 0)/\partial \eta$ is independent of ξ. In other words, the second term in (15) can be approximated by the conventional law of the wake,

$$\eta \frac{\partial F(\xi, 0)}{\partial \eta} = \frac{2\Pi}{\kappa} \sin^2 \frac{\pi \xi}{2} \tag{29}$$

According to Coles (Fernholz and Finley, 1996), the wake strength Π increases with Reynolds number and tends to a constant for large Reynolds number. To be consistent with (22) where an assumption of large Reynolds number is employed, one can assume

$$\Pi = \text{constant} \tag{30}$$

in this paper. Substituting (23) and (29) into (15) produces the conventional log–wake law (3) except that the additive constant B varies with Reynolds number.

2.3.4 Boundary correction

Strictly speaking, boundary layers do not have edges; the mean velocity is only asymptotic to the free stream velocity at the so-called boundary layer edges, i.e., $u \rightarrow U$ at $\xi = y/\delta = 1$. However, in practice the assumptions of

$$u(\xi = 1) = U \tag{31}$$

and

$$\left. \frac{\mathrm{d}u}{\mathrm{d}\xi} \right|_{\xi=1} = 0 \tag{32}$$

are good approximations. To meet the zero velocity gradient requirement (32), one must modify (3) by adding a boundary correction function. Guo and Julien (2003) have shown that a cubic

correction is a good approximation for pipe axis. Similarly, this paper modifies (3) by adding the same correction function,

$$-\frac{\xi^3}{3\kappa} \tag{33}$$

2.3.5 The modified log–wake law and its defect form

Combining (3) and (33) leads to the following velocity profile model:

$$\frac{u}{u_*} = \left(\frac{1}{\kappa} \ln \frac{y u_*}{\nu} + B \right) + \frac{2\Pi}{\kappa} \sin^2 \frac{\pi \xi}{2} - \frac{\xi^3}{3\kappa} \tag{34}$$

Similar to pipe flows (Guo and Julien, 2003), this paper calls the above equation the modified log–wake law (MLWL), which should be valid from the overlap region till the boundary layer edge. Equation (34) is different from the conventional log–wake law in two aspects: it meets the zero velocity gradient at the boundary layer edge; and the additive constant B accounts for the effect of the Reynolds number.

To eliminate the effect of Reynolds number in (34), one can introduce the freestream velocity U at $\xi = 1$ to the modified log–wake law. From (34), one obtains

$$\frac{U}{u_*} = \frac{1}{\kappa} \ln \frac{\delta u_*}{\nu} + B + \frac{2\Pi}{\kappa} - \frac{1}{3\kappa} \tag{35}$$

Subtracting (34) from (35) gives the velocity defect form of the modified log–wake law

$$\frac{U - u}{u_*} = -\frac{1}{\kappa} \left(\ln \xi - 2\Pi \cos^2 \frac{\pi \xi}{2} + \frac{1 - \xi^3}{3} \right) \tag{36}$$

As aforementioned, both κ and Π should be universal constants under the assumption of large Reynolds number.

3 Test of the modified log–wake law

This section first examines the universality of the modified log–wake law by plotting all data points according to the defect form (36). It then tests the applicability of (36) to describe individual velocity profiles in terms of the inner variable $y u_*/\nu$. The 70 experimental velocity profiles by Osterlund (1999) in ZPG boundary layers with Reynolds numbers $900 \leq \text{Re}_\delta \leq 10{,}000$ will be used in this test. The complete descriptions of the experimental apparatus and measured velocity profiles can be found on the web site http://www2.mech.kth.se/~jens/zpg/. Note that Osterlund (1999) measured the wall shear stress by using an oil film interferometry, which is independent of the logarithmic law.

In the data of Osterlund (1999), the freestream velocity U, the shear velocity u_* and the measured velocity profile $u(\xi)$ are given. The boundary layer thickness δ in this paper is interpolated by $u(y = \delta) = 0.999U$. According to the defect form, all 70 measured velocity profiles are plotted in Fig. 1 where except for the viscous sublayer and the buffer layer, all data points fall almost on a single curve down into the overlap layer. This reveals that the velocity defect in the outer region including the overlap region is independent of Reynolds number. Furthermore, it
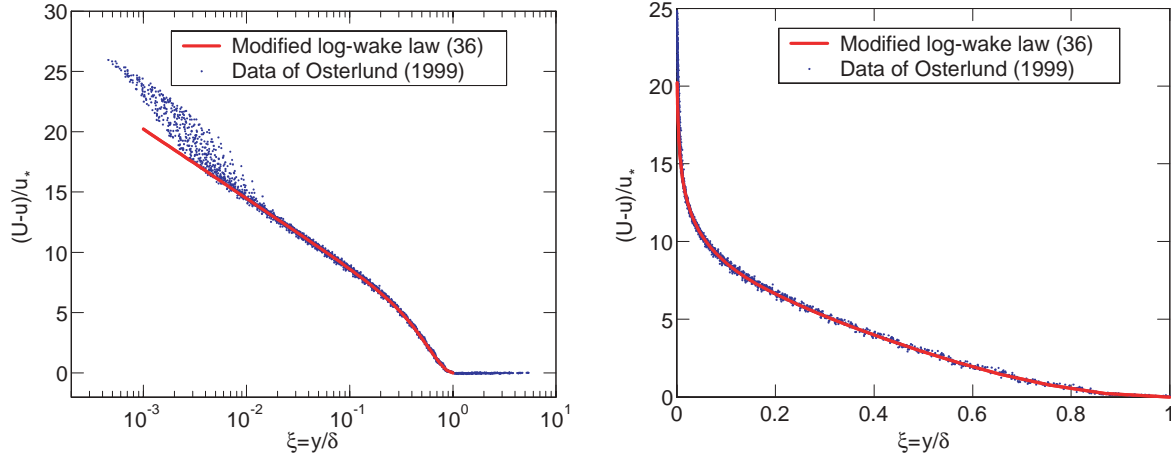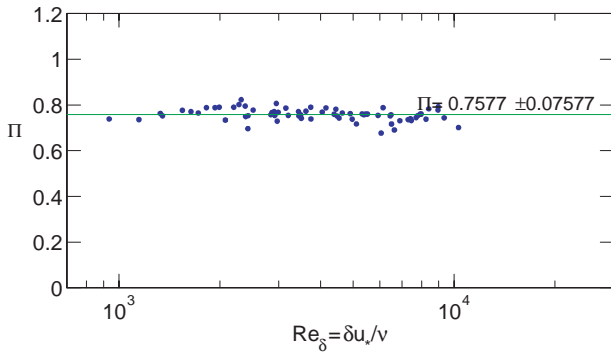
Figure 1   Verification of the universality of the velocity defect law.

Figure 2   Verification of the universal constant Π.

implies that the model parameters κ and Π in the modified log–wake law (36) are universal constants. A preliminary analysis suggests that

$$\kappa = 0.4 \quad \text{and} \quad \Pi = 0.7577 \tag{37}$$

allows Eq. (36) to fit the experimental profiles shown in Fig. 1 very well. Figure 2 further confirms Eq. (37) by plotting the wake strength Π versus Reynolds number Re_δ, in which the values of Π for individual profiles are estimated by the measured value of u/u_* at $yu_*/\nu = 100$ at assuming $\kappa = 0.4$.

In terms of the inner variable yu_*/ν, one can rewrite (36) as

$$\frac{u}{u_*} = \frac{U}{u_*} + \frac{1}{\kappa}\left(\ln\xi - 2\Pi\,\cos^2\frac{\pi\xi}{2} + \frac{1-\xi^3}{3}\right) \tag{38}$$

in which

$$\xi = \frac{yu_*/\nu}{\mathrm{Re}_\delta} \tag{39}$$

Obviously, the Reynolds number $\mathrm{Re}_\delta = \delta u_*/\nu$ is a profile parameter, which suggests that the modified log–wake law is Reynolds number dependent in terms of the inner variables. Figure 3(a–g) compare (38), in which the constants in (37) are used, with all 70 experimental profiles individually and display excellent agreement for almost all profiles. Note that the dotted lines (MLWL) are covered with the solid when $yu_*/\nu \geq 30$. These figures lead to the following conclusions: (a) the basic structure of the modified log–wake law is correct; (b) the modified log–wake law can replicate the experimental data from the overlap region till the boundary layer edge, say $30 \leq yu_*/\nu$ and $y/\delta \leq 1$; (c) the modified log–wake law tends to a straight line in a semilog plot in the overlap region and then coincides with the logarithmic law; and (d) the zero velocity gradient at the boundary layer edge can be clearly seen from all profiles in Fig. 3(a–g) which imply that the boundary correction is necessary.

4   Skin friction and the additive constant in the logarithmic law

One can compute the velocity profile by using the velocity defect law (36), which does not require the additive constant B. Nevertheless, if the modified log–wake law (34) is preferred, the additive constant B can be defined by studying the skin friction factor c_f, which is defined as

$$\tau_w = \frac{c_f}{2}\rho U^2 \quad \text{or} \quad \sqrt{\frac{2}{c_f}} = \frac{U}{u_*} \tag{40}$$

Substituting (21) into (35) gives

$$\frac{U}{u_*} = \left(\frac{1}{\kappa} + a\right)\ln\mathrm{Re}_\delta + \left(b + \frac{2\Pi}{\kappa} - \frac{1}{3\kappa}\right) \tag{41}$$

which can be rearranged as

$$\sqrt{\frac{2}{c_f}} = \frac{U}{u_*} = \frac{1}{\kappa_1}\ln\mathrm{Re}_\delta + B_1 \tag{42}$$

where

$$\frac{1}{\kappa_1} = \frac{1}{\kappa} + a \quad \text{and} \quad B_1 = b + \frac{2\Pi}{\kappa} - \frac{1}{3\kappa} \tag{43}$$

are determined experimentally. Figure 4 displays the experimental skin friction factor c_f of Osterlund (1999) versus the Reynolds number Re_δ according to (42). A least-squares curve fitting reveals that (42) with the constants

$$\kappa_1 = 0.3820 \quad \text{and} \quad B_1 = 6.6040 \tag{44}$$

can represent the experimental data with a correlation coeffcient 0.999. Substituting (37) and (44) into (43) produces

$$a = 0.1176 \quad \text{and} \quad b = 3.6544 \tag{45}$$
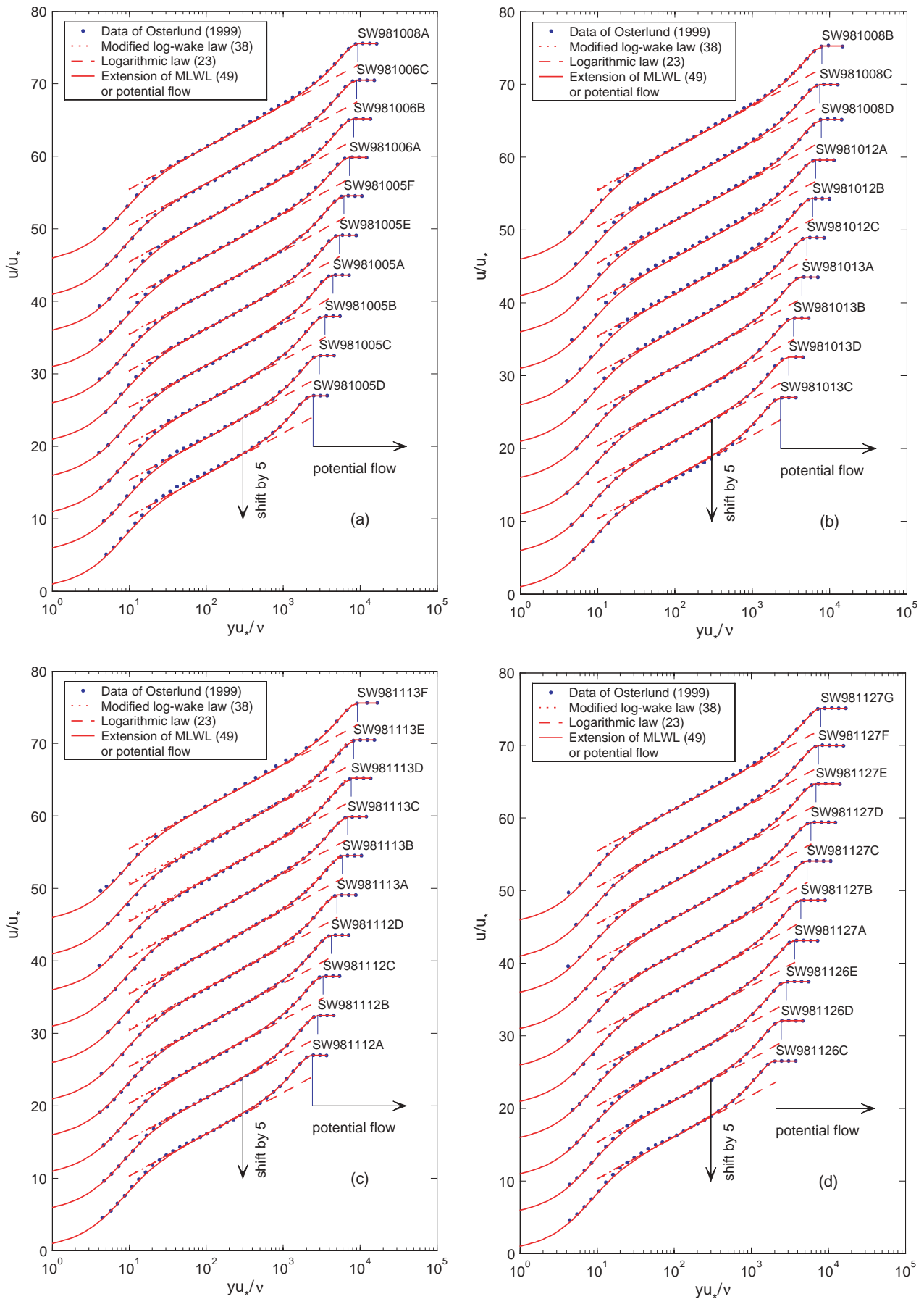
Figure 3   Comparison of modified log–wake law with individual experimental velocity profiles.
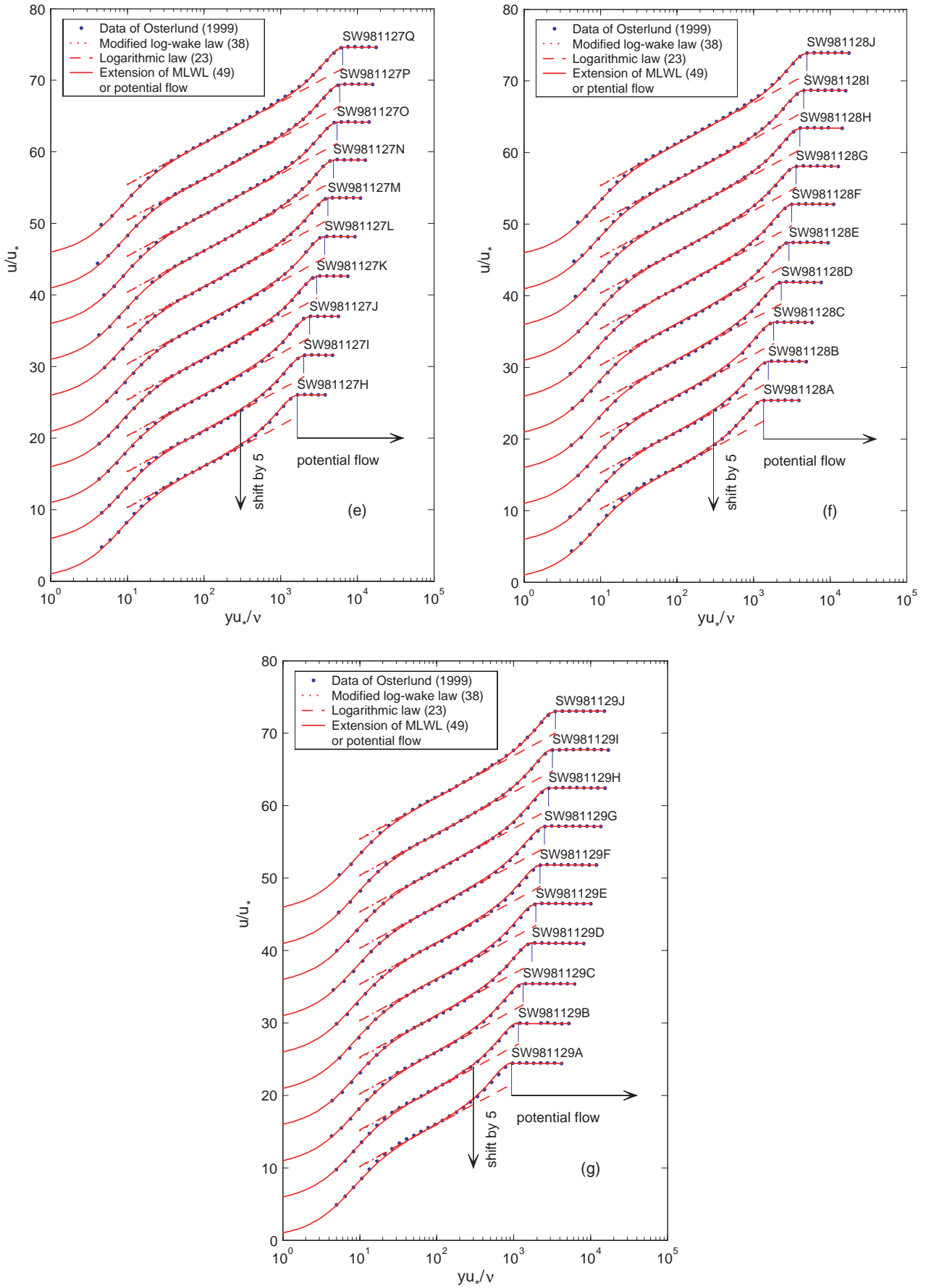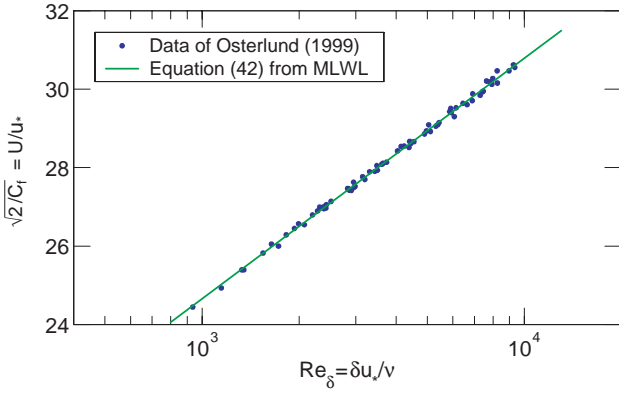
Figure 3   (*Continued*)

Figure 4   Skin friction factor c_f versus Reynolds number Re_δ.

which specify the additive constant B through (21). Like the modified log–wake law, the logarithmic law (23), to which (21) and (45) are applied, also fits the experimental data in the overlap region very well for most profiles, as shown by the dashed lines in Fig. 3(a–g), which validate the hypothesis of the dependence of Reynolds number in (21).

The skin friction factor c_f is often correlated with the momentum thickness Reynolds number Re_θ,

$$c_f = f(Re_\theta); \quad Re_\theta = \frac{\theta U}{\nu} \tag{46}$$

in which θ = the momentum thickness. Applying (38) to the definition of the momentum thickness θ, one can derive a relation between Re_δ and Re_θ (see the Appendix at the end of this paper). Applying (A7) in the Appendix to (42) leads to

$$\sqrt{\frac{2}{c_f}} = \frac{1}{\kappa_1}\ln Re_\theta - \frac{1}{\kappa_1}\ln\left(\alpha\sqrt{\frac{c_f}{2}} + \beta\right) + B_1 \tag{47}$$

in which the constants in (44) are used for the values of κ_1 and B_1, and

$$\alpha = -26.538 \quad \text{and} \quad \beta = 3.7693 \tag{48}$$

are obtained from (A5) and (A6) in the Appendix. Equation (47) agrees with the experimental data, as shown in Fig. 5, where the empirical law of Osterlund *et al.* (2000) is also plotted. The excellent agreement validates the modified log–wake law (36) not
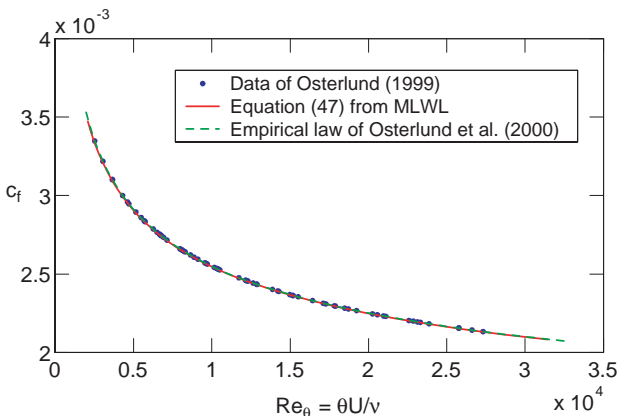

Figure 5   Skin friction factor c_f versus Reynolds number Re_θ.

only for velocity profiles but also for the momentum thickness and the skin friction factor. Note that Osterlund *et al.* (2000) obtained a small von Karman constant from the same data set since they overlooked the effect of Reynolds number on the additive constant B.

Finally, combining (42) and (47) provides a method to estimate the wall shear stress τ_w and the boundary layer thickness δ from a measured velocity profile. That is, (a) calculate the momentum thickness θ by applying a measured velocity profile to the definition of the momentum thickness; (b) estimate the friction factor c_f from (47); (c) compute the wall shear stress τ_w or the shear velocity u_* from (40); and (d) obtain the boundary layer thickness δ from (42). With this method, the wall shear stress and boundary layer thickness can be defined in an experimental program.

5   Extension of the modified log–wake law

From the above analysis, one can see that the modified log–wake law indeed agrees with the experimental data in the outer region. It is noteworthy that the modified log–wake law can be extended to the inner region by applying van Driest's mixing-length model (Schlichting, 1979, p. 604) for the law of the wall or the primary function $F(\xi, 0)$. In other words, replacing the logarithmic law in (34) with van Driest's mixing-length model, one can get a complete velocity profile model for the entire boundary layer,

$$\frac{u}{u_*} = \int_0^{y^+} \frac{2\,dy^+}{1 + \{1 + 4\kappa^2 y^{+2}[1 - \exp(-y^+/A)]\}^{1/2}} + \frac{2\Pi}{\kappa}\sin^2\frac{\pi\xi}{2} - \frac{\xi^3}{3\kappa} \tag{49}$$

in which $\kappa = 0.4$, $\Pi = 0.7577$, $y^+ = yu_*/\nu$ and A is a damping factor,

$$A \approx 5B = 5(a\ln Re_\delta + b) \tag{50}$$

in which (21) has been used. Note that van Driest suggested $A = 26$ to reproduce the additive constant $B \approx 5.0$. In this paper, the additive constant B is a function of Reynolds number, the damping factor A is then modified as Eq. (50). Figure 3(a–g) reveal that the extension equation (49) can indeed clone the entire boundary layer velocity profile, including the inner and outer regions. In brief, the extension of the modified log–wake law (49) agrees with experimental profiles well, satisfies all boundary conditions at the wall and at the boundary layer edge, and connects to the constant potential velocity smoothly, as indicated by the solid lines in Fig. 3(a–g).

6   Conclusions

Based on the boundary layer equations, dimensional analysis, perturbation technique, a recent understanding of the overlap layer and an analogy to the transverse velocity profile in laminar boundary layers, this paper proposes a modified log–wake law for the mean velocity profile of ZPG turbulent boundary

layers. The proposed law consists of three terms: a logarithmic term in which the von Karman constant is about 0.4 while the additive constant increases with Reynolds number; a sine-square term with a constant wake strength about 0.76; and a cubic correction term. The logarithmic law reflects the effect of the wall shear stress and is dominant in the overlap region; the sine-square function approximates the transverse velocity and then reflects the effect of convective inertia; and the cubic correction makes the conventional log–wake law satisfy the zero velocity gradient requirement at the boundary layer edge.

The proposed velocity profile law provides excellent agreement with 70 recent experimental profiles not only for the mean velocity profiles but also for the skin friction factor. Specifically the comparison shows that: (a) the modified log–wake law is independent of Reynolds number in terms of its defect form; that is, the von Karman constant and the wake strength are indeed universal constants for large Reynolds number; (b) the modified log–wake law is Reynolds number dependent when the inner variables are used; (c) the proposed logarithmic law, with a variable additive constant, has been validated by the data in the overlap region; and (d) the friction factor derived from the modified log–wake law is accurate in terms of the momentum thickness.

Finally, applying van Driest's mixing-length model, in which the damping factor varies with Reynolds number, for the law of the wall, the modified log–wake law can be extended to the entire boundary layer from the wall till the boundary layer edge.

Appendix: The displacement thickness δ_1 and the momentum thickness θ

The displacement thickness δ_1 and the momentum thickness θ are two important parameters in boundary layer analysis. They can be estimated from the proposed modified log–wake law (38) which gives

$$\int_0^1 \frac{u}{u_*} d\xi = -\frac{1}{\kappa}\left(\Pi + \frac{3}{4}\right) + \frac{U}{u_*} \tag{A1}$$

and

$$\int_0^1 \left(\frac{u}{u_*}\right)^2 d\xi$$
$$= \frac{1}{\kappa^2}\left(\frac{81}{56} + \frac{3\Pi}{2} - \frac{2\Pi}{\pi^2} + \frac{8\Pi}{\pi^4} + \frac{2\Pi \text{Si}(\pi)}{\pi} + \frac{3\Pi^2}{2}\right)$$
$$- \frac{2}{\kappa}\left(\frac{3}{4} + \Pi\right)\frac{U}{u_*} + \left(\frac{U}{u_*}\right)^2 \tag{A2}$$

Applying (A1) to the definition of the displacement thickness δ_1, one derives

$$\frac{\delta_1}{\delta} = \int_0^1 \left(1 - \frac{u}{U}\right) d\xi$$
$$= 1 - \frac{u_*}{U}\int_0^1 \frac{u}{u_*} d\xi = \frac{1}{\kappa}\left(\Pi + \frac{3}{4}\right)\frac{u_*}{U} \tag{A3}$$

Similarly, applying (A1) and (A2) to the definition of the momentum thickness θ gives

$$\frac{\theta}{\delta} = \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\xi$$
$$= \frac{u_*}{U}\int_0^1 \frac{u}{u_*} d\xi - \left(\frac{u_*}{U}\right)^2 \int_0^1 \left(\frac{u}{u_*}\right)^2 d\xi$$
$$= \alpha\left(\frac{u_*}{U}\right)^2 + \beta\left(\frac{u_*}{U}\right) \tag{A4}$$

in which

$$\alpha = -\frac{81}{56\kappa^2} - \left(\frac{3}{4} + \frac{\text{Si}(\pi)}{\pi} - \frac{1}{\pi^2} + \frac{4}{\pi^4}\right)\frac{2\Pi}{\kappa^2} - \frac{3\Pi^2}{2\kappa^2}$$
$$= -\frac{1.4464 + 2.5585\Pi + 1.5\Pi^2}{\kappa^2} = -26.538 \tag{A5}$$

$$\beta = \frac{1}{\kappa}\left(\frac{3}{4} + \Pi\right) = \frac{0.75 + \Pi}{\kappa} = 3.7693 \tag{A6}$$

where the constants in (37) are applied. Furthermore, one can show

$$\text{Re}_\delta = \frac{\delta u_*}{\nu} = \frac{\theta U}{\nu}\frac{\delta}{\theta}\frac{u_*}{U} = \text{Re}_\theta\left(\alpha\sqrt{\frac{c_f}{2}} + \beta\right)^{-1} \tag{A7}$$

Notation

A = Van Driest's damping factor
a, b, c = Constants in the power law (16)
B = Additive constant in the logarithmic law
B_1 = Additive constant in the friction equation (42)
c_f = Skin friction factor
F, f = Functional symbols
Re_δ = Reynolds number based on the boundary layer thickness, $\delta u_*/\nu$
Re_θ = Reynolds number based on the momentum thickness, $\theta U/\nu$
U = Freestream velocity
u = Time-averaged velocity along the wall
u_* = Shear velocity
V = Transverse velocity at the boundary layer edge
v = Time-averaged velocity normal to the wall
W = Wake function
x = Coordinate along the wall
y = Coordinate normal to the wall
y^+ = Inner variable, yu_*/ν
α, β = Constants in the friction equation (47)
δ = Boundary layer thickness
δ_1 = Displacement thickness
θ = Momentum thickness
η = Transverse velocity distribution function
κ = The von Karman constant in the logarithmic law
κ_1 = The von Karman constant in the friction law (42)
ν = Kinematic viscosity of fluid
ν_t = Eddy viscosity
ξ = Relative distance from the wall, y/δ
Π = The Coles wake strength

ρ = Fluid density
τ = Local shear stress
τ_t = Turbulent shear stress
τ_w = Wall shear stress
χ = Proportional constant in the transverse velocity
     function

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