

Subspace Selection for Projection Maximization With Matroid Constraints

Zhenliang Zhang, *Member, IEEE*, Yuan Wang, Edwin K. P. Chong, *Fellow, IEEE*, and Ali Pezeshki, *Member, IEEE*

Abstract—Suppose that there is a ground set which consists of a large number of vectors in a Hilbert space. Consider the problem of selecting a subset of the ground set such that the projection of a vector of interest onto the subspace spanned by the vectors in the chosen subset reaches the maximum norm. This problem is generally NP-hard, and alternative approximation algorithms such as *forward regression* and *orthogonal matching pursuit* have been proposed as heuristic approaches. In this paper, we investigate bounds on the performance of these algorithms by introducing the notions of *elemental curvatures*. More specifically, we derive lower bounds, as functions of these elemental curvatures, for performance of the aforementioned algorithms with respect to that of the optimal solution under uniform and nonuniform *matroid* constraints, respectively. We show that if the elements in the ground set are mutually orthogonal, then these algorithms are optimal in terms of achieving the largest projection when the matroid is uniform and they achieve at least a fraction of 1/2 of the optimal solution when the matroid is nonuniform.

Index Terms—Greedy algorithm, orthogonal matching pursuit, curvature, compressed sensing.

I. INTRODUCTION

CONSIDER the Hilbert space $L^2(\mu)$ of square integrable random variables with μ the probability measure. We define the inner product of $a, b \in L^2(\mu)$ as follows:

$$\langle a, b \rangle \equiv \int_{\Omega} ab^* d\mu = \mathbb{E}_{\mu}[ab^*],$$

where Ω denotes the sample space of the underlying probability space, and $\int \cdot d\mu$ represents abstract integration with respect to the measure μ . We use $\|a\|$ to denote the norm of $a \in L^2(\mu)$ associated with this inner product, i.e., $\|a\| = \sqrt{\langle a, a \rangle}$.

Let X be a *ground set* of vectors and η be the vector of interest in $L^2(\mu)$. Let I be a non-empty collection of subsets of X , or

equivalently, a subset of the power set 2^X . For any set $E \in I$, we use $\text{span}(E)$ to denote the subspace spanned by the vectors in E . We use $\mathcal{P}_{\eta}(E)$ to denote the projection of η onto $\text{span}(E)$. The goal is to choose an element E in I such that the square norm of $\mathcal{P}_{\eta}(E)$ is maximized, i.e.,

$$\begin{aligned} & \text{maximize } \|\mathcal{P}_{\eta}(E)\|^2 \\ & \text{subject to } E \in I. \end{aligned} \quad (1)$$

A. Motivating Examples

The above formulation has vast applications in statistical signal processing [1], [2] such as maximizing the quadratic covariance bound, sensor selection for minimizing the mean squared error, and sparse approximation for compressive sensing. Here we briefly introduce a few examples.

1) Quadratic Covariance Bound.

Let μ_{θ} be the underlying probability measure associated with parameter θ lying on the parameter space Θ . The problem of interest is to estimate $g(\theta)$, where $g: \Theta \rightarrow \mathbb{R}$ is a bounded known function. Let $\hat{g} \in L^2(\mu_{\theta})$ be an unbiased estimator of $g(\theta)$ and $\eta = \hat{g} - g(\theta) \in L^2(\mu_{\theta})$ be the estimation error, which is the vector of interest. For any set E of *score functions*, the variance of any unbiased estimator is lower bounded by the square norm of the projection of estimation error η onto $\text{span}(E)$. This fact is also known as quadratic covariance bound [3], [4]:

$$\text{Variance}[\hat{g}] = \|\eta\|^2 \geq \|\mathcal{P}_{\eta}(E)\|^2, \quad (2)$$

where $\|\eta\|^2 = \mathbb{E}_{\mu_{\theta}}[\eta\eta^*]$ and $\mathbb{E}_{\mu_{\theta}}$ denotes the expectation with respect to the measure μ_{θ} . The well-known Cramer-Rao bounds [5], Bhattacharyya bounds [6], and Barankin bounds [7] are essentially special cases of the quadratic covariance bound by substituting E with specific sets of score functions. For example, the score function for Cramer-Rao bounds is simply $\partial \ln d(x; \theta) / \partial \theta$, where $d(x; \theta)$ denotes the probability density function of measurement x . While these established bounds provide insightful understandings for the performance of unbiased estimators, the corresponding score functions do not necessarily provide the tightest bounds for the estimator variance. Moreover, derivation of these bounds such as Cramer-Rao bounds requires the inverse or pseudo-inverse *Fisher information matrix*, which can be computationally impractical for large number/dimension of unknown parameters [8]. Last, a necessary condition to compute these bounds is that the probability density

Manuscript received October 15, 2015; revised July 22, 2016; accepted November 1, 2016. Date of publication December 1, 2016; date of current version January 6, 2017. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Ozgur Yilmaz.

Z. Zhang was with the Department of Electrical and Computer Engineering, Colorado State University, Fort Collins, CO 80523-1373 USA. He is now with Intel Labs, Hillsboro, OR 97124 USA and also with the Department of Electrical Engineering, Columbia University, New York, NY 10027 USA (e-mail: zhenliang.zhang@gmail.com).

Y. Wang was with the Department of Biostatistics, University of Texas MD Anderson Cancer Center, Houston, TX 77030 USA. He is now with the Department of Mathematics and Statistics, Washington State University, Pullman, WA 99164 USA (e-mail: ywang@math.wsu.edu).

E. K. P. Chong and A. Pezeshki are with the Department of Electrical and Computer Engineering and Department of Mathematics, Colorado State University, Fort Collins, CO 80523-1373 USA (e-mail: edwin.chong@colostate.edu; ali.pezeshki@colostate.edu).

Digital Object Identifier 10.1109/TSP.2016.2634544

function and its partial derivatives are well-defined. For these reasons, other score functions might be more suitable for providing the lower bound. Suppose that there exists a large set X of candidate score functions in $L^2(\mu_\theta)$. We aim to choose an optimal subset $E \subset X$ which maximizes $\|\mathcal{P}_\eta(E)\|^2$ and hence provides the tightest bound for variances of unbiased estimators.

2) Linear Minimum Mean Squared Error Estimator.

Suppose that there is a large set of sensors, each of which makes a zero-mean and square-integrable random sensor observation. These sensor observations are not necessarily independent. The goal is to select a subset of the sensors such that the mean squared error for estimating the parameter of interest η is minimized. It is well-known that the orthogonality principle implies that the Linear Minimum Mean Squared Error (LMMSE) estimator, denoted by η_{LMMSE} , is the projection of η onto the subspace spanned by a selected subset E [1]. The problem of interest is how to choose E from the set X of all sensor observations such that the mean squared error $\mathbb{E}[(\eta_{\text{LMMSE}} - \eta)^2]$ is minimized, i.e., the projection of η onto $\text{span}(E)$ is maximized. Another approach to this sensor selection problem is to maximize the information gain and apply submodularity to bound the performance of greedy algorithms [9]–[11]. When the criterion is mean squared error, the objective function is in general not submodular, resulting in difficulty to quantify the performance of the greedy algorithms.

3) Sparse Approximation for Compressive Sensing.

Compressive sensing is the problem of recovering a sparse signal using linear compressing measurements (see, e.g., [13]–[18]). Let $\eta \in \mathbb{R}^d$ be the measurement signal. We assume that $\eta = \mathbf{H}x$ where $\mathbf{H} \in \mathbb{R}^{d \times n}$ is the measurement matrix. The goal is to find K non-zero components in the n -dimensional vector x with $K \ll d < n$ such that $\mathbf{H}x$ can exactly recover or well-approximate η , i.e.,

$$\begin{aligned} & \text{minimize } \|\eta - \mathbf{H}x\|^2 \\ & \text{subject to } \|x\|_0 \leq K, \end{aligned}$$

where $\|x\|_0$ denotes the L_0 -norm of x . The geometrical interpretation of the above problem is to select K columns of matrix \mathbf{H} such that the norm of the projection of η onto the subspace spanned by the chosen columns is maximized. Adaptive algorithms such as those based on partially observable Markov decision processes have been proposed to find the optimal solution [19]. The computation complexity for adaptive algorithms is in general quite high despite the reduction brought by approximation methods such as rollout. We note that the objective function here differs from the usual reconstruction error for sparse recovery (e.g., minimizing L_1 -norm of x). Moreover, we focus on the case where of a fixed vector of interest rather than deriving uniform sparse recovery conditions with strong assumptions on the measurement matrix H .

All the above applications are special cases of the projection maximization problem defined in (1). In general, problem (1)

Algorithm 1: Forward Regression.

Input : Ground set X and an associated matroid (X, I) ;
vector of interest η .

Output: An element $E \in I$.

```

1 begin
2    $E \leftarrow \emptyset$ ;
3   for  $\ell = 1$  to  $K$  do
4      $s^* = \arg \max_{s \in X \setminus E, E \cup \{s\} \in I} \|\mathcal{P}_\eta(E \cup \{s\})\|^2$ ;
5     Update  $E \leftarrow E \cup \{s^*\}$ ;
6   end
7 end
```

Algorithm 2: Orthogonal Matching Pursuit.

Input : Ground set X and an associated matroid (X, I) ;
vector of interest η .

Output: An element $E \in I$.

```

1 begin
2    $E \leftarrow \emptyset$ ;
3   Residue  $r = \eta$ ;
4   for  $\ell = 1$  to  $K$  do
5      $s^* = \arg \max_{s \in X \setminus E, E \cup \{s\} \in I} |\langle r, s \rangle|$ ;
6     Update  $E \leftarrow E \cup \{s^*\}$ ;
7     Update  $r \leftarrow r - \mathcal{P}_\eta(E)$ ;
8   end
9 end
```

is a combinatorial optimization problem and it is NP-hard to obtain the optimal solution. Alternative algorithms such as forward regression [12] and orthogonal matching pursuit [20]–[24] have been studied intensively to approximate the optimal solution of (1). Each of these two algorithms starts with an empty set, and then incrementally adds one element to the current solution by optimizing a local criterion, while the updated solution still belongs to the set of feasible solutions I . They are known as greedy approaches due to the nature of local optimality, although the local criteria are different. Many other variations of greedy approaches have been proposed and investigated, e.g., stagewise orthogonal matching pursuit [25], regularized orthogonal matching pursuit [26], [27], CoSaMP [28], information-based greedy approach [29], and super greedy algorithm [30]). Compared with forward regression and orthogonal matching pursuit, these variations might exhibit better properties in terms of uniformity, stability, running time, etc. However, the local criteria for these variations usually vary over different stages, which makes it non-trivial to apply the submodularity framework directly. In this paper, we only focus on the performance analysis of forward regression and orthogonal matching pursuit. Details of these two algorithms are given in Algorithms 1 and 2, respectively. The concept of a matroid captures the structure of the independent set I , which in turn guarantees the feasibility of using an iterative algorithm on the independent set. Its formal definition will be given in Section II. Notice that neither algorithm achieves the maximum projection in general. The main purpose of this paper is to quantify their performance with respect to that of the optimal

solution. We note that another frequently used approach is through convex relaxation schemes based on sparse-eigenvalue or restricted isometry property [31], although the objective there is usually to minimize the difference between the actual and estimated coefficients of sparse vectors (this corresponds to L_0 -norm minimization while (1) deals with L_2 -norm).

B. Main Contributions

The main purpose of this paper is to provide performance bounds for forward regression and orthogonal matching pursuit with respect to the optimal solution. To derive the bounds, we will define several notions of elemental curvatures, which are inspired by the elemental curvature introduced in [32]. These elemental curvatures are quantitative measures of the diminishing-return property. Their formal definitions are given in Section II. We also illustrate from a geometric perspective how these elemental curvatures are related with principal angles, which are in turn related with the restricted isometry property and mutual incoherence [33]. It turns out that the (near-)optimality of the two aforementioned algorithms is closely related with the mutual (near-)orthogonality of the vectors in the ground set and the structure of the matroid. Our approach allows the derivation of sharp approximation bounds for these two algorithms, in general situations (where the matroid might be uniform or non-uniform). To the best of our knowledge, the non-uniform matroid situation has never been investigated in any previous papers. More specifically, in the special case where the vectors in the ground set are mutually orthogonal, these two algorithms are solutions to (1) when the matroid is uniform and they achieve at least a fraction of $1/2$ of the optimal solution when the matroid is non-uniform, i.e., they achieve at least $1/2$ -approximation of the optimal solution. In the case where the matroid is uniform and the vectors in the ground set are not mutually orthogonal, we show that forward regression and orthogonal matching pursuit achieve at least a fraction of $(1 - (1 - \frac{1}{K})^K)$ and $(1 - (1 - \frac{\sin^2 \phi}{K})^K)$ of the optimal solution, respectively, where K is the number of iterations of the algorithm, \hat{K} is a number that depends on K and the forward and backward elemental curvatures, and ϕ is the principal angle. In the non-uniform matroid case, we have also derived these performance bounds as functions of elemental curvatures and principal angle.

II. CURVATURES, MATROID, AND RELATED WORK

In this section, we first introduce two new notions of curvature and review the definition of the matroid. Then we review the related literature to our study. Last, we investigate the notions of curvature from a geometric perspective.

As we shall see later from this geometric perspective, curvatures are essentially metrics to capture the mutual near-orthogonality of the vectors in the ground set. Without loss of generality, throughout the paper we assume that all elements in X are normalized, i.e., $\|t\|^2 = 1$ for any $t \in X$. Let $t^\perp(E)$ and $\bar{t}(E)$ be the normalized orthogonal and parallel components of t with respect to $\text{span}(E)$ (simplified as t^\perp and \bar{t} unless otherwise specified):

$$t = t^\perp \sin \varphi + \bar{t} \cos \varphi,$$

where φ denotes the angle between t and $\text{span}(E)$.

We define the **forward elemental curvature**, denoted by $\hat{\kappa}$, as follows:

$$\hat{\kappa} = \max_{E,s,t} \frac{\|\mathcal{P}_\eta(E \cup \{s, t\})\|^2 - \|\mathcal{P}_\eta(E \cup \{s\})\|^2}{\|\mathcal{P}_\eta(E \cup \{t\})\|^2 - \|\mathcal{P}_\eta(E)\|^2} \quad (3)$$

subject to $E \subset X, s, t \in X \setminus E, \text{card}(E) \leq 2K - 2,$
and $\|\mathcal{P}_\eta(\{s^\perp(E)\})\| \leq \|\mathcal{P}_\eta(\{t^\perp(E)\})\|.$

Similarly, we define the **backward elemental curvature**, denoted by $\bar{\kappa}$ as follows:

$$\bar{\kappa} = \max_{E,s,t} \frac{\|\mathcal{P}_\eta(E \cup \{s, t\})\|^2 - \|\mathcal{P}_\eta(E \cup \{s\})\|^2}{\|\mathcal{P}_\eta(E \cup \{t\})\|^2 - \|\mathcal{P}_\eta(E)\|^2} \quad (4)$$

subject to $E \subset X, \text{card}(E) \leq 2K - 2, s, t \in X \setminus E,$
and $\|\mathcal{P}_\eta(\{s^\perp(E)\})\| \geq \|\mathcal{P}_\eta(\{t^\perp(E)\})\|.$

The **OMP elemental curvature** is defined as follows:

$$\tilde{\kappa} = \max_{E,s,t} \frac{\|\mathcal{P}_\eta(E \cup \{s, t\})\|^2 - \|\mathcal{P}_\eta(E \cup \{s\})\|^2}{\|\mathcal{P}_\eta(E \cup \{t\})\|^2 - \|\mathcal{P}_\eta(E)\|^2}. \quad (5)$$

subject to $E \subset X, \text{card}(E) \leq 2K - 2, s, t \in X \setminus E,$
and $|\langle \eta^\perp(E), s \rangle| \geq |\langle \eta^\perp(E), t \rangle|.$

Notice that all these curvatures are ratios of *differences* of the discrete function, analogous to second-order derivative of a continuous function. In particular, if all the elements in X are mutually orthogonal, then $\hat{\kappa} = \bar{\kappa} = \tilde{\kappa} = 1$. Moreover, it is easy to show that the objective function in (1) is always monotone: Suppose that $S \subset T \subset X$. Then, by definition, $\text{span}(S)$ is a subspace of $\text{span}(T)$. Thus we have

$$\|\mathcal{P}_\eta(S)\|^2 \leq \|\mathcal{P}_\eta(T)\|^2,$$

which indicates that these curvatures are always non-negative. These curvatures are essentially ratios of the step-wise gains between consecutive stages, which can be seen as quantitative measures of the diminishing-return property (introduced formally in the next subsection). The diminishing-return property, meaning that gains decrease toward a later stage of a heuristic algorithm, allows us to bound the performance of the greedy strategy with respect to that of the optimal solution. We also note that some other papers introduced different notions of curvatures that involve gains for multiple stages. We adopt the terminology ‘‘elemental’’ from [32] to highlight that our definitions only involve step-wise gains. Also, note that the denominators for the forward and backward elemental curvatures are different. More specifically, the added elements of the nominator and denominator are identical for the forward elemental curvature but different for backward elemental curvature.

Next we state the definition of **matroid**. Let I be a collection of subsets of X . We call (X, I) a matroid [34] if it has the *hereditary* property: For any $S \subset T \subset X, T \in I$ implies that $S \in I$; and the *augmentation* property: For any $S, T \in I$, if T has a larger cardinality than S , then there exists $j \in T \setminus S$ such that $S \cup \{j\} \in I$. Furthermore, we call (X, I) a *uniform* matroid if $I = \{S \subset X : \text{card}(S) \leq K\}$ for a given K , where $\text{card}(S)$ denotes the cardinality of S . Otherwise, (X, I) is a *non-uniform*

matroid. The structure of a matroid captures the feasible combinatorial solutions within the power set of the ground set. Take the sensor selection problem as an example, a uniform matroid constraint means that we can choose any combination of K sensors from all the sensors for the solution; a non-uniform matroid constraint means that only certain combinations of K sensors are feasible solutions. Similarly, in many compressed sensing applications such as [35], we might have some prior knowledge that not all combinations of sparsity locations are feasible solutions.

We note that K corresponds to the maximal cardinality of all the elements in I , even if the matroid is non-uniform. The augmentation property guarantees that starting from the empty set, there exists at least one path for the iterative algorithm to evolve from the empty set to the set with the maximal cardinality K . As we shall see later, most of the objective functions are monotone in the sense that larger sets lead to better gains, which means that the solutions in the monotone cases given by the greedy algorithms always have cardinality K .

A. Related Work

We first review the notion of submodular set function. Let X be a ground set and $f : 2^X \rightarrow \mathbb{R}$ be a function defined on the power set 2^X . We call that f is submodular if

- 1) f is non-decreasing: $f(A) \leq f(B)$ for all $A \subset B$;
- 2) $f(\emptyset) = 0$ where \emptyset denotes the empty set (note that we can always substitute f by $f - f(\emptyset)$ if $f(\emptyset) \neq 0$);
- 3) f has the diminishing-return property: For all $A \subset B \subset X$ and $j \in X \setminus B$, we have $f(A \cup \{j\}) - f(A) \geq f(B \cup \{j\}) - f(B)$.

The optimization problem that aims to find a set in the matroid to maximize a submodular function is in general not tractable. Many papers have studied the greedy algorithm as an alternative: starting with an empty set, incrementally add one more element that maximizes the local gain of the objective function to the current solution, while the updated solution still lies in the matroid. Existing studies have shown that the greedy algorithm approximates the optimal solution well. More specifically, Nemhauser *et al.* [36] showed that the greedy algorithm achieves at least a $(1 - e^{-1})$ -approximation for a uniform matroid. Fisher *et al.* [37] proved that the greedy algorithm provides at least a $1/2$ -approximation of the optimal solution for a non-uniform matroid. Moreover, let κ_t be the total curvature of function f , which is defined as

$$\kappa_t = \max_{j \in X} \left\{ 1 - \frac{f(X) - f(X \setminus \{j\})}{f(\{j\}) - f(\emptyset)} \right\}.$$

Conforti and Cornuéjols [38] showed that the greedy algorithm achieves at least $\frac{1}{\kappa_t}(1 - e^{-\kappa_t})$ and $\frac{1}{1+\kappa_t}$ -approximations of the optimal solution for uniform and non-uniform matroids, respectively. Note that $\kappa_t \in [0, 1]$ for any submodular function, and the greedy algorithm is optimal when $\kappa_t = 0$. Vondrák [39] showed that the *continuous greedy algorithm* achieves at least a $\frac{1}{\kappa_t}(1 - e^{-\kappa_t})$ -approximation for any matroid. On the other hand, Wang *et al.* [32] provided approximation bounds for the greedy algorithm as a function of elemental curvature, which

generalizes the notion of diminishing return and is defined as

$$\kappa_e = \max_{E \subset X, i, j \in X \setminus E, i \neq j} \frac{f(E \cup \{i, j\}) - f(E \cup \{j\})}{f(E \cup \{i\}) - f(E)}.$$

Note that the objective function is submodular if and only if $\kappa_e \leq 1$. When $\kappa_e < 1$, the lower bound for greedy approximation is greater than $(1 - e^{-1})$. If $\kappa_e > 1$, then the objective function is not submodular. In this case, lower bound for the greedy algorithm is derived as a function of the elemental curvature. In [40] and [41], Zhang *et al.* generalized the notions of total curvature and elemental curvature to *string submodular functions* where the objective function value depends on the order of the elements in the set. This framework is further extended to approximate dynamic programming problems by Liu *et al.* in [42].

We use $|i\rangle$ to denote the orthonormal basis elements of the Hilbert space, $i = 0, 1, \dots$. The objective function in (1) is not submodular in general. For example, let $\eta = |0\rangle$, $s = |1\rangle$, and $t = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$. Then we have

$$\begin{aligned} \|\mathcal{P}_\eta(\{t\})\|^2 - \|\mathcal{P}_\eta(\emptyset)\|^2 &= \frac{1}{4}; \\ \|\mathcal{P}_\eta(\{s, t\})\|^2 - \|\mathcal{P}_\eta(\{s\})\|^2 &= 1 > \frac{1}{4}. \end{aligned}$$

Evidently the diminishing return property does not hold in this case. In fact, the diminishing return property does not always hold even if all the elements in the ground set are mutually orthogonal. Therefore, the results from classical submodularity theory (e.g., [36] and [37]) are not directly applicable to our problem. To address this issue, several notions of *approximation submodularity* are introduced to bound the greedy algorithm performance. Cevher and Krause [43] showed that the greedy algorithm achieves a good approximation for sparse approximation problems using the approach of approximation submodularity. Das and Kempe [44] improved the approximation bound by introducing the notion of *submodularity ratio*. These are powerful results, but with limited extension to non-uniform matroid structures. In this paper, we will use the aforementioned notions of curvature to bound the performance of forward regression and orthogonal matching pursuit with respect to the optimal solution even if the matroid is non-uniform. Specifically, we have shown that if the vectors in the ground set are mutually orthogonal, then these two algorithms are solutions to (1) when the matroid is uniform and they achieve at least $1/2$ -approximations of the optimal solution when the matroid is non-uniform. In the case where the matroid is uniform and the vectors in the ground set are not mutually orthogonal, we have shown that the forward regression and orthogonal matching pursuit achieve at least $(1 - (1 - \frac{1}{K})^K)$ and $(1 - (1 - \frac{\sin^2 \phi}{K})^K)$ -approximations of the optimal solution, respectively, where K is the number of iterations of the algorithm, \hat{K} is a number that depends on K and the forward and backward elemental curvatures, and ϕ is the principal angle. In the non-uniform matroid case, we have also derived these performance bounds as functions of elemental curvatures and principal angle.

B. Geometric Interpretation of Curvatures

To understand the curvatures from a geometric perspective, we define the *principal angle* as follows:

$$\phi = \min_{E \subset X, |E| \leq 2K-2, s \in X \setminus E} \arccos \|\mathcal{P}_s(E)\|,$$

where $\phi \in [0, \pi/2]$. Geometrically speaking, this is saying that the angle between the subspace spanned by any subset E (with cardinality less than or equal to $2K - 2$) and any element in the set $X \setminus E$ is not smaller than ϕ . Note that if all the elements in X are mutually orthogonal, then $\phi = \pi/2$.

We now investigate the relationship between the principal angle and two widely used conditions in compressed sensing to quantify the performance of recovery algorithms, namely restricted isometry and mutual incoherence. Let $\mathbf{H} = [h_1, h_2, \dots, h_m]$ be the matrix associated with $E = \{h_1, h_2, \dots, h_m\}$. It is easy to see that

$$\begin{aligned} \cos \phi &= \max_{E \subset X, |E| \leq 2K-2, s \in X \setminus E} \|\mathcal{P}_s(E)\| \\ &= \max_{E \subset X, |E| \leq 2K-2, s \in X \setminus E} \|\mathbf{H}(\mathbf{H}^T \mathbf{H})^{-1} \mathbf{H}^T s\| \\ &\leq \max_{E \subset X, |E| \leq 2K-2, s \in X \setminus E} \|\mathbf{H}(\mathbf{H}^T \mathbf{H})^{-1}\| \|\mathbf{H}^T s\|. \end{aligned}$$

The last inequality is by the Cauchy-Schwarz inequality. Moreover, we have

$$\begin{aligned} \|\mathbf{H}(\mathbf{H}^T \mathbf{H})^{-1}\| &= \sup_{\|x\|=1} \|\mathbf{H}(\mathbf{H}^T \mathbf{H})^{-1} x\| \\ &= \sqrt{\lambda_{\max}((\mathbf{H}(\mathbf{H}^T \mathbf{H})^{-1})^T \mathbf{H}(\mathbf{H}^T \mathbf{H})^{-1})} \\ &= \sqrt{\lambda_{\max}(\mathbf{H}^T \mathbf{H})^{-1}} \\ &= \left(\sqrt{\lambda_{\min}(\mathbf{H}^T \mathbf{H})} \right)^{-1}. \end{aligned}$$

and

$$\|\mathbf{H}^T s\| = \left(\sum_{i=1}^m \langle h_i, s \rangle^2 \right)^{\frac{1}{2}}.$$

Thus, we have

$$\begin{aligned} \cos \phi &\leq \\ &\max_{E \subset X, |E| \leq 2K-2, s \in X \setminus E} \left(\sqrt{\lambda_{\min}(\mathbf{H}^T \mathbf{H})} \right)^{-1} \left(\sum_{i=1}^m \langle h_i, s \rangle^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (6)$$

Here $\lambda_{\min}(\mathbf{H}^T \mathbf{H})$ denotes the minimum eigenvalue of the correlation matrix $\mathbf{H}^T \mathbf{H}$. We know that for a matrix A that satisfies the restricted isometry property with constant δ_K , the following condition holds: for any submatrix A_K that consists of K columns of A , we have

$$1 - \delta_K \leq \lambda_{\min}(A_K^* A_K) \leq \lambda_{\max}(A_K^* A_K) \leq 1 + \delta_K.$$

We also know that the summation term in (6) is monotone with respect to m and therefore the maximum is achieved when $m = \|E\| = 2K - 2$. The inner product term in the summation

is upper bounded by the mutual incoherence μ . Therefore, we have

$$\cos \phi \leq \frac{\mu \sqrt{2K-2}}{\sqrt{1 - \delta_{2K-2}}}. \quad (7)$$

In the literature on compressed sensing, there have been many well-known results giving conditions on the restricted isometry constant and mutual coherence for exact recovery. For example, Candes [45] showed that the condition $\delta_{2K} < \sqrt{2} - 1$ guarantees the exact recovery for K -sparse signal using L_1 minimization. Cai *et al.* [46] showed that if $K < \frac{1}{2} \left(1 + \frac{1}{\mu}\right)$, then there is a unique solution to the sparse recovery problem. Substituting the restricted isometry constant and the inequality in (7), we can show that the cosine of the principal angle scales in the order of $K^{\frac{1}{2}}$. Therefore, for typical choices of restricted isometry constant and mutual coherence, the principal angle is usually close to $\pi/2$. We will assume that $\phi \in (\pi/3, \pi/2]$ throughout the rest of this paper for simplicity.

We have now established the relationship between the principal angle, restricted isometry constant, and mutual coherence. Next we present a result that bridges curvatures and principal angle.

Theorem 1: If $\phi > \pi/3$, then the forward and backward elemental curvatures are both upper bounded as:

$$\max(\hat{\kappa}, \bar{\kappa}) \leq \frac{1}{1 - 2 \cos \phi}.$$

The proof is given in Appendix A. This result is important in the cases where the curvatures are difficult to calculate. We can use the principal angle, or an upper bound for the principal angle such as (6) to bound the curvature, which in turn provides performance bounds for forward regression and orthogonal matching pursuit.

We can also provide an upper bound for OMP elemental curvature using principal angles. Note that $|\langle \eta^\perp, s \rangle| \geq |\langle \eta^\perp, t \rangle|$ implies that

$$\frac{|\langle \eta, t^\perp \rangle|}{|\langle \eta, s^\perp \rangle|} = \frac{|\langle \eta^\perp, t^\perp \rangle|}{|\langle \eta^\perp, s^\perp \rangle|} \leq \frac{1}{\sin \phi}.$$

We can show that $\tilde{\kappa}$ is upper bounded as

$$\tilde{\kappa} \leq \frac{(\sin^{-2} \phi + \langle t^\perp, s^\perp \rangle)^2}{1 - \langle t^\perp, s^\perp \rangle^2}.$$

Similar to the technique in Theorem 1, we can further bound the curvature using (13).

Next we give a concrete example to demonstrate these connections. Consider an equiangular tight frame consisting of a finite sequence of N vectors in an M -dimensional Hilbert space. It has been shown in [47] that for any $\delta_K < 1$, the equiangular tight frame exhibits the restricted isometry property with constant δ_K if

$$K \leq 1 + \delta_K \left(\frac{M(N-1)}{N-M} \right)^{\frac{1}{2}}. \quad (8)$$

We also know that the mutual coherence of the equiangular tight frame achieves the *Welch bound* :

$$\mu = \left(\frac{N - M}{M(N - 1)} \right)^{\frac{1}{2}}.$$

Combining this with (7), we have

$$\cos \phi \leq \left(\frac{(N - M)(2K - 2)}{M(N - 1)(1 - \delta_{2K-2})} \right)^{\frac{1}{2}}. \quad (9)$$

Note that when $N \geq 2M$, we have $1 \leq \frac{N-1}{N-M} \leq 2$, which means that the maximum permissible choice of K in (8) scales on the order of $M^{\frac{1}{2}}$. Hence, the upper bound in (9) scales on the order of $M^{-\frac{1}{4}}$, which means that as M grows, the angle goes to $\pi/2$ at the rate of $M^{-\frac{1}{4}}$. Then combining this scaling law and (9) with Theorem 1, we obtain an upper bound close to 1 for the elemental curvatures.

Next we study the performance of forward regression and orthogonal matching pursuit with uniform and non-uniform matroid constraints.

III. RESULTS FOR UNIFORM MATROID

In this section, we will focus on the case where the matroid is uniform, i.e., $I = \{S \subset X : \text{card}(S) \leq K\}$ for a given K . We consider two scenarios depending on the mutual orthogonality of elements in X .

A. Orthogonal Scenario

We call the set X mutually orthogonal if any two non-identical elements in X are orthogonal: $\langle s|t \rangle = 0$ for any $s \neq t \in X$. It is easy to show that forward regression and orthogonal matching pursuit are equivalent given that X is mutually orthogonal. It turns out that the optimality of these two algorithms is closely related with the mutual orthogonality of X .

Theorem 2: Suppose that X is mutually orthogonal. If (X, I) is a uniform matroid, then forward regression and orthogonal matching pursuit are solutions to (1).

The proof is given in Appendix B. Theorem 2 implies that the optimality of forward regression and orthogonal matching pursuit is guaranteed if we find an orthonormal basis for X . The Gram-Schmidt process can be used to generate an orthonormal basis using the elements in X . However, this is, in general, intractable especially when $\text{card}(X)$ is large. Moreover, the problem of optimally selecting K elements in X is different from the problem of optimally selecting K orthogonalized elements after applying the Gram-Schmidt process.

Mutual orthogonality depends on the definition of inner product in the Hilbert space. For example, the Hilbert space defined on Gaussian measures has an orthonormal basis: Hermite polynomials. Some other well-known examples include Charlier polynomials for Poisson measures, Laguerre polynomials for Gamma measures, Legendre and Fourier polynomials for uniform measures.

The physical meaning of mutual orthogonality differs depending on the context of the problem. Take the quadratic covariance bound problem for example and consider the uniform

distribution parameterized by its mean θ : Uniform $[-\pi + \theta, \pi + \theta]$. The Cramer-Rao bound is not applicable here because the derivative of the probability density function is not well-defined. On the other hand, the Fourier basis $\{\cos(m(x - \theta))\}_{m=1}^{\infty}$ is a well-defined orthonormal basis. These basis functions can be considered as energy eigenstates for a quantum particle in an infinite potential well. Another example is the Bhattacharya bound with the following Bhattacharya score functions:

$$\left\{ \frac{\partial \ln d(x, \theta)}{\partial \phi}, \frac{\partial^2 \ln d(x, \theta)}{\partial \phi^2}, \dots, \frac{\ln \partial^k d(x, \theta)}{\partial \phi^k} \dots \right\},$$

where $d(x, \theta)$ denotes the probability density function for the measurement x . In general, these score functions are not orthonormal. Moreover, the projection of the estimator error onto the first order partial derivative is not necessarily the largest, meaning that the Fisher score is not necessarily the optimal. However, in the Gaussian measure case, the Bhattacharya score functions turn out to be the Hermit polynomials and therefore are mutually orthogonal. For the LMMSE problem, mutual orthogonality means that all the sensor measurements are mutually *uncorrelated*. Therefore, if all the sensors generate independent measurement signals, then forward regression and orthogonal matching pursuit are optimal in the uniform matroid case. For the sparse approximation problem, mutual orthogonality says that all the columns in the measurement matrix are mutually orthogonal, which cannot be true in the case of the under-determined system.

B. Non-orthogonal Scenario

When X is not mutually orthogonal, forward regression and orthogonal matching pursuit are in general not optimal. We give a counter example for forward regression; a similar counter example can be given for orthogonal matching pursuit. Let $X = \{s_1, s_2, s_3\}$ where $s_1 = \frac{\sqrt{2}}{2}(|0\rangle + |1\rangle)$, $s_2 = \frac{\sqrt{2}}{2}(|1\rangle + |2\rangle)$, and $s_3 = \frac{\sqrt{2}}{2}(|2\rangle + |3\rangle)$. Suppose that $\eta = |0\rangle + 2|1\rangle + 2|2\rangle + |3\rangle$, and the objective is to choose a subset E of X with $\text{card}(E) \leq 2$ such that the projection of η onto $\text{span}(E)$ is maximized. Obviously, the optimal solution is to choose s_1 and s_3 and the maximum projection is

$$\|\mathcal{P}_\eta(\{s_1, s_3\})\|^2 = \langle \eta, s_1 \rangle^2 + \langle \eta, s_3 \rangle^2 = 9.$$

Forward regression, however, is fooled into picking s_2 first because along s_2 it has the largest projection. After that, it chooses either s_1 or s_3 . By the Gram-Schmidt process, the normalized orthogonal component of s_1 with respect to s_2 is given by

$$s_1^\perp = \frac{s_1 - \langle s_1, s_2 \rangle s_2}{\|s_1 - \langle s_1, s_2 \rangle s_2\|} = \sqrt{\frac{4}{3}} \left(\frac{\sqrt{2}}{2} |0\rangle + \frac{\sqrt{2}}{4} |1\rangle - \frac{\sqrt{2}}{4} |2\rangle \right).$$

Therefore,

$$\begin{aligned} \|\mathcal{P}_\eta(\{s_1, s_2\})\|^2 &= \|\mathcal{P}_\eta(\{s_1^\perp, s_2\})\|^2 \\ &= \langle \eta, s_2 \rangle^2 + \langle \eta, s_1^\perp \rangle^2 = 8 + \frac{2}{3}. \end{aligned}$$

Apparently, forward regression is not optimal. Moreover, if X is not mutually orthogonal, then the two algorithms yield different results, which we discuss in separate subsections.

1) *Forward Regression*: We first study forward regression when the matroid is uniform with the maximal cardinality of the sets in I equal to K . We use G_K to denote the solution using forward regression and OPT to denote the optimal solution. We will often use $f(E)$ to represent $\|\mathcal{P}_\eta(E)\|^2$ throughout the rest of the paper to simplify notation.

Theorem 3 (Uniform matroid): The forward regression algorithm achieves at least a $(1 - (1 - \frac{1}{\hat{K}})^K)$ -approximation of the optimal solution:

$$f(G_K) \geq \left(1 - \left(1 - \frac{1}{\hat{K}}\right)^K\right) f(\text{OPT}), \quad (10)$$

where $\hat{K} = \sum_{i=1}^K \min(\bar{\kappa}, \hat{\kappa})^{i-1}$ and the curvatures $\bar{\kappa}$ and $\hat{\kappa}$ are defined in (3) and (4), respectively.

The proof is given in Appendix C. When $\min(\bar{\kappa}, \hat{\kappa}) \leq 1$, the forward regression algorithm achieves at least a $(1 - 1/e)$ -approximation of the optimal solution.

2) *Orthogonal Matching Pursuit*: We first compare the step-wise gains in the objective function between orthogonal matching pursuit and forward regression. Recall that η^\perp and $\bar{\eta}$ represent the normalized orthogonal and parallel components of η with respect to $\text{span}(E)$:

$$\eta = \eta^\perp \sin \varphi + \bar{\eta} \cos \varphi,$$

where φ denotes the angle between η and $\text{span}(E)$. The orthogonal matching pursuit algorithm aims to find an element t to maximize $|\langle \eta^\perp, t \rangle|$. The forward regression algorithm aims to find an element s to maximize $|\langle \eta, s^\perp \rangle|$, where s^\perp denotes the normalized orthogonal component of s with respect to $\text{span}(E)$. Suppose that the angle between s and $\text{span}(E)$ is $\delta(s)$. Note that $\delta(s)$ is lower bounded by the principal angle ϕ by definition. By the fact that

$$\begin{aligned} & \max_{s \in X \setminus E} \langle \eta^\perp, s \rangle^2 \\ &= \max_{s \in X \setminus E} \langle \eta^\perp, s^\perp \sin \delta(s) \rangle^2 \\ &\geq \sin^2 \phi \max_{s \in X \setminus E} \langle \eta^\perp, s^\perp \rangle^2 \\ &\geq \sin^2 \phi \max_{s \in X \setminus E} \langle \eta, s^\perp \rangle^2, \end{aligned} \quad (11)$$

even though orthogonal matching pursuit is not the ‘‘greediest’’ algorithm, its step-wise gain is still within a certain range of that of forward regression, captured by the principal angle. With this observation, we can derive a performance bound for orthogonal matching pursuit. Again, we assume that the matroid is uniform with the maximal cardinality of the sets in I equal to K . We use T_K to denote the solution using orthogonal matching pursuit.

Theorem 4 (Uniform matroid): The orthogonal matching pursuit algorithm achieves at least a $(1 - (1 - \frac{\sin^2 \phi}{\hat{K}})^K)$ -approximation of the optimal solution:

$$f(T_K) \geq \left(1 - \left(1 - \frac{\sin^2 \phi}{\hat{K}}\right)^K\right) f(\text{OPT}), \quad (12)$$

where $\hat{K} = \sum_{i=1}^K \min(\bar{\kappa}, \hat{\kappa})^{i-1}$ and the curvatures $\bar{\kappa}$ and $\hat{\kappa}$ are defined in (3) and (4), respectively.

The proof is given in Appendix D. Notice that the difference between Theorem 3 and Theorem 4 is only the principal angle term $\sin^2 \phi$. It is easy to see that the lower bound in (12) is always lower than that in (10), but this does not necessarily mean that $f(T_k) \leq f(G_k)$.

IV. RESULTS FOR NON-UNIFORM MATROID

For non-uniform matroids, the two algorithms are not necessarily optimal even when X is mutually orthogonal. As a counter example, suppose that $X = \{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$ and $I = \{\{|0\rangle\}, \{|1\rangle\}, \{|2\rangle\}, \{|3\rangle\}, \{|0\rangle, |1\rangle\}, \{|2\rangle, |3\rangle\}\}$. It is easy to verify that (X, I) is a non-uniform matroid. Let $\eta = \sqrt{1+\epsilon}|0\rangle + |2\rangle + |3\rangle$ be the vector of interest, where $\epsilon > 0$. Forward regression ends up with $\{|0\rangle, |1\rangle\}$ while the optimal solution is $\{|2\rangle, |3\rangle\}$. However, notice that

$$\frac{\|\mathcal{P}_\eta(\{|0\rangle, |1\rangle\})\|^2}{\|\mathcal{P}_\eta(\{|2\rangle, |3\rangle\})\|^2} = (1 + \epsilon)/2 > 1/2.$$

In this section, we will show that $1/2$ is essentially a lower bound for the performance of these two algorithms with respect to that of the optimal solution when the ground set is mutually orthogonal and the matroid is non-uniform. This bound surprisingly matches the bound in [37]. However, a significant distinction is that in our paper the submodularity of the objective function is no longer necessary (which is required by [37]).

Next we derive performance bounds for forward regression and orthogonal matching pursuit in the situation where (X, I) is a non-uniform matroid.

A. Forward Regression

In this section, we state the result for forward regression with the non-uniform matroid constraint.

Theorem 5 (Non-uniform matroid): The forward regression algorithm achieves at least a $\frac{1}{1 + a(\bar{\kappa}, \bar{\kappa})b(\hat{\kappa})}$ -approximation of the optimal solution:

$$f(G_K) \geq \frac{1}{1 + a(\hat{\kappa}, \bar{\kappa})b(\hat{\kappa})} f(\text{OPT}),$$

where $a(\hat{\kappa}, \bar{\kappa}) = \max(\hat{\kappa}, \bar{\kappa})$ if $\max(\hat{\kappa}, \bar{\kappa}) \leq 1$ and $a(\hat{\kappa}, \bar{\kappa}) = \max(\hat{\kappa}, \bar{\kappa})^K$ if $\max(\hat{\kappa}, \bar{\kappa}) > 1$; $b(\hat{\kappa}) = \hat{\kappa}^{K-1}$ if $\hat{\kappa} > 1$ and $b(\hat{\kappa}) = 1$ if $\hat{\kappa} \leq 1$. Note that the curvatures $\bar{\kappa}$ and $\hat{\kappa}$ are defined in (3) and (4), respectively.

The proof is given in Appendix E.

B. Orthogonal Matching Pursuit

Next we derive the bound for orthogonal matching pursuit for the case where (X, I) is a non-uniform matroid.

Theorem 6 (Non-uniform matroid): The orthogonal matching pursuit achieves at least a $\frac{1}{1 + a(\hat{\kappa}, \bar{\kappa}, \bar{\kappa})b(\hat{\kappa})\sin^{-2} \phi}$ -approximation of the optimal solution:

$$f(T_K) \geq \frac{1}{1 + a(\hat{\kappa}, \bar{\kappa}, \bar{\kappa})b(\hat{\kappa})\sin^{-2} \phi} f(\text{OPT}),$$

TABLE I
THE PERFORMANCE LOWER BOUND FOR **FORWARD REGRESSION** WITH
RESPECT TO THE OPTIMAL SOLUTION IN DIFFERENT SCENARIOS

	Uniform Matroid	Non-uniform Matroid
Orthogonal	1	1/2
Non-Orthogonal	$1 - \left(1 - \frac{1}{\sum_{i=1}^K \min(\bar{\kappa}_i, \tilde{\kappa}_i)^{i-1}}\right)^K$	$\frac{1}{1+a(\hat{\kappa}, \bar{\kappa})b(\hat{\kappa})}$

TABLE II
THE PERFORMANCE LOWER BOUND FOR **ORTHOGONAL MATCHING**
PURSUIT WITH RESPECT TO THE OPTIMAL SOLUTION IN
DIFFERENT SCENARIOS

	Uniform Matroid	Non-uniform Matroid
Orthogonal	1	1/2
Non-Orthogonal	$1 - \left(1 - \frac{\sin^2 \phi}{\sum_{i=1}^K \min(\bar{\kappa}_i, \tilde{\kappa}_i)^{i-1}}\right)^K$	$\frac{1}{1+a(\hat{\kappa}, \bar{\kappa})b(\hat{\kappa})\sin^{-2} \phi}$

where $a(\hat{\kappa}, \bar{\kappa}, \tilde{\kappa}) = \max(\hat{\kappa}, \bar{\kappa}, \tilde{\kappa})$ if $\max(\hat{\kappa}, \bar{\kappa}, \tilde{\kappa}) \leq 1$ and $a(\hat{\kappa}, \bar{\kappa}, \tilde{\kappa}) = \max(\hat{\kappa}, \bar{\kappa}, \tilde{\kappa})^K$ if $\max(\hat{\kappa}, \bar{\kappa}, \tilde{\kappa}) > 1$; $b(\hat{\kappa}) = \hat{\kappa}^{K-1}$ if $\hat{\kappa} > 1$ and $b(\hat{\kappa}) = 1$ otherwise. Note that the curvatures $\bar{\kappa}$, $\hat{\kappa}$, and $\tilde{\kappa}$ are defined in (3), (4), and (5), respectively.

The proof is given in Appendix F. Note that when X is mutually orthogonal, $\sin \phi = \max(\hat{\kappa}, \bar{\kappa}, \tilde{\kappa}) = 1$. An immediate result follows.

Corollary 1: Suppose that X is mutually orthogonal. Then,

- 1) Forward regression is equivalent to orthogonal matching pursuit;
- 2) If I is a non-uniform matroid, then forward regression achieves at least a 1/2-approximation of the optimal solution, by which we mean that

$$f(G_K) \geq \frac{1}{2}f(\text{OPT}).$$

Recall that when X is mutually orthogonal, we have shown in Section II-A that these two algorithms are optimal when (X, I) is a uniform matroid. For a non-uniform matroid, they are not necessarily optimal. However, these two algorithms achieve at least 1/2-approximations of the optimal solution. Our results extend those in [37] from a submodular function to a more general class of objective functions.

We have stated that the near-optimality of the greedy strategies is closely related to the near-orthogonality of the vectors in the ground set and the structure of the matroid. Here, in Tables I and II, we summarize all the lower bounds with respect to the optimal solution for forward regression and orthogonal matching pursuit, respectively. Specifically for the non-orthogonal case, the level of near-orthogonality is captured by the principal angle, which in turn determines the curvatures and the lower bounds as functions of these curvatures.

Suppose that X is not mutually orthogonal but close in the sense that the principal angle ϕ almost equal to $\pi/2$. We use $\delta = \pi/2 - \phi$ to denote the gap between ϕ and $\pi/2$. Moreover, we assume that the curvatures are larger but close to 1 and δ is sufficiently small such that we only have to keep first order terms for Taylor expansions (for example, the equiangular tight frame discussed in Section II):

$$\frac{1}{1 - 2 \cos(\pi/2 - \delta)} \approx 1 + 2\delta,$$

and

$$\hat{\kappa}^{K-1} \approx 1 + (K-1)|\hat{\kappa} - 1|.$$

Then, in the case of non-uniform matroid constraints, the lower bounds in Theorems 5 and 6 for the aforementioned algorithms scale as

$$\frac{1}{2 + 2(2K-1)\delta},$$

which indicates that the lower bound scales inverse linearly with cardinality constraints K and the principal angle gap δ with $\pi/2$. Fortunately, K is mostly a small number (for example, the number of sparsity locations in compressive sensing problem).

V. CONCLUSION

In this paper, we have studied the subspace selection problem for maximizing the projection of a vector of interest. We have introduced several new notions of elemental curvatures, upper bounded by functions of principal angle. We then derived explicit lower bounds for the performance of forward regression and orthogonal matching pursuit in the cases of uniform and non-uniform matroids. Moreover, we showed that if the elements in the ground sets are mutually orthogonal, then these algorithms are essentially optimal under the uniform matroid constraint and they achieve at least 1/2 approximations of the optimal solution under the non-uniform matroid constraint.

APPENDIX A PROOF OF THEOREM 1

Proof: First consider a subset E of X , and two elements s and t in the set $X \setminus E$. Assuming that all the elements in X are normalized, we know that $|\langle s, t \rangle| \leq \cos \phi$ by the definition of the principal angle. We decompose the two elements into their normalized parallel and orthogonal components with respect to $\text{span}(E)$, respectively. Let us assume that ϕ_1 and ϕ_2 are the angles between s , t and $\text{span}(E)$, respectively, then we have

$$s = \cos \phi_1 \bar{s} + \sin \phi_1 s^\perp,$$

$$\text{and } t = \cos \phi_2 \bar{t} + \sin \phi_2 t^\perp,$$

where \bar{s} and \bar{t} represent the normalized parallel components of s and t with respect to $\text{span}(E)$, s^\perp and t^\perp represent the normalized orthogonal components of s and t with respect to $\text{span}(E)$. We know that

$$\begin{aligned} \langle s, t \rangle &= \langle \cos \phi_1 \bar{s} + \sin \phi_1 s^\perp, \cos \phi_2 \bar{t} + \sin \phi_2 t^\perp \rangle \\ &= \cos \phi_1 \cos \phi_2 \langle \bar{s}, \bar{t} \rangle + \sin \phi_1 \sin \phi_2 \langle s^\perp, t^\perp \rangle. \end{aligned}$$

From the above we obtain

$$\begin{aligned} |\langle s^\perp, t^\perp \rangle| &= \left| \frac{\langle s, t \rangle - \cos \phi_1 \cos \phi_2 \langle \bar{s}, \bar{t} \rangle}{\sin \phi_1 \sin \phi_2} \right| \\ &\leq \frac{\cos \phi + \cos^2 \phi}{\sin^2 \phi} < 1, \end{aligned} \quad (13)$$

where the last inequality comes from the assumption that $\phi > \pi/3$.

For the numerator and denominator in the definitions of curvature, using Pythagoras' theorem, it is easy to show that the numerator and denominator of $\hat{\kappa}$ can be written as:

$$\|\mathcal{P}_\eta(E \cup \{t\})\|^2 - \|\mathcal{P}_\eta(E)\|^2 = \|\mathcal{P}_\eta(\{t^\perp\})\|^2 = \langle \eta, t^\perp \rangle^2,$$

and

$$\begin{aligned} \|\mathcal{P}_\eta(E \cup \{s, t\})\|^2 - \|\mathcal{P}_\eta(E \cup \{s\})\|^2 \\ = \|\mathcal{P}_\eta(\{\hat{t}^\perp\})\|^2 = \langle \eta, \hat{t}^\perp \rangle^2, \end{aligned}$$

where \hat{t}^\perp denotes the orthonormal component of t with respect to $\text{span}(E \cup \{s\})$. By the Gram–Schmidt process, we know that

$$\hat{t}^\perp = \frac{t^\perp - \langle t^\perp, s^\perp \rangle s^\perp}{\sqrt{1 - \langle t^\perp, s^\perp \rangle^2}}.$$

Therefore, we obtain

$$\langle \eta, \hat{t}^\perp \rangle = \frac{\langle \eta, t^\perp \rangle - \langle t^\perp, s^\perp \rangle \langle \eta, s^\perp \rangle}{\sqrt{1 - \langle t^\perp, s^\perp \rangle^2}}.$$

Hence, using (13) we can provide an upper bound of the forward elemental curvature using ϕ :

$$\hat{\kappa} \leq \frac{(1 + \langle t^\perp, s^\perp \rangle)^2}{1 - \langle t^\perp, s^\perp \rangle^2} = \frac{1 + \langle t^\perp, s^\perp \rangle}{1 - \langle t^\perp, s^\perp \rangle} \leq \frac{1}{1 - 2 \cos \phi}.$$

Using a similar argument, we can provide an upper bound for the backward elemental curvature with the same form. The proof is complete. \blacksquare

APPENDIX B PROOF OF THEOREM 2

Proof: Let $E = \{e_1, \dots, e_K\}$ be a subset and η be the vector of interest. By the Hilbert projection theorem and Pythagoras' theorem, we have

$$\|\mathcal{P}_\eta(E)\|^2 = \sum_{i=1}^K \langle \eta, e_i \rangle^2.$$

It is easy to see that the optimal solution is to choose K largest projections among all vectors in X , which is the same as what the forward regression does. The insight of this result is closely related with *principle component analysis*. \blacksquare

APPENDIX C PROOF OF THEOREM 3

Proof: For any $M, N \in I$ and $|M| \leq K$ and $|N| = K$, let $J = (M \cup N) \setminus M = \{j_1, \dots, j_r\}$ where $r \leq |N|$. We can permute the elements in J such that the elements are ordered to use the forward elemental curvature. More specifically, starting from M , we iteratively find the element \bar{j}_i in J that gives the largest gain in the projection, i.e.,

$$\bar{j}_i = \underset{j \in J \setminus \{\bar{j}_1, \dots, \bar{j}_{i-1}\}}{\operatorname{argmin}} \|\mathcal{P}_\eta(\{j^\perp(M \cup \{\bar{j}_1, \dots, \bar{j}_{i-1}\})\})\|,$$

where $j^\perp(M \cup \{\bar{j}_1, \dots, \bar{j}_{i-1}\})$ denotes the normalized orthogonal component of j with respect to $\text{span}(M \cup \{\bar{j}_1, \dots, \bar{j}_{i-1}\})$.

Using the definition of forward elemental curvature, we have

$$\begin{aligned} f(M \cup N) - f(M) \\ = \sum_{i=1}^r (f(M \cup \{\bar{j}_1, \dots, \bar{j}_i\}) - f(M \cup \{\bar{j}_1, \dots, \bar{j}_{i-1}\})) \\ \leq \sum_{i=1}^r \hat{\kappa}^{i-1} (f(M \cup \{\bar{j}_i\}) - f(M)) \end{aligned}$$

Therefore, there exists $\bar{j} \in X$ such that

$$\begin{aligned} f(M \cup N) - f(M) \\ \leq \sum_{i=1}^{|N|} \hat{\kappa}^{i-1} (f(M \cup \{\bar{j}\}) - f(M)) \\ = \sum_{i=1}^{|N|} \hat{\kappa}^{i-1} (f(M \cup \{\bar{j}\}) - f(M)). \end{aligned}$$

We use G_k to denote the forward regression solution with cardinality k and OPT to denote the optimal solution. Using the properties of the forward regression algorithm and the monotone property, we have

$$\begin{aligned} f(G_i) - f(G_{i-1}) \\ \geq \frac{1}{\sum_{i=1}^K \hat{\kappa}^{i-1}} (f(G_{i-1} \cup \text{OPT}) - f(G_{i-1})) \\ \geq \frac{1}{\sum_{i=1}^K \hat{\kappa}^{i-1}} (f(\text{OPT}) - f(G_{i-1})). \end{aligned}$$

Therefore, by recursion, we have

$$\begin{aligned} f(G_K) &\geq \frac{1}{\sum_{i=1}^K \hat{\kappa}^{i-1}} f(\text{OPT}) + \left(1 - \frac{1}{\sum_{i=1}^K \hat{\kappa}^{i-1}}\right) f(G_{K-1}) \\ &= \frac{1}{\sum_{i=1}^K \hat{\kappa}^{i-1}} f(\text{OPT}) \sum_{i=0}^{K-1} \left(1 - \frac{1}{\sum_{i=1}^K \hat{\kappa}^{i-1}}\right)^i \\ &= f(\text{OPT}) \left(1 - \left(1 - \frac{1}{\sum_{i=1}^K \hat{\kappa}^{i-1}}\right)^K\right). \end{aligned}$$

Using a similar argument, we can show that

$$f(G_K) \geq f(\text{OPT}) \left(1 - \left(1 - \frac{1}{\sum_{i=1}^K \bar{\kappa}^{i-1}}\right)^K\right).$$

Combining these two inequalities, the proof is complete. \blacksquare

APPENDIX D PROOF OF THEOREM 4

Proof: For any $M, N \in I$ and $|M| \leq K$ and $|N| = K$, let $J = (M \cup N) \setminus M = \{j_1, \dots, j_r\}$ where $r \leq |N|$. We can permute the elements in J such that the elements are ordered to use the forward elemental curvature. More specifically, let

$$\bar{j}_i = \underset{j \in J \setminus \{\bar{j}_1, \dots, \bar{j}_{i-1}\}}{\operatorname{argmin}} \|\mathcal{P}_\eta(\{j^\perp(M \cup \{\bar{j}_1, \dots, \bar{j}_{i-1}\})\})\|,$$

where $j^\perp(M \cup \{\bar{j}_1, \dots, \bar{j}_{i-1}\})$ denotes the normalized orthogonal component of j with respect the span($M \cup \{\bar{j}_1, \dots, \bar{j}_{i-1}\}$). Using the definition of forward elemental curvature, we have

$$\begin{aligned} & f(M \cup N) - f(M) \\ &= \sum_{i=1}^r (f(M \cup \{\bar{j}_1, \dots, \bar{j}_i\}) - f(M \cup \{\bar{j}_1, \dots, \bar{j}_{i-1}\})) \\ &\leq \sum_{i=1}^r \hat{\kappa}^{i-1} (f(M \cup \{\bar{j}_i\}) - f(M)) \end{aligned}$$

Therefore, there exists $\bar{j} \in X$ such that

$$\begin{aligned} & f(M \cup N) - f(M) \\ &\leq \sum_{i=1}^{|\bar{N}|} \hat{\kappa}^{i-1} (f(M \cup \{\bar{j}\}) - f(M)) \\ &= \sum_{i=1}^{|\bar{N}|} \hat{\kappa}^{i-1} (f(M \cup \{\bar{j}\}) - f(M)). \end{aligned}$$

Using a similar argument, we can show that

$$\begin{aligned} & f(M \cup N) - f(M) \\ &\leq \sum_{i=1}^{|\bar{N}|} \bar{\kappa}^{i-1} (f(M \cup \{\bar{j}\}) - f(M)). \end{aligned}$$

Using the properties of the forward regression algorithm, the monotone property, and (11), we have

$$\begin{aligned} & f(T_i) - f(T_{i-1}) \\ &\geq \sin^2 \phi (f(T_{i-1} \cup \{g^*\}) - f(T_{i-1})) \\ &\geq \frac{\sin^2 \phi}{\hat{K}} (f(T_{i-1} \cup \text{OPT}) - f(T_{i-1})) \\ &\geq \frac{\sin^2 \phi}{\hat{K}} (f(\text{OPT}) - f(T_{i-1})). \end{aligned}$$

Therefore, by recursion, we have

$$\begin{aligned} f(G_K) &\geq \frac{\sin^2 \phi}{\hat{K}} f(\text{OPT}) + \left(1 - \frac{\sin^2 \phi}{\hat{K}}\right) f(G_{K-1}) \\ &= \frac{\sin^2 \phi}{\hat{K}} f(\text{OPT}) \sum_{i=0}^{K-1} \left(1 - \frac{\sin^2 \phi}{\hat{K}}\right)^i \\ &= f(\text{OPT}) \left(1 - \left(1 - \frac{\sin^2 \phi}{\hat{K}}\right)^K\right). \end{aligned}$$

APPENDIX E PROOF OF THEOREM 5

Proof: We use a similar approach as that of the proof of Theorem 3. Let $G_i = \{g_1, \dots, g_i\}$ where g_j denotes the element added in the forward regression algorithm at stage j . Let $\text{OPT} = \{o_1, \dots, o_K\}$ and assume that the elements are already

reordered such that we can use the forward elemental curvature. We know that

$$\begin{aligned} & f(G_K \cup \text{OPT}) - f(G_K) \\ &\leq \sum_{i=1}^K \hat{\kappa}^{i-1} (f(G_K \cup \{o_i\}) - f(G_K)) \\ &\leq \begin{cases} \sum_{i=1}^K (f(G_K \cup \{o_i\}) - f(G_K)), & \text{if } \hat{\kappa} \leq 1 \\ \hat{\kappa}^{K-1} \sum_{i=1}^K (f(G_K \cup \{o_i\}) - f(G_K)), & \text{if } \hat{\kappa} > 1. \end{cases} \end{aligned}$$

Before proceeding, we state a lemma that assists in handling the non-uniform matroid constraint. Let G_i and T_i be the forward regression and orthogonal matching pursuit solutions up to step i , respectively. Note that the cardinalities of G_i and T_i are i .

Lemma 1: Any $E \subset X$ with cardinality K can be ordered into $\{e_1, \dots, e_K\}$ such that for $i = 1, \dots, K$, we have

$$f(G_{i-1} \cup \{e_i\}) - f(G_{i-1}) \leq f(G_i) - f(G_{i-1}) \quad (14)$$

and

$$f(T_{i-1} \cup \{e_i\}) - f(T_{i-1}) \leq f(T_{i-1} \cup \{g^*\}) - f(T_{i-1}), \quad (15)$$

where g^* denotes the element added to T_{i-1} using the forward regression algorithm.

We prove this lemma using induction in descending order on the index i . First consider the sets E and G_{K-1} , and notice that $|E| = K > |G_{K-1}|$. By the augmentation property of matroids, there exists an element in E , denoted by e_K , such that $G_{K-1} \cup \{e_K\} \in I$. It is easy to see that $f(G_K) - f(G_{K-1}) \geq f(G_{K-1} \cup \{e_K\}) - f(G_{K-1})$. Suppose that $f(G_k) - f(G_{k-1}) \geq f(G_{k-1} \cup \{e_k\}) - f(G_{k-1})$ for all $k \geq i$; we want to show that the inequality holds for the index $i-1$. Consider G_{i-2} and $E \setminus \{e_i, \dots, e_K\}$, where e_k denotes the element in E such that the claim holds for $k = i, \dots, K$. Again by the augmentation property of matroids, there exists an element in $E \setminus \{e_i, \dots, e_K\}$, denoted by e_{i-1} , such that $G_{i-2} \cup \{e_{i-1}\} \in I$. By the property of the forward regression algorithm, we know that $f(G_{i-1}) - f(G_{i-2}) \geq f(G_{i-2} \cup \{e_{i-1}\}) - f(G_{i-2})$. This concludes the induction proof for (14). The proof for (15) follows a similar argument and it is omitted for the sake of brevity.

We state another lemma that will be used for the proof.

Lemma 2: For $i = 1, 2, \dots, K$, we have

$$f(G_i) - f(G_{i-1}) \leq \bar{\kappa} (f(G_{i-1}) - f(G_{i-2})).$$

We prove this lemma as follows: Let $G_i = \{g_1, \dots, g_i\}$ where g_j denotes the element added in the forward regression algorithm at step j . We know that $G_{i-2} \cup \{g_i\} \in I$ because of the hereditary property of the matroid. Moreover, the projection of η gains more by adding g_{i-1} than g_i at stage $i-1$ by the property of the forward regression algorithm. Then, by the definition of the backward elemental curvature, we obtain the desired result.

From Lemma 1, we know that OPT can be ordered into $\{\hat{o}_1, \dots, \hat{o}_K\}$, such that

$$f(G_{i-1} \cup \{\hat{o}_i\}) - f(G_{i-1}) \leq f(G_i) - f(G_{i-1}),$$

for $i = 1, \dots, K$. Moreover, we know that $G_{i-2} \cup \{g_i\} \in I$ because of the hereditary property of the matroid. Moreover, we know that the projection of η gains more by adding g_{i-1} than g_i at stage $i - 1$ by the property of the forward regression algorithm. Using Lemmas 1 and 2 and the definitions of forward and backward elemental curvatures, we obtain (16) shown at the bottom of the page, therefore, by recursion we have

$$\begin{aligned} & f(G_K \cup \{\hat{o}_i\}) - f(G_K) \\ & \leq \max(\hat{\kappa}, \bar{\kappa})^{K-i+1} (f(G_i) - f(G_{i-1})), \end{aligned}$$

for $i = 1, \dots, K$. Hence, we obtain

$$\begin{aligned} & \sum_{i=1}^K (f(G_K \cup \{\hat{o}_i\}) - f(G_K)) \\ & \leq \sum_{i=1}^K \max(\hat{\kappa}, \bar{\kappa})^{K-i+1} (f(G_i) - f(G_{i-1})) \\ & \leq \begin{cases} \max(\hat{\kappa}, \bar{\kappa}) f(G_K), & \text{if } \max(\hat{\kappa}, \bar{\kappa}) \leq 1 \\ \max(\hat{\kappa}, \bar{\kappa})^K f(G_K), & \text{if } \max(\hat{\kappa}, \bar{\kappa}) > 1. \end{cases} \end{aligned}$$

Therefore, we have

$$f(\text{OPT}) \leq (1 + a(\hat{\kappa}, \bar{\kappa})b(\hat{\kappa}))f(G_K),$$

where $a(\hat{\kappa}, \bar{\kappa}) = \max(\hat{\kappa}, \bar{\kappa})$ if $\max(\hat{\kappa}, \bar{\kappa}) \leq 1$ and $a(\hat{\kappa}, \bar{\kappa}) = \max(\hat{\kappa}, \bar{\kappa})^K$ if $\max(\hat{\kappa}, \bar{\kappa}) > 1$; $b(\hat{\kappa}) = \hat{\kappa}^{K-1}$ if $\hat{\kappa} > 1$ and $b(\hat{\kappa}) = 1$ otherwise. ■

APPENDIX F PROOF OF THEOREM 6

Proof: Let $\text{OPT} = \{o_1, \dots, o_K\}$ be ordered such that the elemental forward curvature can be used. We know that

$$\begin{aligned} & f(T_K \cup \text{OPT}) - f(T_K) \\ & \leq \begin{cases} \sum_{i=1}^K (f(T_K \cup \{o_i\}) - f(T_K)), & \text{if } \hat{\kappa} \leq 1 \\ \hat{\kappa}^{K-1} \sum_{i=1}^K (f(T_K \cup \{o_i\}) - f(T_K)), & \text{if } \hat{\kappa} > 1. \end{cases} \end{aligned}$$

Using Lemma 1 and (11), we know that OPT can be ordered as $\{\hat{o}_1, \dots, \hat{o}_K\}$, such that

$$\begin{aligned} & f(T_{i-1} \cup \{\hat{o}_i\}) - f(T_{i-1}) \\ & \leq f(T_{i-1} \cup \{g^*\}) - f(T_{i-1}) \\ & \leq \frac{f(T_i) - f(T_{i-1})}{\sin^2 \phi}, \end{aligned}$$

for $i = 1, \dots, K$. Next we state a lemma and its proof that we will use.

Lemma 3: For $i = 1, 2, \dots, K$, we have

$$f(T_i) - f(T_{i-1}) \leq \tilde{\kappa}(f(T_{i-1}) - f(T_{i-2})).$$

Proof of Lemma 3: For $i = 1, \dots, K$, let $T_i = \{t_1, \dots, t_i\}$ where t_j denotes the element added in the orthogonal matching pursuit algorithm at stage j . We know that $T_{i-2} \cup \{t_i\} \in I$ because of the hereditary property of the matroid. Therefore, we have $|\langle \eta^\perp | t_{i-1} \rangle| \geq |\langle \eta^\perp | t_i \rangle|$ by the property of orthogonal matching pursuit. Then, by the definition of the OMP elemental curvature, we obtain the inequality in the lemma.

By the definitions of forward and backward elemental curvatures, we obtain (17) shown at the bottom of this page, therefore, by Lemma 3 and recursion, we have

$$\begin{aligned} & f(T_K \cup \{\hat{o}_i\}) - f(T_K) \\ & \leq \frac{\max(\hat{\kappa}, \bar{\kappa}, \tilde{\kappa})^{K-i+1}}{\sin^2 \phi} (f(T_i) - f(T_{i-1})), \end{aligned}$$

for $i = 1, \dots, K$.

From this we obtain

$$\begin{aligned} & \sum_{i=1}^K (f(T_K \cup \{\hat{o}_i\}) - f(T_K)) \\ & = \sum_{i=1}^K \frac{\max(\hat{\kappa}, \bar{\kappa}, \tilde{\kappa})^{K-i+1}}{\sin^2 \phi} (f(T_i) - f(T_{i-1})) \\ & \leq \begin{cases} \sin^{-2} \phi \max(\hat{\kappa}, \bar{\kappa}, \tilde{\kappa}) f(T_K), & \text{if } \max(\hat{\kappa}, \bar{\kappa}, \tilde{\kappa}) \leq 1 \\ \sin^{-2} \phi \max(\hat{\kappa}, \bar{\kappa}, \tilde{\kappa})^K f(T_K), & \text{if } \max(\hat{\kappa}, \bar{\kappa}, \tilde{\kappa}) > 1. \end{cases} \end{aligned}$$

Therefore, we have

$$f(\text{OPT}) \leq (1 + \sin^{-2} \phi a(\hat{\kappa}, \bar{\kappa}, \tilde{\kappa})b(\hat{\kappa}))f(T_K),$$

where $a(\hat{\kappa}, \bar{\kappa}, \tilde{\kappa}) = \max(\hat{\kappa}, \bar{\kappa}, \tilde{\kappa})$ if $\max(\hat{\kappa}, \bar{\kappa}, \tilde{\kappa}) \leq 1$ and $a(\hat{\kappa}, \bar{\kappa}, \tilde{\kappa}) = \max(\hat{\kappa}, \bar{\kappa}, \tilde{\kappa})^K$ if $\max(\hat{\kappa}, \bar{\kappa}, \tilde{\kappa}) > 1$; $b(\hat{\kappa}) = \hat{\kappa}^{K-1}$ if $\hat{\kappa} > 1$ and $b(\hat{\kappa}) = 1$ otherwise. ■

ACKNOWLEDGMENT

The authors would like to thank Louis L. Scharf for several insightful discussions. The authors also thank the anonymous reviewers for the careful reading of the manuscript and many constructive comments that have improved the presentation.

$$f(G_K \cup \{\hat{o}_i\}) - f(G_K) \leq \begin{cases} \hat{\kappa}(f(G_{K-1} \cup \{\hat{o}_i\}) - f(G_{K-1})), & \text{if } \|\mathcal{P}_\eta(\hat{o}_i^\perp)\| \geq \|\mathcal{P}_\eta(\hat{g}_K^\perp)\| \\ \bar{\kappa}(f(G_K) - f(G_{K-1})), & \text{if } \|\mathcal{P}_\eta(\hat{o}_i^\perp)\| \leq \|\mathcal{P}_\eta(\hat{g}_K^\perp)\|. \end{cases} \quad (16)$$

$$f(T_K \cup \{\hat{o}_i\}) - f(T_K) \leq \begin{cases} \hat{\kappa}(f(T_{K-1} \cup \{\hat{o}_i\}) - f(T_{K-1})), & \text{if } \|\mathcal{P}_\eta(\hat{o}_i^\perp)\| \geq \|\mathcal{P}_\eta(\hat{t}_K^\perp)\| \\ \bar{\kappa}(f(T_K) - f(T_{K-1})), & \text{if } \|\mathcal{P}_\eta(\hat{o}_i^\perp)\| \leq \|\mathcal{P}_\eta(\hat{t}_K^\perp)\|. \end{cases} \quad (17)$$

REFERENCES

- [1] L. L. Scharf, *Statistical Signal Processing*. Reading, MA, USA: Addison-Wesley, 1991.
- [2] E. L. Lehmann and G. Casella, "Theory of point estimation," *Springer Texts in Statistics*, 2nd ed. New York, NY, USA: Springer, 2003.
- [3] L. T. McWhorter and L. L. Scharf, "Properties of quadratic covariance bounds," in *Proc. 27th Asilomar, Conf. Signals, Syst. Comput.*, Nov. 1993, vol. 2, pp. 1176–1180.
- [4] K. Todros and J. Tabrikian, "General classes of performance lower bounds for parameter estimation—Part I: Non-Bayesian bounds for unbiased estimators," *IEEE Trans. Inf. Theory*, vol. 56, no. 10, pp. 5045–5063, Oct. 2010.
- [5] C. R. Rao, "Information and accuracy attainable in the estimation of statistical parameters," *Bull. Calcutta Math. Soc.*, vol. 37, pp. 81–91, 1945.
- [6] A. Bhattacharyya, "On some analogues of the amount of information and their use in statistical estimation," *Shankya*, vol. 8, no. 3, pp. 201–218, 1947.
- [7] E. W. Barankin, "Locally best unbiased estimates," *Ann. Math. Stat.*, vol. 20, pp. 477–501, 1946.
- [8] A. O. Hero, M. Usman, A. C. Sauve, and J. A. Fessler, "Recursive algorithms for computing the Cramer-Rao bound," *IEEE Trans. Signal Process.*, vol. 45, no. 3, pp. 803–807, Mar. 1997.
- [9] J. L. Williams, "Information theoretic sensor management," Ph.D. dissertation, Dept. Elect. Eng. Comput. Sci., Massachusetts Inst. Technol., 2007.
- [10] A. Krause and C. Guestrin, "Near-optimal observation selection using submodular functions," in *Proc. Natl. Conf. Artif. Intell.*, Vancouver, BC, Canada, Jul. 2007, vol. 22, pp. 1650–1654.
- [11] M. Shamaiah, S. Banerjee, and H. Vikalo, "Greedy sensor selection: Leveraging submodularity," in *Proc. 49th IEEE Conf. Decis. Control*, Atlanta, GA, USA, Dec. 2010, pp. 2572–2577.
- [12] A. Miller, *Subset Selection in Regression*, 2nd ed. Boston, MA, USA: Chapman & Hall, 2002.
- [13] E. J. Candes, "Compressive sampling," in *Proc. Int. Congr. Math.*, 2006, vol. 3, pp. 1433–1452.
- [14] D. L. Donoho, "Compressed sensing," *IEEE Trans. Inf. Theory*, vol. 52, no. 4, pp. 1289–1306, 2006.
- [15] R. G. Baraniuk, "Compressive sensing," *IEEE Signal Process. Mag.*, vol. 24, no. 4, pp. 118–121, Jul. 2007.
- [16] Y. C. Eldar and G. Kutyniok, *Compressed Sensing: Theory and Applications*, vol. 95. Cambridge, U.K.: Cambridge Univ. Press, 2012.
- [17] W. Xu and B. Hassibi, "Efficient compressive sensing with deterministic guarantees using expander graphs," in *Proc. IEEE Inf. Theory Workshop*, Sep. 2007, pp. 414–419.
- [18] Y. Chi, L. L. Scharf, A. Pezeshki, and A. R. Calderbank, "Sensitivity to basis mismatch in compressed sensing," *IEEE Trans. Signal Process.*, vol. 59, no. 5, pp. 2182–2195, May 2011.
- [19] R. Zehedi, L. W. Krakow, E. K. P. Chong, and A. Pezeshki, "Adaptive estimation of time-varying sparse signals," *IEEE Access*, vol. 1, pp. 449–464, 2013.
- [20] J. Tropp, "Greedy is good: Algorithmic results for sparse approximation," *IEEE Trans. Inf. Theory*, vol. 50, no. 10, pp. 2231–2242, Oct. 2004.
- [21] Y. Chi and R. Calderbank, "Coherence-based performance guarantees for orthogonal matching pursuit," in *Proc. Allerton Conf. Control, Commun. Comput.*, 2012.
- [22] S. Kunis and H. Rauhut, "Random sampling of sparse trigonometric polynomials, II. orthogonal matching pursuit versus basis pursuit," *Found. Comput. Math.*, vol. 8, no. 6, pp. 737–763, 2007.
- [23] T. Cai and L. Wang, "Orthogonal matching pursuit for sparse signal recovery with noise," *IEEE Trans. Inf. Theory*, vol. 57, no. 7, pp. 4680–4688, Jul. 2011.
- [24] M. A. Davenport and M. B. Wakin, "Analysis of orthogonal matching pursuit using the restricted isometry property," *IEEE Trans. Inf. Theory*, vol. 56, no. 9, pp. 4395–4401, Sep. 2010.
- [25] D. L. Donoho, Y. Tsaig, I. Drori, and J. L. Starck, "Sparse solution of underdetermined systems of linear equations by stagewise orthogonal matching pursuit," *IEEE Trans. Inf. Theory*, vol. 58, no. 2, pp. 1094–1121, Feb. 2012.
- [26] D. Needell and R. Vershynin, "Signal recovery from incomplete and inaccurate measurements via regularized orthogonal matching pursuit," *IEEE J. Sel. Topics Signal Process.*, vol. 4, no. 2, pp. 310–316, Apr. 2010.
- [27] D. Needell and R. Vershynin, "Uniform uncertainty principle and signal recovery via regularized orthogonal matching pursuit," *Found. Comput. Math.*, vol. 9, no. 3, pp. 317–334, 2009.
- [28] D. Needell and J. A. Tropp, "CoSaMP: Iterative signal recovery from incomplete and inaccurate samples," *Appl. Comput. Harmon. Anal.*, vol. 26, no. 3, pp. 301–321, 2009.
- [29] G. Braun, S. Pokutta, and Y. Xie, "Info-greedy sequential adaptive compressed sensing," *IEEE J. Sel. Topics Signal Process.*, vol. 9, no. 4, pp. 601–611, Jun. 2015.
- [30] E. Liu and V. N. Temlyakov, "The orthogonal super greedy algorithm and applications in compressed sensing," *IEEE Trans. Inf. Theory*, vol. 58, no. 4, pp. 2040–2047, Apr. 2012.
- [31] E. Candes, J. Romberg, and T. Tao, "Stable signal recovery from incomplete and inaccurate measurements," *Commun. Pure Appl. Math.*, vol. 59, no. 8, pp. 1207–1223, 2006.
- [32] Z. Wang, W. Moran, X. Wang, and Q. Pan, "Approximation for maximizing monotone non-decreasing set functions with a greedy method," *J. Combinatorial Optim.*, no. 31, pp. 29–43, Jan. 2014, doi: 10.1007/s10878-014-9707-3.
- [33] D. L. Donoho, M. Elad, and V. N. Temlyakov, "Stable recovery of sparse overcomplete representations in the presence of noise," *IEEE Trans. Inf. Theory*, vol. 52, no. 1, pp. 6–18, Jan. 2006.
- [34] W. Tutte, "Lectures on matroids," *J. Res. Nat. Bureau Standards*, vol. 69B, nos. 1–2, pp. 1–47, 1965.
- [35] L. Zhang, J. Luo, and D. Guo, "Neighbor discovery for wireless networks via compressed sensing," *Perform. Eval.*, vol. 70, no. 78, pp. 457–471, Jul. 2013.
- [36] G. L. Nemhauser, L. A. Wolsey, and M. L. Fisher, "An analysis of approximations for maximizing submodular set functions—I," *Math. Program.*, vol. 14, no. 1, pp. 265–294, 1978.
- [37] M. L. Fisher, G. L. Nemhauser, and L. A. Wolsey, "An analysis of approximations for maximizing submodular set functions—II," in *Polyhedral Combinatorics*. New York, NY, USA: Springer, 1978, pp. 73–87.
- [38] M. Conforti and G. Cornuejols, "Submodular set functions, matroids and the greedy algorithm: Tight worst-case bounds and some generalizations of the Rado-Edmonds theorem," *Discrete Appl. Math.*, vol. 7, no. 3, pp. 251–274, 1984.
- [39] J. Vondrák, "Submodularity and curvature: The optimal solution," *RIMS Kokyuroku Bessatsu B*, vol. 23, pp. 253–266, 2010.
- [40] Z. Zhang, Z. Wang, E. K. P. Chong, A. Pezeshki, and W. Moran, "Near optimality of greedy strategies for string submodular functions with forward and backward curvature constraints," in *Proc. 52nd IEEE Conf. Decision Control*, Florence, Italy, Dec. 10–13, 2013, pp. 5156–5161.
- [41] Z. Zhang, E. K. P. Chong, A. Pezeshki, and W. Moran, "String submodular functions with curvature constraints," *IEEE Trans. Autom. Control*, vol. 61, no. 3, pp. 601–616, Mar. 2015.
- [42] Y. Liu, E. K. P. Chong, A. Pezeshki, and W. Moran, "Bounds for general approximate dynamic programming based on string submodularity and curvature," in *Proc. 53rd IEEE Conf. Decision Control*, Los Angeles, CA, USA, Dec. 15–17, 2014, pp. 6653–6658.
- [43] V. Cevher and A. Krause, "Greedy dictionary selection for sparse representation," *IEEE J. Sel. Topics Signal Process.*, vol. 5, no. 5, pp. 979–988, Sep. 2011.
- [44] A. Das and D. Kempe, "Submodular meets spectral: Greedy algorithms for subset selection, sparse approximation and dictionary selection," in *Proc. 28th Int. Conf. Mach. Learn.*, Bellevue, WA, USA 2011.
- [45] E. J. Candes, "The restricted isometry property and its implications for compressed sensing," *Comptes Rendus Mathématique*, vol. 346, no. 9, pp. 589–592, 2008.
- [46] T. T. Cai, L. Wang, and G. Xu, "Stable recovery of sparse signals and an oracle inequality," *IEEE Trans. Inf. Theory*, vol. 56, no. 7, pp. 3516–3522, Jul. 2010.
- [47] M. Fickus, D. G. Mixon, and J. C. Tremain, "Steiner equiangular tight frames," *Linear Algebra Appl.*, vol. 436, no. 5, pp. 1014–1027, 2012.



Zhenliang Zhang (S'09–M'13) received the B.S. degree in physics major from Special Class for Gifted Young, University of Science and Technology of China, Hefei, China, in 2008 and the Ph.D. degree from the Department of Electrical and Computer Engineering, Colorado State University, Fort Collins, CO, USA, in August 2013.

He is currently a Research Scientist in Intel laboratory, Hillsboro, OR, USA. He was a senior engineer in Qualcomm Corporate R&D, New Jersey Research Center. He is also an Adjunct Professor in the Department of Electrical Engineering, Columbia University. He was a Visiting Professor in Shenzhen Institutes of Advanced Technology, Chinese Academy of Sciences. His research interests include primarily in the areas of signal processing, optimization, and network science. He is an associate editor of IEEE ACCESS.



Yuan Wang received the B.S. degree in Statistics from Special Class for Gifted Young, University of Science and Technology of China, Hefei, China, in 2008 and the Ph.D. degree in statistics from Colorado State University, Fort Collins, CO, USA, in 2013. She is currently an Assistant Professor at Washington State University, Pullman, WA, USA. Her research interests focus on information fusion, passive radar detection, and object-oriented data analysis including tree-structured data, imaging data, and functional data.



Edwin K. P. Chong (F'04) received the B.E. degree with First Class Honors from the University of Adelaide, Adelaide, SA, Australia, in 1987 and the M.A. and Ph.D. degrees from Princeton University, Princeton, NJ, USA, in 1989 and 1991, respectively, where he held an IBM Fellowship. He joined the School of Electrical and Computer Engineering, Purdue University in 1991, where he was named a University Faculty Scholar in 1999. Since August 2001, he has been a Professor of electrical and computer engineering and Professor of mathematics at Colorado State University, Fort Collins, CO, USA. He coauthored the best-selling book, *An Introduction to Optimization* (4th ed., Wiley-Interscience, 2013).

He was the founding chairman of the IEEE Control Systems Society Technical Committee on Discrete Event Systems, and served as an IEEE Control Systems Society Distinguished Lecturer. He is currently a Senior Editor of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL, and has also served on the editorial boards of *Computer Networks*, *Journal of Control Science and Engineering*, and *IEEE Expert Now*. He has served as a member of the IEEE Control Systems Society Board of Governors and as the Vice President for Financial Activities until 2014. He currently serves as President-Elect. He was the General Chair for the 2011 Joint 50th IEEE Conference on Decision and Control and European Control Conference. He received the NSF CAREER Award in 1995 and the ASEE Frederick Emmons Terman Award in 1998. He received the 2004 Best Paper Award for a paper in the journal *Computer Networks*. In 2010, he received the IEEE Control Systems Society Distinguished Member Award.



Ali Pezeshki (S'95–M'05) received the B.Sc. and M.Sc. degrees in electrical engineering from University of Tehran, Tehran, Iran, in 1999 and 2001, respectively, and the Ph.D. degree in electrical engineering from Colorado State University, Fort Collins, CO, USA, in 2004. In 2005, he was a Postdoctoral Research Associate with the Electrical and Computer Engineering Department, Colorado State University. From January 2006 to August 2008, he was a Postdoctoral Research Associate with The Program in Applied and Computational Mathematics, Princeton

University.

In August 2008, he joined the Faculty of Colorado State University, where he is currently an Associate Professor in the Department of Electrical and Computer Engineering, and the Department of Mathematics. His research interests include statistical signal processing, coding theory, applied harmonic analysis, and bioimaging.