

FUSION FRAMES: EXISTENCE AND CONSTRUCTION

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ABSTRACT. Fusion frame theory is an emerging mathematical theory that provides a natural framework for performing hierarchical data processing. A *fusion frame* is a frame-like collection of subspaces in a Hilbert space, thereby generalizing the concept of a frame for signal representation. In this paper, we study the existence and construction of fusion frames. We first present a complete characterization of a special class of fusion frames, called *Parseval fusion frames*. The value of Parseval fusion frames is that the inverse fusion frame operator is equal to the identity and therefore signal reconstruction can be performed with minimal complexity. We then introduce two general methods – the *spatial complement* and the *Naimark complement* – for constructing a new fusion frame from a given fusion frame. We then establish existence conditions for fusion frames with desired properties. In particular, we address the following question: Given $M, N, m \in \mathbb{N}$ and $\{\lambda_j\}_{j=1}^M$, does there exist a fusion frame in \mathbb{R}^M with N subspaces of dimension m for which $\{\lambda_j\}_{j=1}^M$ are the eigenvalues of the associated fusion frame operator? We address this problem by providing an algorithm which computes such a fusion frame for almost any collection of parameters $M, N, m \in \mathbb{N}$ and $\{\lambda_j\}_{j=1}^M$. Moreover, we show how this procedure can be applied, if subspaces are to be added to a given fusion frame to force it to become Parseval.

1. INTRODUCTION

1.1. **Fusion Frames.** Recent advances in hardware technology have enabled the economic production and deployment of sensing and computing networks consisting of a large number of low-cost components, which through collaboration enable reliable and efficient operation. Across different disciplines there is a fundamental shift from centralized information processing to distributed or network-wide information processing. Data communication is shifting from point-to-point communication to packet transport over wide area networks where network management is distributed and the reliability of individual links is less critical. Radar imaging is moving away from single platforms to multiple platforms that cooperate to achieve better performance. Wireless sensor networks are emerging as a new technology with the potential to enable cost-effective and reliable surveillance. These applications typically involve a large number of data streams, which need to be integrated at a central processor. Low

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communication bandwidth and limited transmit/computing power at each single node in the network give rise to the need for decentralized data analysis, where data reduction/processing is performed in two steps: local processing at neighboring nodes followed by the integration of locally processed data streams at a central processor.

Fusion frames (or *frames of subspaces*) [21] are a recent development that provide a natural mathematical framework for two-stage (or, more generally, hierarchical) data processing. The notion of a fusion frame was introduced in [21] with the main ideas already contained in [18]. A fusion frame is a frame-like collection of subspaces in a Hilbert space. In frame theory, a signal is represented by a collection of *scalars*, which measure the amplitudes of the projections of the signal onto the frame vectors, whereas in fusion frame theory the signal is represented by the projections of the signal onto the fusion frame subspaces. In a two-stage data processing setup, these projections serve as locally processed data, which can be combined to reconstruct the signal of interest.

Given a Hilbert space \mathcal{H} and a family of closed subspaces $\{\mathcal{W}_i\}_{i \in I}$ with associated positive weights v_i , $i \in I$, a *fusion frame* for \mathcal{H} is a collection of weighted subspaces $\{(\mathcal{W}_i, v_i)\}_{i \in I}$ such that there exist constants $0 < A \leq B < \infty$ satisfying

$$A\|f\|^2 \leq \sum_{i \in I} v_i^2 \|P_i f\|^2 \leq B\|f\|^2 \quad \text{for any } f \in \mathcal{H},$$

where P_i is the orthogonal projection onto \mathcal{W}_i . The constants A and B are called *fusion frame bounds*. A fusion frame is called *tight*, if A and B can be chosen to be equal, and *Parseval* if $A = B = 1$. If $v_i = 1$ for all $i \in I$, for the sake of brevity, we sometimes write $\{\mathcal{W}_i\}_{i \in I}$ instead of $\{(\mathcal{W}_i, 1)\}_{i \in I}$.

Any signal $f \in \mathcal{H}$ can be reconstructed [21] from its fusion frame measurements $\{v_i P_i f\}_{i \in I}$ by performing

$$f = \sum_{i \in I} v_i S^{-1}(v_i P_i f), \quad (1.1)$$

where $S = \sum_{i \in I} v_i P_i$ is the *fusion frame operator* known to be positive and self-adjoint.

Remark 1.1. *If we wish to perform dimension reduction, we can regard $\{v_i U_i^* f\}_{i \in I}$ as fusion frame measurements (cf. [40]), where U_i is a left-orthogonal basis for \mathcal{W}_i , i.e., $P_i = U_i U_i^*$ and $U_i^* U_i = I$. In this case, the reconstruction formula takes the form*

$$f = \sum_{i \in I} v_i S^{-1} U_i (v_i U_i^* f). \quad (1.2)$$

Remark 1.2. *Reconstruction of a sparse signal from its fusion frame measurements is considered in [10].*

1.2. Applications of Fusion Frames. Frame theory has been established as a powerful mathematical framework for robust and stable representation of signals. It has found numerous applications in sampling theory [31], data quantization [9], quantum measurements [32], coding [2, 45], image processing [11, 25], wireless communications [34, 36, 44], time-frequency analysis [29, 30, 48], speech recognition [1], and bioimaging [27]. The reader is referred to survey papers [38, 39] and the references therein for more examples. Fusion frame theory is a generalization of frame theory that is more suited for applications where two-stage (local

and global) signal/data analysis is required. To highlight this we give three signal processing applications wherein fusion frames arise naturally. We also discuss the connection between fusion frames and two pressing questions in pure mathematics.

Distributed Sensing. Consider a large number of small and inexpensive sensors that are deployed in an area of interest to measure various physical quantities or to keep the area under surveillance. Due to practical and economical factors, such as low communication bandwidth, limited signal processing power, limited battery life, or the topography of the surveillance area, the sensors are typically deployed in clusters, where each cluster includes a unit with higher computational and transmission power for local data processing. A typical large sensor network can thus be viewed as a redundant collection of subnetworks forming a set of subspaces (e.g. see [22, 40, 42]). The local subspace information are passed to a central processing station for joint processing. A similar local-global signal processing principle is applicable to modeling of human visual cortex as discussed in [43].

Parallel Processing. If a frame system is simply too large to handle effectively (from either computational complexity or numerical stability standpoints), we can divide it into multiple small subsystems for simple and perhaps parallelizable processing. Fusion frames provide a natural framework for splitting a large frame system into smaller subsystems and then recombining the subsystems. Splitting of a large frame system into smaller subsystems for parallel processing was first considered in [4] and predates the introduction of fusion frames.

Packet Encoding. Information bearing symbols are typically encoded into a number of packets and then transmitted over a communication network, e.g., the internet. The transmitted packet may be corrupted during the transmission or completely lost due to buffer overflows. By introducing redundancy in encoding the symbols, we can increase the reliability of the communication scheme. Fusion frames, as redundant collections of subspaces, can be used to produce a redundant representation of a source symbol. In the simplest form, each fusion frame projection can be viewed as a packet that carries some new information about the symbol. The packets can be decoded jointly at the destination to recover the transmitted symbol. The use of fusion frames for packet encoding is considered in [5].

The Kadison-Singer Problem and Optimal Packings. The Kadison-Singer Problem [26] has been among the most famous unsolved problems in analysis since 1959. It turns out that this problem is, roughly speaking, equivalent to the following question (cf. [26]). Can a frame be partitioned such that the spans of the partitions as a fusion frame lead to a ‘good’ lower fusion frame bound? The reader is referred to [26] for details. Therefore, advances in the design of fusion frames will have direct impact in providing new angles for a renewed attack to the Kadison-Singer Problem. In addition, there is a close connection between Parseval fusion frames and Grassmannian packings. In fact, as shown in [40], Parseval fusion frames consisting of equi-distance and equi-dimensional subspaces are optimal Grassmannian packings. Therefore, new methods for constructing such fusion frames also provide ways to construct optimal packings. We note that the frame counterpart of this connection also exists (cf. [45]).

1.3. Main Contribution: Construction of Fusion Frames with Desired Properties.

The value of fusion frames for signal processing is that the interplay between local-global processing and redundant representation provides resilience to noise and erasures due to,

for instance, sensor failures or buffer overflows [5, 20, 40, 42]. It also provides robustness to subspace perturbations [21], which may be due to imprecise knowledge of sensor network topology. In most cases, extra structure on fusion frames is required to guarantee satisfactory performance. For instance, our recent work [40, 42] shows that in order to minimize the mean-squared error in the linear minimum mean-squared error estimation of a random vector from its fusion frame measurements in white noise the fusion frame needs to be Parseval or tight. The Parseval property is also desirable for managing signal processing complexity. It means that the fusion frame operator S is equal to the identity operator and hence the operator inversion required for signal reconstruction is trivial. To provide maximal robustness against erasures of one fusion frame subspace the fusion frame subspaces must also be equi-dimensional. If maximal robustness with respect to two or more subspace erasures is desired then the fusion frame subspaces must all have the same pairwise chordal distance as well. Other examples of optimality of *structured* fusion frames for signal reconstruction can be found in [5, 20, 40, 42, 21].

Remark 1.3. *We note that signal reconstruction in a frame system in the presence of erasures has been studied by several authors. The results indicate that robustness to erasures of frame coefficients also require the frame system to have specific properties and structure, such as Parseval, equiangular, or equal-norm property. The reader is referred to [7, 8, 17, 33, 37, 46, 47] and the references therein for a collection of relevant results.*

A natural question is: How can one construct fusion frames with desired properties? More specifically, how can one construct fusion frames for which a set of parameters such as

- 1) *eigenvalues of the fusion frame operator,*
- 2) *dimensions of the subspaces,*
- 3) *chordal distances between subspaces, and/or*
- 4) *weights assigned to the subspaces*

can be prescribed?

In this paper, we present a complete answer to the above question under the first design criterion and provide partial answers for the construction of fusion frames under the second and third design criteria. Our main contributions are as follow.

- In Section 2, we provide a complete characterization of *Parseval* fusion frames in terms of the existence of special isometries defined on an encompassing Hilbert space.
- In Section 3, we present two general ways for constructing a new fusion frame from a given fusion frame, by exploiting the notions of spatial complement and Naimark complement, and establish the relationship between the parameters of the two fusion frames. In particular, we show how the weights, subspace dimensions, fusion frame bounds, eigenvalues of the fusion frame operator, and the chordal distance between the subspaces for the new fusion frame can be determined from those of the original fusion frame prior to construction.

- In Section 4, we establish existence conditions and develop simple algorithms for constructing fusion frames with desired fusion frame operators.¹ Our construction produces frames with desired frame operators as a special case.

We note that the construction of frames with arbitrary frame operators has been studied by several authors (see, e.g., [3, 13, 24, 19]). However, the fusion frame counterparts are much less exploited. In fact, even establishing existence conditions for fusion frames is a deep and involved problem. Frame potentials, introduced in (cf. [3]), have proven to be a valuable tool in asserting the existence of tight frames. Two recent papers [14, 41] have introduced and studied *fusion frame potentials* to address the existence of fusion frames, but with limited success. The problem here is that minimizers of the fusion frame potential are not necessarily tight fusion frames. Also, the fusion frame potential is a very complex notion and it requires some deep ideas to make it work. However, until recently, no general construction method was known for the construction of fusion frames with desired properties. A significant advance for the construction of equi-dimensional tight fusion frames was presented in [15]. The authors have provided a complete characterization of triples (M, N, m) for which tight fusion frames exist. Here M is the total dimension of the Hilbert space, N is the number of subspaces, and m is the dimension of the fusion frame subspaces. They have also developed an elegant and simple algorithm which can produce a tight fusion frame for most (M, N, m) triples.

Our paper is concerned with a more general question than that answered in [15], that is, the construction of a fusion frame (not necessarily tight) for which the fusion frame operator can possess any desired set of eigenvalues. This includes fusion frames with desired bounds as a special case, as the fusion frame bounds are simply the smallest and largest eigenvalues of the associated fusion frame operator. More specifically, given $M, N, m \in \mathbb{N}$, and a set of real positive values $\{\lambda_j\}_{j=1}^M$, we establish existence conditions for fusion frames whose fusion frame operators have eigenvalues $\{\lambda_j\}_{j=1}^M$ and develop a simple algorithm that produces such a fusion frame. The answer to this problem has profound practical and theoretical implications. From a signal analysis standpoint, it provides a flexible mathematical framework where the representation system can be tailored to satisfy data processing demands. From a theoretical standpoint, it provides a deep understanding of the boundaries of fusion frame theory viewed as a generalization of frame theory.

We note that our solution provides an answer to the construction of frames with arbitrary frame operators as a special case, which has been an open problem until now. Construction of frames with arbitrary frame operators is studied in [23] (See also [13])

2. CHARACTERIZATION OF PARSEVAL FUSION FRAMES

In this section, we provide a characterization of *Parseval* fusion frames in terms of the existence of special isometries defined on an encompassing Hilbert space. This characterization may be viewed as the fusion frame counterpart to *Naimark's theorem* [12, 16, 28, 35], where Parseval frames are characterized as frame systems generated by an orthogonal projection of

¹Throughout this paper whenever we say a fusion frame with a desired fusion frame operator we mean a fusion frame for which the fusion frame operator has a desired set of eigenvalues. A similar language is used to refer to a frame for which the frame operator has a desired set of eigenvalues.

an orthonormal basis from a larger Hilbert space. However, these characterizations cannot be easily exploited for constructing Parseval frames or Parseval fusion frames. The difficulty arises from the uncontrollable nature of the projection of the larger Hilbert space. For fusion frames, the construction of appropriate isometries are particularly difficult. In fact, these problems are equivalent to serious unsolved problems in operator theory concerning the construction of projections which sum to a given operator. Nonetheless, these isometries are illuminating for understanding Parseval fusion frames.

The following theorem states the main result of this section, which can be regarded as a quantitative version of [21, Thm. 3.1].

Theorem 2.1. *For a complete family of subspaces² $\{\mathcal{W}_i\}_{i \in I}$ of \mathcal{H} and positive weights $\{v_i\}_{i \in I}$, the following conditions are equivalent.*

- (i) $\{(\mathcal{W}_i, v_i)\}_{i \in I}$ is a Parseval fusion frame for \mathcal{H} .
- (ii) There exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$, an orthonormal basis $\{e_j\}_{j \in J}$ for \mathcal{K} , a partition $\{J_i\}_{i \in I}$ of J , and isometries $L_i : \mathcal{E}_i := \text{span}\{e_j\}_{j \in J_i} \rightarrow \mathcal{W}_i$, $i \in I$, such that

$$P = \sum_{i \in I} v_i L_i$$

is an orthogonal projection of \mathcal{K} onto \mathcal{H} .

Proof. (i) \Rightarrow (ii). For every $i \in I$, let $\{e_{ij}\}_{j \in J_i}$ be an orthonormal basis for \mathcal{W}_i . Since $\{(\mathcal{W}_i, v_i)\}_{i \in I}$ is a Parseval fusion frame for \mathcal{H} , by [21, Thm. 2.3], the family $\{v_i e_{ij}\}_{i \in I, j \in J_i}$ is a Parseval frame for \mathcal{H} . This implies (cf. [12, 28, 35]) that there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ with an orthonormal basis $\{\tilde{e}_{ij}\}_{i \in I, j \in J_i}$ so that the orthogonal projection P of \mathcal{K} onto \mathcal{H} satisfies

$$P(\tilde{e}_{ij}) = v_i e_{ij}, \quad i \in I, j \in J_i.$$

Setting $\mathcal{E}_i = \text{span}\{\tilde{e}_{ij}\}_{j \in J_i}$, the map

$$L_i := \frac{1}{v_i} P|_{\mathcal{E}_i} : \mathcal{E}_i \rightarrow \mathcal{W}_i$$

is an isometry for all $i \in I$, and

$$P = \sum_{i \in I} v_i L_i$$

is an orthogonal projection of \mathcal{K} onto \mathcal{H} .

(ii) \Rightarrow (i). Since $P = \sum_{i \in I} v_i L_i$ is an orthogonal projection of \mathcal{K} onto \mathcal{H} , $\{P e_j\}_{j \in J}$ is a Parseval frame for \mathcal{H} . Further, since $L_i := 1/v_i \cdot P|_{\mathcal{E}_i} : \mathcal{E}_i \rightarrow \mathcal{W}_i$ is an isometry, it follows that $\{1/v_i \cdot P e_j\}_{j \in J_i}$ is an orthonormal basis for \mathcal{W}_i , $i \in I$. Applying these observations and

²A family of subspaces is called *complete* in \mathcal{H} , if their span equals \mathcal{H} .

denoting by P_i the orthogonal projection onto \mathcal{W}_i , for all $f \in \mathcal{H}$, we have

$$\begin{aligned}
 \sum_{i \in I} v_i^2 \|P_i f\|^2 &= \sum_{i \in I} v_i^2 \left\| \sum_{j \in J_i} \left\langle f, \frac{1}{v_i} P e_j \right\rangle \frac{1}{v_i} P e_j \right\|^2 \\
 &= \sum_{i \in I} v_i^2 \sum_{j \in J_i} \left| \left\langle f, \frac{1}{v_i} P e_j \right\rangle \right|^2 \\
 &= \sum_{i \in I} \sum_{j \in J_i} |\langle f, P e_j \rangle|^2 \\
 &= \|f\|^2.
 \end{aligned}$$

Thus $\{(\mathcal{W}_i, v_i)\}_{i \in I}$ is a Parseval fusion frame as claimed. \square

Considering this theorem and its proof, we can derive an interesting corollary which links the construction of Parseval fusion frames to the construction of special Parseval frames. In fact, the question of existence of Parseval fusion frames is equivalent to the question of existence of Parseval frames for which certain subsets of frame vectors are orthonormal. The answer to this question is not known, but the connection between the two problems may provide insights into the construction Parseval fusion frames.

Corollary 2.2. *For a family of subspaces $\{\mathcal{W}_i\}_{i \in I}$ of \mathcal{H} and positive weights $\{v_i\}_{i \in I}$, the following conditions are equivalent.*

- (i) $\{(\mathcal{W}_i, v_i)\}_{i \in I}$ is a Parseval fusion frame for \mathcal{H} .
- (ii) There exists a Parseval frame $\{e_{ij}\}_{i \in I, j \in J_i}$ for \mathcal{H} such that $\{1/v_i \cdot e_{ij}\}_{j \in J_i}$ is an orthonormal basis for \mathcal{W}_i for all $i \in I$.

3. CONSTRUCTION OF NEW FUSION FRAMES FROM EXISTING ONES

In this section, we present two general ways, namely the spatial complement and the Naimark complement, for constructing a new fusion frame from a given fusion frame and establish the relationship between the parameters of the two fusion frames. A special case of the construction methods presented here is reported in [15]. The result of [15] deals only with the construction of Parseval fusion frames in a finite dimensional Hilbert space and does not investigate the relation between the new and the original fusion frame parameters.

3.1. The Spatial Complement. Taking the spatial complement appears to be a natural way for generating a new fusion frame from a given fusion frame. We begin by defining the notion of an *orthogonal fusion frame to a given fusion frame*, which is central to our discussion.

Definition 3.1. *Let $\{(\mathcal{W}_i, v_i)\}_{i \in I}$ be a fusion frame for \mathcal{H} . If the family $\{(\mathcal{W}_i^\perp, v_i)\}_{i \in I}$, where \mathcal{W}_i^\perp is the orthogonal complement of \mathcal{W}_i , is also a fusion frame, then we call $\{(\mathcal{W}_i^\perp, v_i)\}_{i \in I}$ the orthogonal fusion frame to $\{(\mathcal{W}_i, v_i)\}_{i \in I}$.*

Theorem 3.2. *Let $\{(\mathcal{W}_i, v_i)\}_{i \in I}$ be a fusion frame for \mathcal{H} with optimal fusion frame bounds $0 < A \leq B < \infty$. Then the following conditions are equivalent.*

- (i) $\bigcap_{i \in I} \mathcal{W}_i = \{0\}$.
- (ii) $B < \sum_{i \in I} v_i^2$.
- (iii) *The family $\{(\mathcal{W}_i^\perp, v_i)\}_{i \in I}$ is a fusion frame for \mathcal{H} with optimal fusion frame bounds $\sum_{i \in I} v_i^2 - B$ and $\sum_{i \in I} v_i^2 - A$.*

Proof. (iii) \Rightarrow (i): Suppose that (i) is false. Then there exists a vector $0 \neq f \in \bigcap_{i \in I} \mathcal{W}_i$. This implies $f \perp \mathcal{W}_i^\perp$ for all $i \in I$, hence $\{\mathcal{W}_i^\perp\}_{i \in I}$ does not span \mathcal{H} . This is a contradiction to (iii).

(i) \Rightarrow (ii): Since B is optimal, by using the fusion frame property, it follows that there exists some $f \in \mathcal{H}$ so that

$$B\|f\|^2 = \left\langle \sum_{i \in I} v_i^2 P_i f, f \right\rangle = \sum_{i \in I} v_i^2 \|P_i f\|^2 \leq \sum_{i \in I} v_i^2 \|f\|^2.$$

Hence

$$B \leq \sum_{i \in I} v_i^2. \quad (3.3)$$

It now suffices to observe that we have equality in (3.3) if and only if

$$f \in \bigcap_{i \in I} \mathcal{W}_i \neq \{0\}.$$

(ii) \Rightarrow (iii): Since $AI \leq \sum_{i \in I} v_i^2 P_i \leq BI$, we have

$$\left(\sum_{i \in I} v_i^2 - B \right) I \leq \sum_{i \in I} v_i^2 (I - P_i) \leq \left(\sum_{i \in I} v_i^2 - A \right) I. \quad (3.4)$$

From (ii), we have $\sum_{i \in I} v_i^2 - B > 0$ and hence

$$\{(\mathcal{W}_i^\perp, v_i)\}_{i \in I} = \{((I - P_i)\mathcal{H}, v_i)\}_{i \in I},$$

is a fusion frame. The fusion frame bounds from (3.4) are optimal. \square

The following theorem shows that all the parameters of the new fusion frame can be determined from those of the generating fusion frame prior to the construction.

Theorem 3.3. *Let $\{(\mathcal{W}_i, v_i)\}_{i \in I}$ be a fusion frame for \mathcal{H} , and let $\{(\mathcal{W}_i^\perp, v_i)\}_{i \in I}$ be its associated orthogonal fusion frame. Then the following conditions hold.*

- (i) *Let S denote the frame operator for $\{(\mathcal{W}_i, v_i)\}_{i \in I}$ with eigenvectors $\{e_j\}_{j \in J}$ and respective eigenvalues $\{\lambda_j\}_{j \in J}$. Then the fusion frame operator for $\{(\mathcal{W}_i^\perp, v_i)\}_{i=1}^N$ possesses the same eigenvectors $\{e_j\}_{j \in J}$ and respective eigenvalues $\{\sum_{i \in I} v_i^2 - \lambda_j\}_{j \in J}$.*
- (ii) *Assume that $\dim \mathcal{H} < \infty$ and $m := \dim \mathcal{W}_i$ for all $i \in I$. Then,*

$$d_c^2(\mathcal{W}_i^\perp, \mathcal{W}_j^\perp) = d_c^2(\mathcal{W}_i, \mathcal{W}_j) + 2m - \dim \mathcal{H} \quad \text{for all } i, j \in \{1, \dots, N\}, i \neq j.$$

where $d_c^2(\mathcal{W}_i, \mathcal{W}_j)$ denotes the squared chordal distance between subspaces \mathcal{W}_i and \mathcal{W}_j and is given by

$$d_c^2(\mathcal{W}_i, \mathcal{W}_j) = \dim \mathcal{H} - \text{tr}[P_i P_j].$$

Proof. (i). For each $j \in J$, we have

$$\sum_{i \in I} v_i^2 P_i e_j = \lambda_j e_j.$$

Hence,

$$\sum_{i \in I} v_i^2 (I - P_i) e_j = \left(\sum_{i \in I} v_i^2 - \lambda_j \right) e_j,$$

which implies the claimed properties for the fusion frame operator S^\perp .

(ii). The orthogonal projection onto \mathcal{W}_i^\perp is given by $I - P_i$. Hence,

$$d_c^2(\mathcal{W}_i^\perp, \mathcal{W}_j^\perp) = \dim \mathcal{H} - \text{tr}[(I - P_i)(I - P_j)].$$

The claim follows from

$$\text{tr}[(I - P_i)(I - P_j)] = \text{tr}[I - P_i - P_j + P_i P_j] = \dim \mathcal{H} - 2m + \text{tr}[P_i P_j]$$

and the definition of $d_c^2(\mathcal{W}_i, \mathcal{W}_j)$. \square

Corollary 3.4. *Let $\{\mathcal{W}_i\}_{i=1}^N$ be an A -tight fusion frame for \mathbb{R}^M such that $\mathcal{W}_k \neq \mathcal{H}$ for some $k \in \{1, \dots, N\}$. Then $\{\mathcal{W}_i^\perp\}_{i=1}^N$ is an $(N - A)$ -tight fusion frame for \mathbb{R}^M . If $m := \dim \mathcal{W}_i$ for all $i \in \{1, \dots, N\}$ and $d^2 := d_c^2(\mathcal{W}_i, \mathcal{W}_j)$ for all $i, j \in \{1, \dots, N\}$, $i \neq j$, then*

$$d_c^2(\mathcal{W}_i^\perp, \mathcal{W}_j^\perp) = d^2 + 2m - M \quad \text{for all } i, j \in \{1, \dots, N\}, i \neq j.$$

Proof. Assume that $\mathcal{W}_k \neq \mathbb{R}^M$. Then by choosing some $0 \neq f \in \mathcal{W}_k^\perp$, we obtain

$$A \|f\|^2 = \sum_{i=1}^N v_i^2 \|P_i f\|^2 = \sum_{i \neq k} v_i^2 \|P_i f\|^2 < \left(\sum_{i=1}^N v_i^2 \right) \|f\|^2.$$

Thus we have $A < \sum_{i=1}^N v_i^2$, and the application of Theorem 3.2 proves the first part of the claim. The second part follows immediately from Theorem 3.3 (ii). \square

A straightforward application of Corollary 3.4 provides a way of constructing tight fusion frames with equ-dimensional subspaces. This construction starts with a given set of equ-dimensional subspaces that do not form a tight fusion frames and fills up the Hilbert space by adding a new set of subspaces, with the same dimension, to produce a tight fusion frame.

Corollary 3.5. *Let $\{\mathcal{W}_i\}_{i=1}^N$ be a family of m -dimensional subspaces of \mathbb{R}^M . Then there exist $N(M - 1)$ m -dimensional subspaces $\{\mathcal{V}_i\}_{i=1}^{N(M-1)}$ of \mathbb{R}^M so that $\{\mathcal{W}_i\}_{i=1}^N \cup \{\mathcal{V}_i\}_{i=1}^{N(M-1)}$ is a tight fusion frame. Moreover, if $N = 1$ and $\dim \mathcal{W}_1 = M - 1$ then the construction is minimal in the sense that it identifies the smallest number of m -dimensional subspaces which need to be added to obtain a tight fusion frame.*

Proof. For each $i = 1, \dots, N$, we choose an orthonormal basis $\{e_j^i\}_{j=1}^M$ for \mathbb{R}^M in such a way that $\{e_j^i\}_{j=1}^m$ is an orthonormal basis for \mathcal{W}_i . Let T_i , $i = 1, \dots, N$, denote the circular shift operator on the orthonormal basis $\{e_j^i\}_{j=1}^M$. Then

$$\{T_i^k \mathcal{W}_i\}_{i=1, k=0}^{N, M-1},$$

is a tight fusion frame for \mathbb{R}^M of m -dimensional subspaces which contains $\{\mathcal{W}_i\}_{i=1}^N$.

Now consider the case where $N = 1$ and $\dim \mathcal{W}_1 = M - 1$. Let $\{\mathcal{V}_i\}_{i=1}^{N_1}$ be any collection of $(M - 1)$ -dimensional subspaces so that $\{\mathcal{W}_1\} \cup \{\mathcal{V}_i\}_{i=1}^{N_1}$ is a tight fusion frame. By Theorem 3.2, we have $1 + N_1 = M$, hence $N_1 = M - 1$, which equals $N(M - 1)$. \square

3.2. The Naimark Complement. Another approach to constructing a new fusion frame from an existing one is to use the notion of *Naimark complement*. This approach however applies to Parseval fusion frames only, as stated in the following theorem.

Theorem 3.6. *Let $\{(\mathcal{W}_i, v_i)\}_{i \in I}$ be a Parseval fusion frame for \mathcal{H} . Then there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a Parseval fusion frame $\{(\mathcal{W}'_i, \sqrt{1 - v_i^2})\}_{i \in I}$ for $\mathcal{K} \ominus \mathcal{H}$ with the following properties.*

- (i) $\dim \mathcal{W}'_i = \dim \mathcal{W}_i$ for all $i \in I$.
- (ii) If $\dim \mathcal{H} < \infty$ and $\dim \mathcal{W}_i = \dim \mathcal{W}_j$ for all $i, j \in I, i \neq j$, then

$$d_c^2(\mathcal{W}'_i, \mathcal{W}'_j) = d_c^2(\mathcal{W}_i, \mathcal{W}_j) \quad \text{for all } i, j \in \{1, \dots, N\}, i \neq j.$$

Proof. For each $i \in I$, let $\{f_{ij}\}_{j \in J_i}$ be an orthonormal basis for \mathcal{W}_i . Then the family

$$\{v_i f_{ij}\}_{i \in I, j \in J_i}$$

is a Parseval frame for \mathcal{H} . By [12, 28, 35], there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$, an orthogonal projection $P : \mathcal{K} \rightarrow \mathcal{H}$, and an orthonormal basis $\{e_{ij}\}_{i \in I, j \in J_i}$ for \mathcal{K} so that

$$P e_{ij} = v_i f_{ij}, \quad i \in I, j \in J_i. \quad (3.5)$$

This implies that $\{(I - P)e_{ij}\}_{i \in I, j \in J_i}$ is a Parseval frame for $\mathcal{K} \ominus \mathcal{H}$. Further,

$$\|(I - P)e_{ij}\| = \sqrt{1 - v_i^2}, \quad i \in I, j \in J_i,$$

and, for $j, j' \in J_i, j \neq j'$, we have

$$\langle (I - P)e_{ij}, (I - P)e_{ij'} \rangle = -\langle P e_{ij}, e_{ij'} \rangle = -\langle v_i f_{ij}, v_i f_{ij'} \rangle = 0.$$

Defining

$$\mathcal{W}'_i = \text{span}\{(I - P)e_{ij} : j \in J_i\},$$

we conclude that $\{(\mathcal{W}'_i, \sqrt{1 - v_i^2})\}_{i \in I}$ is a Parseval fusion frame for $\mathcal{K} \ominus \mathcal{H}$.

(i). By construction,

$$\dim \mathcal{W}'_i = |J_i| = \dim \mathcal{W}_i \quad \text{for all } i \in I.$$

(ii). Set $M := \dim \mathcal{H}$, $L := \dim \mathcal{K}$, $I := \{1, \dots, N\}$, and $m := \dim \mathcal{W}_i$ for all $i \in \{1, \dots, N\}$. For the sake of brevity, we define $E_i := ((I - P)e_{i1}, \dots, (I - P)e_{im}) \in \mathbb{R}^{M \times m}$ and $F_i := (v_i f_{i1}, \dots, v_i f_{im}) \in \mathbb{R}^{M \times m}$. Then, for every $i, i' \in \{1, \dots, N\}, i \neq i'$, we obtain

$$\text{tr}[P_i P_{i'}] = \text{tr}[F_i F_i^T F_{i'} F_{i'}^T] = \text{tr}[(F_{i'}^T F_i)(F_i^T F_{i'})] = \text{tr}[\langle v_i f_{i'j}, v_i f_{ik} \rangle_{j,k} \langle v_i f_{ij}, v_i f_{i'k} \rangle_{j,k}].$$

By employing (3.5),

$$\text{tr}[P_i P_{i'}] = \text{tr}[\langle P e_{i'j}, P e_{ik} \rangle_{j,k} \langle P e_{ij}, P e_{i'k} \rangle_{j,k}]. \quad (3.6)$$

Now letting P'_i denote the orthogonal projection onto \mathcal{W}'_i , for each $i, i' \in \{1, \dots, N\}, i \neq i'$, the definition of \mathcal{W}'_i implies

$$\text{tr}[P'_i P'_{i'}] = \text{tr}[E_i E_i^T E_{i'} E_{i'}^T] = \text{tr}[(E_{i'}^T E_i)(E_i^T E_{i'})]$$

and

$$(E_{i'}^T E_i) = (\langle (I - P)e_{i'j}, (I - P)e_{ik} \rangle)_{j,k}.$$

Utilizing the choice of $\{e_{ij}\}$ and careful dealing with the inner products on \mathcal{K} , \mathcal{H} , and $\mathcal{K} \ominus \mathcal{H}$, for each j, k ,

$$\langle (I - P)e_{i'j}, (I - P)e_{ik} \rangle = \langle e_{i'j}, e_{ik} \rangle - \langle Pe_{i'j}, Pe_{ik} \rangle = -\langle Pe_{i'j}, Pe_{ik} \rangle.$$

Combining the above three equations,

$$\text{tr}[P_i' P_i'] = \text{tr}[\langle Pe_{i'j}, Pe_{ik} \rangle_{j,k} \langle Pe_{ij}, Pe_{i'k} \rangle_{j,k}].$$

Comparison with (3.6) completes the proof. \square

Definition 3.7. Let $\{(\mathcal{W}_i, v_i)\}_{i \in I}$ be a tight fusion frame for \mathcal{H} . We refer to the tight fusion frame $\{(\mathcal{W}'_i, \sqrt{1 - v_i^2})\}_{i \in I}$ for $\mathcal{K} \ominus \mathcal{H}$ from Theorem 3.6 as the Naimark fusion frame associated with $\{(\mathcal{W}_i, v_i)\}_{i \in I}$. The rationale for this terminology is that this is the fusion frame version of the Naimark theorem [12, 28, 35].

Corollary 3.8. Let $\{\mathcal{W}_i\}_{i=1}^N$ be an A -tight fusion frame for \mathbb{R}^M . Then there exists some $L \geq M$ and a $\sqrt{1 - 1/A^2}$ -tight fusion frame for \mathbb{R}^{L-M} which satisfies $\dim \mathcal{W}'_i = \dim \mathcal{W}_i$ for all $i \in \{1, \dots, N\}$. If, in addition, $d^2 := d_c^2(\mathcal{W}_i, \mathcal{W}_j)$ for all $i, j \in \{1, \dots, N\}$, $i \neq j$, then

$$d_c^2(\mathcal{W}'_i, \mathcal{W}'_j) = d^2 \quad \text{for all } i, j \in \{1, \dots, N\}, i \neq j.$$

Proof. This follows immediately from Theorem 3.6. \square

We note that Theorem 3.6 is not always constructive, since it requires the knowledge of a larger Hilbert space from which the given Parseval frame is derived by an orthogonal projection of an orthonormal basis.

4. EXISTENCE AND CONSTRUCTION OF A FUSION FRAME WITH A DESIRED FUSION FRAME OPERATOR

We now focus on the existence and construction of fusion frames whose fusion frame operators possess a desired set of eigenvalues. We answer the following questions: (1) Given a set of eigenvalues, does there exist a fusion frame whose fusion frame operator possesses those eigenvalues? (2) If such a fusion frame exists how can it be constructed?

Let $\lambda_1 \geq \dots \geq \lambda_M > 0$, $M \in \mathbb{N}$, be real positive values satisfying a factorization as

$$(FAC) \quad \sum_{j=1}^M \lambda_j = Nm \in \mathbb{N}.$$

We wish to construct a fusion frame $\{\mathcal{W}_i\}_{i=1}^N$, $\mathcal{W}_i \subseteq \mathbb{R}^M$, such that

(FF1) $\dim \mathcal{W}_i = m$ for all $i = 1 \dots, N$, and

(FF2) the associated fusion frame operator has $\{\lambda_j\}_{j=1}^M$ as its eigenvalues.

4.1. The Integer Case. We first consider the simple case where $\lambda_j \in \mathbb{N}$ for all $i = 1, \dots, M$. This case is central to developing intuition about the construction algorithms to be developed.

Proposition 4.1. *If the positive integers $N \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_M > 0$, $N \in \mathbb{N}$, and $m \in \mathbb{N}$ satisfy (FAC), then the fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ constructed via the (FFCIE) algorithm outlined in Figure 1 satisfies both (FF1) and (FF2).*

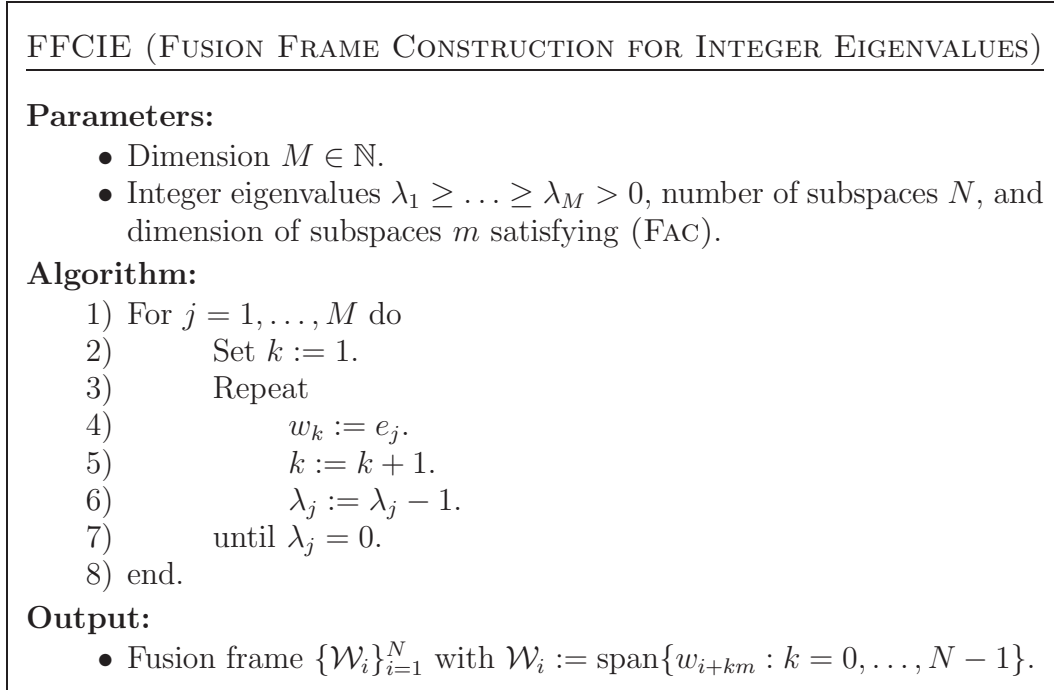


FIGURE 1. The FFCIE Algorithm for constructing a fusion frame with a fusion frame operator with prescribed integer eigenvalues.

Proof. If the set of vectors

$$\{w_{i+km} : k = 0, \dots, N - 1\}$$

is pairwise orthogonal for each $i = 1, \dots, N$, then (FF1) and (FF2) follow automatically. Now fix $i \in \{1, \dots, N\}$. By construction, it is sufficient to show that, for each $0 \leq k \leq N - 2$, the vectors w_{i+km} and $w_{i+(k+1)m}$ are orthogonal. Again by construction, the only possibility for this to fail is that there exists some $j_0 \in \{1, \dots, M\}$ satisfying $\lambda_{j_0} > N$. But this was excluded by the hypothesis. \square

The algorithm outlined in Figure 1 shuffles the intended eigenvalues in terms of associated unit vectors $e_1, \dots, e_M \in \mathbb{R}^M$ as basis vectors into the subspaces of the fusion frame to be constructed. Considering a matrix $W \in \mathbb{R}^{Nm \times M}$ with the vectors w_1, \dots, w_{Nm} as rows, intuitively (FFCIE) fills this matrix up from top to bottom, row by row in such a way that the ℓ_2 norm of the rows is 1, the ℓ_2 norm of column j is λ_j , $j = 1, \dots, M$, and the columns are orthogonal. The vectors w_k are then assigned to subspaces in such a way that the vectors

assigned to each subspace forms an orthonormal system. We note that the generated vectors $w_k, k = 1 \dots, Nm$ are as sparse as possible, providing optimal fast computation abilities.

We wish to note that the condition $N \geq \lambda_1$ is necessary for (FFCIE). The question whether or not this is necessary in general is much more involved and will not be discussed here.

4.2. The General Case. We now discuss the general case where the desired eigenvalues for the fusion frame operator are real positive values that satisfy (FAC).

4.2.1. The Algorithm. As a first step we generalize (FFCIE) (see Figure 1) by introducing Lines 4) – 9), which deal with the non-integer parts. The construction algorithm for real eigenvalues, called (FFCRE), is outlined in Figure 2.

FFCRE (FUSION FRAME CONSTRUCTION FOR REAL EIGENVALUES)

Parameters:

- Dimension $M \in \mathbb{N}$.
- Eigenvalues $\lambda_1 \geq \dots \geq \lambda_M > 0$, number of subspaces N , and dimension of subspaces m satisfying (FAC).

Algorithm:

- 1) For $j = 1, \dots, M$ do
- 2) Set $k := 1$.
- 3) Repeat
- 4) If $\lambda_j < 2$ and $\lambda_j \neq 1$ then
- 5) $w_k := \sqrt{\frac{\lambda_j}{2}} \cdot e_j + \sqrt{1 - \frac{\lambda_j}{2}} \cdot e_{j+1}$.
- 6) $w_{k+1} := \sqrt{\frac{\lambda_j}{2}} \cdot e_j - \sqrt{1 - \frac{\lambda_j}{2}} \cdot e_{j+1}$.
- 7) $k := k + 2$.
- 8) $\lambda_j := 0$.
- 9) $\lambda_{j+1} := \lambda_{j+1} - (2 - \lambda_j)$.
- 10) else
- 11) $w_k := e_j$.
- 12) $k := k + 1$.
- 13) $\lambda_j := \lambda_j - 1$.
- 14) end;
- 15) until $\lambda_j = 0$.
- 16) end;

Output:

- Fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ with $\mathcal{W}_i := \text{span}\{w_{i+km} : k = 0, \dots, N\}$.

FIGURE 2. The FFCRE algorithm for constructing a fusion frame with a desired fusion frame operator.

The principle for constructing the row vectors w_k which generate the subspaces \mathcal{W}_i of the fusion frame is similar to that in (FFCIE), that is, again the matrix W which contains the vectors w_k , $k = 1 \dots, Nm$ as rows is filled up from top to bottom, row by row in such a way that the ℓ_2 norm of the rows is 1, the ℓ_2 norm of column j is λ_j , $j = 1, \dots, M$, and the columns are orthogonal. The vectors w_k are then assigned to subspaces in such a way that the vectors assigned to each subspace form an orthonormal system. However, here the task is more delicate since the λ_j 's are not all integers. This forces the introduction of (2×2) -submatrices of the type

$$\begin{pmatrix} \sqrt{\frac{\lambda_j}{2}} & \sqrt{1 - \frac{\lambda_j}{2}} \\ \sqrt{\frac{\lambda_j}{2}} & -\sqrt{1 - \frac{\lambda_j}{2}} \end{pmatrix}.$$

These submatrices have orthogonal columns and unit norm (ℓ_2 norm) rows and allow us to handle non-integer eigenvalues. This construction was introduced in [15] for constructing tight fusion frames.

Before we prove that (FFCRE) indeed produces fusion frames with desired operators we consider a special case, in which the construction coincides with the construction of frames with desired frame operators. Our intention is to highlight the applicability of (FFCRE) to the construction of frames with arbitrary frame operators and to present a simple example that demonstrates how the algorithm works. A detail analysis of the algorithm and the proof of its correctness are provided in Subsection 4.2.5.

4.2.2. A Special Case and An Example. In the special case where $m = 1$ a fusion frame reduces to a frame and (FFCRE) simplifies to an algorithm for constructing frames with desired fusion frame operators. This algorithm, which we refer to as (FCRE), is outlined in Figure 3.

We now present an example to demonstrate the application of (FCRE) as a special case of (FFCRE).

Example 4.2. Let $M = 3$, $m = 1$ (special case of frame construction), $N = 8$, and $\lambda_1 = \frac{11}{4}$, $\lambda_2 = \frac{11}{4}$, $\lambda_3 = \frac{10}{4}$. Then, the algorithm constructs the following matrix W . Notice that indeed the ℓ_2 norm of the rows is 1, the ℓ_2 norm of the column j is λ_j , $j = 1, \dots, M$, and the columns are orthogonal.

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ \sqrt{3/8} & \sqrt{5/8} & 0 \\ \sqrt{3/8} & -\sqrt{5/8} & 0 \\ 0 & 1 & 0 \\ 0 & \sqrt{1/4} & \sqrt{3/4} \\ 0 & \sqrt{1/4} & -\sqrt{3/4} \\ 0 & 0 & 1 \end{bmatrix}$$

The eigenvalues of the frame operator of the constructed frame $\{w_{k,j}\}_{k=1}^8$ are indeed $\frac{11}{4}$, $\frac{11}{4}$, and $\frac{10}{4}$ as a simple computation shows. This also follows from Theorem 4.8 or Corollary 4.9 presented later in this subsection.

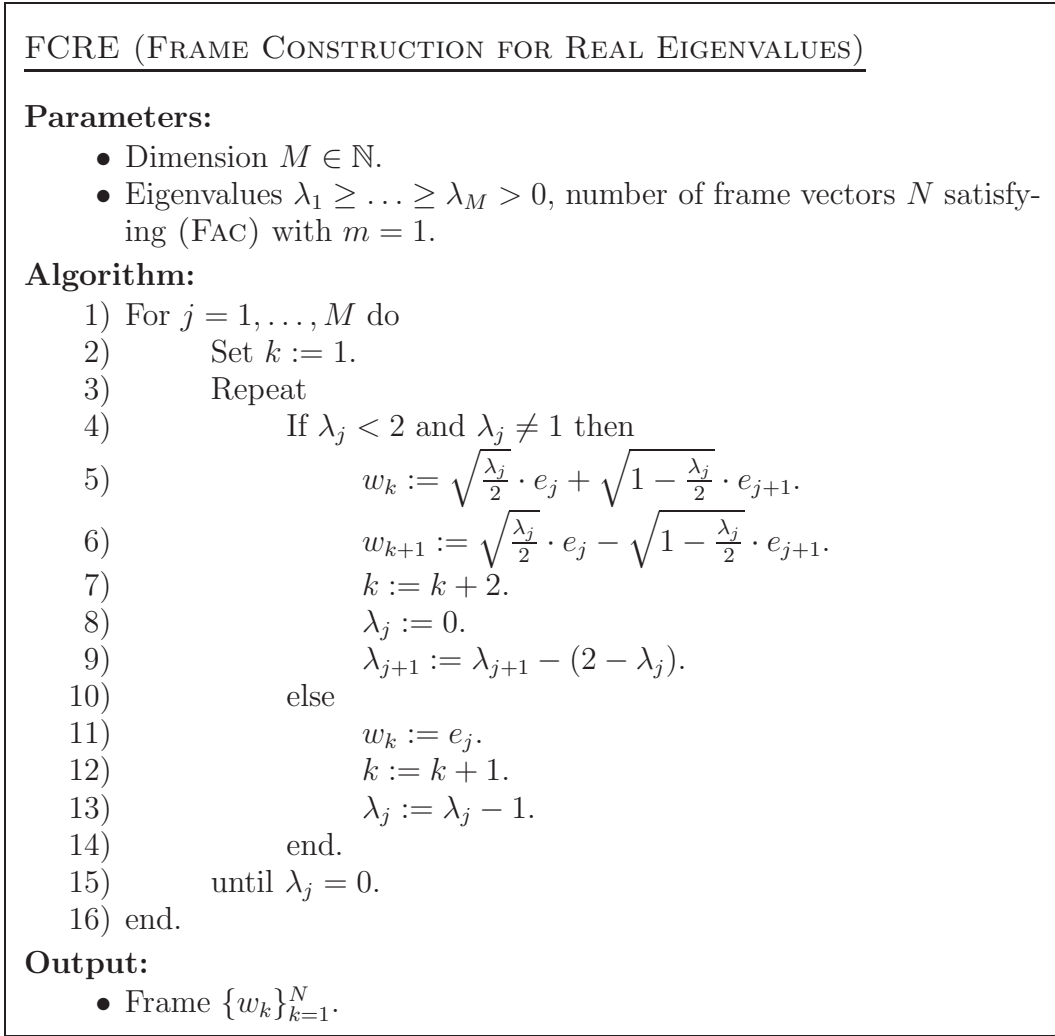


FIGURE 3. The FCRE algorithm for constructing a frame with a desired frame operator.

From now on we concentrate on the analysis of (FFCRE), keeping in mind that our analysis also applies to (FCRE) as a special case.

4.2.3. *Feasibility Checks.* Before proving that (FFCRE) indeed produces a fusion frame satisfying (FF1) and (FF2), we investigate the feasibility of the solution furnished by the algorithm.

Lemma 4.3. *For all $k = 1, \dots, Nm$,*

$$\|w_k\|_2^2 = 1.$$

Proof. This follows immediately from Lines 5), 6), and 11) of (FFCRE). □

Denoting the λ_j 's in Lines 4) – 6) of (FFCRE) by $\tilde{\lambda}_j$'s to distinguish them from the eigenvalues λ_j , $j = 1 \dots, M$, the only two problems which could occur while running (FFCRE) are:

- (P1) $\lambda_{j+1} - (2 - \tilde{\lambda}_j) < 0$ in Line 9) for some $j = 1, \dots, M - 1$,
(P2) using e_{M+1} in Lines 5) – 9) when performing the step for $j = M$.

The following result shows that these cannot happen.

Proposition 4.4. *If $\lambda_j \geq 2$ for all $j = 1 \dots, M$, then (P1) and (P2) cannot happen.*

Proof. (P1). Since $\lambda_j \geq 2$ for all $j = 1 \dots, M$, we have

$$\lambda_{j+1} \geq 2 \geq 2 - \tilde{\lambda}_j \quad \text{for all } j = 1, \dots, M - 1.$$

(P2). Suppose the algorithm is executed until Line 16) with $j = M - 1$. Let $K + 1$ denote the value which k has reached at this point, and denote the coefficients of the vectors w_k by $w_k = (w_{k1}, \dots, w_{kM})$. This means that so far we have constructed w_{kj} for $k = 1, \dots, K$, $j = 1 \dots, M - 1$. Then, by construction,

$$\sum_{k=1}^K w_{kj}^2 = \lambda_j \quad \text{for all } 1 \leq j \leq M - 1. \quad (4.7)$$

We have to distinguish between two cases:

Case 1. $w_{K-2,M} = 0$ and $w_{K-1,M} = 0$. Then, by (4.7) and Lemma 4.3,

$$\sum_{j=1}^{M-1} \lambda_j = \sum_{j=1}^{M-1} \sum_{k=1}^K w_{kj}^2 = \sum_{k=1}^K \sum_{j=1}^{M-1} w_{kj}^2 = \sum_{k=1}^K 1 = K.$$

Since

$$\sum_{j=1}^M \lambda_j = \lambda_M + \sum_{j=1}^{M-1} \lambda_j = \lambda_M + K$$

is an integer, it follows that λ_M is an integer as well. Hence during the step $j = M$ only the Block 11) – 14) as opposed to the Block 5) – 9) will be executed. Thus (P2) does not happen.

Case 2. $w_{K-2,M} = \sqrt{1 - \frac{\tilde{\lambda}_{M-1}}{2}}$ and $w_{K-1,M} = -\sqrt{1 - \frac{\tilde{\lambda}_{M-1}}{2}}$. In this case,

$$\sum_{j=1}^{M-1} \lambda_j + (2 - \tilde{\lambda}_{M-1}) = \sum_{j=1}^{M-1} \sum_{k=1}^K w_{kj}^2 = \sum_{k=1}^K \sum_{j=1}^{M-1} w_{kj}^2 = \sum_{k=1}^K 1 = K,$$

an integer. Since $\sum_{j=1}^M \lambda_j$ is an integer as well, so is

$$\sum_{j=1}^M \lambda_j - \left(\sum_{j=1}^{M-1} \lambda_j + (2 - \tilde{\lambda}_{M-1}) \right) = \lambda_M - (2 - \tilde{\lambda}_{M-1}).$$

Hence, as before, in the step $j = M$ only the Block 11) – 14) as opposed to the Block 5) – 9) will be executed; here $\lambda_M - (2 - \tilde{\lambda}_{M-1})$ times. Thus, in this situation, (P2) does not occur. \square

4.2.4. *Terminology and Lemmata.* In preparation for a detailed analysis of (FFCRE), which is presented in Subsection 4.2.5, we need to establish some terminology and a few results.

Definition 4.5. *An entry of a vector w_k , $k \in \{1, \dots, Nm\}$ of the form $\pm\sqrt{1 - \frac{\tilde{\lambda}_j}{2}}$ (entered in Line 5) or 6) of (FFCRE)) will be termed a terminal point. An initial point will be an entry of the form $\pm\sqrt{\tilde{\lambda}_j/2}$ (entered in Line 5) or 6)).*

Considering the matrix $W \in \mathbb{R}^{Nm \times M}$ with the vectors w_1, \dots, w_{Nm} as rows, the initial points start non-zero entries in a row with more than one non-zero entry, whereas the terminal points end such non-zero entries. It is obvious from algorithm (FFCRE) that column n of W has no initial points if and only if

$$\sum_{j=1}^n \lambda_j \text{ is an integer,}$$

and it has no terminal points if and only if

$$\sum_{j=1}^{n-1} \lambda_j \text{ is an integer.}$$

Let $N(j)$ denote the number of non-zero terms in each column j , $j = 1, \dots, M$ of the matrix W , that is, let $N(j)$ denote the number of non-zero entries of the vector $w_{\cdot j}$. The following proposition determines exactly the value of $N(j)$ depending on the occurrence of initial and/or terminal points. We remind the reader of the definition of $\tilde{\lambda}_j$ right before Proposition 4.4.

Lemma 4.6. *The following conditions hold for the previously defined values $N(j)$, $j = 1, \dots, M$.*

- (i) $N(j) = \lambda_j$, if $w_{\cdot j}$ contains no initial or terminal points.
- (ii) $N(j) = \lfloor \lambda_j \rfloor + 1$, if $w_{\cdot j}$ contains terminal, but no initial points.
- (iii) $N(j) = \lfloor \lambda_j \rfloor + 2$, if $w_{\cdot j}$ contains initial, but no terminal points.
- (iv) If $w_{\cdot j}$ contains both initial and terminal points, then
 - (a) if $\tilde{\lambda}_j \geq \tilde{\lambda}_{j-1}$, then $N(j) = \lfloor \lambda_j \rfloor + 2$,
 - (b) if $\tilde{\lambda}_j < \tilde{\lambda}_{j-1}$ then $N(j) = \lfloor \lambda_j \rfloor + 3$.
- (v) If λ_{j_0} is the first non-integer value, then $N(j_0) = \lfloor \lambda_{j_0} \rfloor + 2$.
- (vi) If λ_{j_1} is the last non-integer value, then $N(j_1) = \lfloor \lambda_{j_1} \rfloor + 1$.

Proof. (i). This is obvious, since in this case only the Block 11) – 14) is performed as opposed to the Block 5) – 9).

(ii). Letting n_j denote the number of ones in the vector $w_{\cdot j}$, it follows that $N(j) = n_j + 2$. Hence

$$\lambda_j = n_j + (2 - \tilde{\lambda}_{j-1}) = n_j + 1 + (1 - \tilde{\lambda}_{j-1}) \quad \text{with } 0 < 1 - \tilde{\lambda}_{j-1} < 1.$$

This implies $\lfloor \lambda_j \rfloor = n_j + 1$, and thus

$$\lfloor \lambda_j \rfloor + 1 = n_j + 2 = N(j).$$

(ii). Letting n_j denote the number of ones in the vector $w_{\cdot,j}$, it follows that $N(j) = n_j + 2$. Since the entries of $w_{\cdot,j}$ are n_j times a 1 as well as the values $\pm\sqrt{1 - \frac{\tilde{\lambda}_{j-1}}{2}}$,

$$\lambda_j = n_j + (2 - \tilde{\lambda}_{j-1}) = n_j + 1 + (1 - \tilde{\lambda}_{j-1}) \quad \text{with } 0 < 1 - \tilde{\lambda}_{j-1} < 1.$$

This implies $\lfloor \lambda_j \rfloor = n_j + 1$, and thus

$$N(j) = n_j + 2 = \lfloor \lambda_j \rfloor + 1.$$

(iii). Now the non-zero entries of the vector $w_{\cdot,j}$ are $\pm\sqrt{\frac{\tilde{\lambda}_j}{2}}$ as well as n_j , say, entries 1. Hence $N(j) = n_j + 2$, and

$$\lambda_j = n_j + \tilde{\lambda}_j \quad \text{with } 0 < \tilde{\lambda}_j < 1.$$

This implies $\lfloor \lambda_j \rfloor = n_j$, and thus

$$N(j) = n_j + 2 = \lfloor \lambda_j \rfloor + 2.$$

(iv). The vector $w_{\cdot,j}$ contains as non-zero entries, the initial points $\pm\sqrt{\frac{\tilde{\lambda}_j}{2}}$ and the terminal points $\pm\sqrt{1 - \frac{\tilde{\lambda}_{j-1}}{2}}$ as well as, say, n_j entries 1, hence $N(j) = kn_j + 4$. Thus

$$\lambda_j = n_j + \tilde{\lambda}_j + (2 - \tilde{\lambda}_{j-1}) = n_j + 2 + (\tilde{\lambda}_{j-1} - \tilde{\lambda}_j).$$

If $\tilde{\lambda}_{j-1} - \tilde{\lambda}_j \geq 0$, then $\lfloor \lambda_j \rfloor = n_j + 2$, which implies

$$N(j) = n_j + 4 = \lfloor \lambda_j \rfloor + 2.$$

If $\tilde{\lambda}_{j-1} - \tilde{\lambda}_j < 0$, then $\lfloor \lambda_j \rfloor = n_j + 1$, which implies

$$N(j) = n_j + 4 = \lfloor \lambda_j \rfloor + 3.$$

(v) and (vi). These are direct consequences from the previous conditions. \square

The following lemma shows an interesting relation between consecutive values of $N(j)$ as j progresses. However, we note that only the previous lemma is required for the proofs of the main theorems that will be presented in Subsection 4.2.5.

Lemma 4.7. *For any $j \in \{1, \dots, M-1\}$, the following conditions hold for the previously defined values $N(j)$ and $N(j+1)$ supposing that they are not integers.*

- (i) *If $w_{\cdot,j}$ contains no initial or terminal points, then $N(j) \geq N(j+1) - 1$.*
- (ii) *If $w_{\cdot,j}$ contains initial, but no terminal points, then*
 - (a) *if $\lambda_j + \lambda_{j+1}$ is an integer, then $N(j) \geq N(j+1) + 1$,*
 - (b) *if $\lambda_j + \lambda_{j-1}$ is not an integer, then $N(j) \geq N(j+1) - 1$.*
- (iii) *If $w_{\cdot,j}$ contains both initial and terminal points, then $N(j) \geq N(j+1) - 1$.*

Proof. Recall that we have $\lambda_j \geq \lambda_{j+1}$.

(i). Since $w_{\cdot,j}$ contains no initial points and λ_{j+1} is not an integer, the vector $w_{\cdot,j+1}$ contains initial, but no terminal points. Thus, by Lemma 4.6,

$$N(j) = \lfloor \lambda_j \rfloor + 1 \geq \lfloor \lambda_{j+1} \rfloor + 2 - 1 = N(j+1) - 1.$$

(ii). By Lemma 4.6, $N(j) = \lfloor \lambda_j \rfloor + 2$. Also $w_{\cdot,j}$ contains initial points, hence the vector $w_{\cdot,j+1}$ contains terminal points.

- (a). Since $\lambda_j + \lambda_{j+1}$ is an integer and $w_{\cdot,j}$ does not contain any terminal points, the vector $w_{\cdot,j+1}$ does not contain initial points. This implies $N(j+1) = \lfloor \lambda_{j+1} \rfloor + 1$.
- (b). Since $\lambda_j + \lambda_{j+1}$ is not an integer and $w_{\cdot,j}$ does not contain any terminal points, the vector $w_{\cdot,j+1}$ does contain initial points. This implies $N(j+1) \leq \lfloor \lambda_{j+1} \rfloor + 3$.
- (iii). By Lemma 4.6, $N(j) \geq \lfloor \lambda_j \rfloor + 2$ and the vector $w_{\cdot,j+1}$ can not contain more than $\lfloor \lambda_{j+1} \rfloor + 3$ non-zero entries. \square

4.2.5. *Main Results: Analysis of (FFCRE)*. We now present the main results concerning (FFCRE). We first show that the algorithm indeed delivers the correct fusion frame, i.e., a fusion frame with the prescribed fusion frame operator. From this result, we deduce that in certain cases a fusion frame can be turned into a tight fusion frame by careful adding of new subsets (compare also with Corollary 3.5).

Theorem 4.8. *Suppose the real values $\lambda_1 \geq \dots \geq \lambda_M$, $N \in \mathbb{N}$, and $m \in \mathbb{N}$ satisfy (FAC) as well as the following conditions.*

- (i) $\lambda_M \geq 2$.
- (ii) *If j_0 is the first integer in $\{1, \dots, M\}$, for which λ_{j_0} is not an integer, then $\lfloor \lambda_{j_0} \rfloor \leq N - 3$.*

Then the fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ constructed by (FFCRE) fulfills (FF1) and (FF2).

Proof. If the set of vectors

$$\{w_{i+km} : k = 0, \dots, N-1\}$$

is pairwise orthogonal for each $i = 1, \dots, N$, then (FF1) and (FF2) follow automatically. Fix $i \in \{1, \dots, N\}$. By construction, it is sufficient to show that, for each $0 \leq k \leq N-2$, the vectors w_{i+km} and $w_{i+(k+1)m}$ are disjointly supported. We distinguish between the following two cases:

Case 1. The vector w_{i+km} is a unit vector, e_n , say. By (ii) and Lemma 4.6, $w_{\cdot,n}$ does not have more than N non-zero elements. When defining the vector $w_{i+(k+1)m}$, already $N-1$ vectors w_ℓ have been defined before its construction. Therefore this definition takes place in a different step of the loop in Line 1). Hence $w_{i+(k+1)m,j} = 0$ for all $j = 1, \dots, n$. This prove the claim in this case.

Case 2. The vector w_{i+km} has two non-zero entries, namely an initial and a terminal point, where the terminal point is at the n th position, say. Again, by (ii) and Lemma 4.6, $w_{\cdot,n}$ does not have more than N non-zero elements. Hence, concluding as before, $w_{i+(k+1)m,j} = 0$ for all $j = 1, \dots, n$. This prove the claim also in this case. \square

Certainly, this theorem also holds in the special case of frames, i.e., 1-dimensional subspaces.

Corollary 4.9. *Suppose the real values $\lambda_1 \geq \dots \geq \lambda_M$ and $N \in \mathbb{N}$ satisfy*

$$\sum_{j=1}^M \lambda_j = N$$

as well as the following conditions.

- (i) $\lambda_M \geq 2$.

- (ii) If j_0 is the first integer in $\{1, \dots, M\}$, for which λ_{j_0} is not an integer, then $\lfloor \lambda_{j_0} \rfloor \leq N - 3$.

Then the eigenvalues of the frame operator of the frame $\{w_k\}_{k=1}^N$ constructed by (FFCRE) are $\{\lambda_j\}_{j=1}^M$.

Proof. This result follows directly from Theorem 4.8 by choosing $m = 1$. \square

Theorem 4.8 is now applied to generate a tight fusion frame from a given fusion frame, satisfying some mild conditions.

Theorem 4.10. Let $\{\mathcal{W}_i\}_{i=1}^N$ be a fusion frame for \mathbb{R}^M with $\dim \mathcal{W}_i = m < M$ for all $i = 1, \dots, N$, and let S be the associated fusion frame operator with eigenvalues $\lambda_1 \geq \dots \geq \lambda_M$ and eigenvectors $\{e_j\}_{j=1}^M$. Further, let A be the smallest positive integer, which satisfies the following conditions.

- (i) $\lambda_1 + 2 \leq A$.
- (ii) $AM = N_0 m$ for some $N_0 \in \mathbb{N}$.
- (iii) $A \leq \lambda_1 + N_0 - (N + 3)$.

Then there exists a fusion frame $\{\mathcal{V}_i\}_{i=1}^{N_0-N}$ for \mathbb{R}^M with $\dim \mathcal{V}_i = m$ for all $i \in \{1, \dots, N_0 - N\}$ so that

$$\{\mathcal{W}_i\}_{i=1}^N \cup \{\mathcal{V}_i\}_{i=1}^{N_0-N}$$

is an A -tight fusion frame.

Proof. The first task is to check whether such a positive integer A exists at all. We use the ansatz $A = nm$ for some $n \in \mathbb{N}$. This immediately satisfies (ii). Now choose n as the smallest positive integer still satisfying

$$\lambda_1 + 2 \leq A.$$

Thus (i) and (ii) are fulfilled (and they will still be fulfilled for all larger $n \in \mathbb{N}$.) For inequality (iii), we require

$$nm \leq \lambda_1 + nM - (N + 3),$$

which we can reformulate as

$$\frac{m}{M} \leq \frac{\lambda_1}{Mn} + 1 - \frac{N + 3}{Mn}.$$

Since $\frac{m}{M} < 1$ by assumption, n can be chosen large enough for this inequality to be satisfied.

Next, we set

$$\mu_j = A - \lambda_j \quad \text{for all } j = 1, \dots, M.$$

In particular, we have $\mu_1 \geq \dots \geq \mu_M$. We claim that the hypotheses of Theorem 4.8 are satisfied by the sequence $\{\mu_j\}_{j=1}^M$. For the proof, we refer to the assumption of the present theorem as (i'), (ii'), and (iii').

(i). By (i'),

$$\mu_1 = A - \lambda_1 \geq 2.$$

Letting $N_1 = N_0 - N$,

$$\sum_{j=1}^M \mu_j = \sum_{j=1}^M (A - \lambda_j) = AM - \sum_{j=1}^M \lambda_j = AM - Nm = N_0 m - Nm = N_1 m.$$

(ii). By (iii'),

$$\mu_1 = A - \lambda_1 \leq (N_0 - N) - 3 = N_1 - 3.$$

From Theorem 4.8 it follows that there exists a fusion frame $\{\mathcal{V}_i\}_{i=1}^{N_1}$ for \mathbb{R}^M whose fusion frame operator S_1 , say, has eigenvectors $\{e_j\}_{j=1}^M$ and respective eigenvalues $\{\mu_j\}_{j=1}^M$. The fusion frame operator for $\{\mathcal{W}_i\}_{i=1}^N \cup \{\mathcal{V}_i\}_{i=1}^{N_1}$ is $S + S_1$, which then possesses as eigenvectors the sequence $\{e_j\}_{j=1}^M$ with associated eigenvalues

$$\lambda_j + \mu_j = \lambda_j + (A - \lambda_j) = A.$$

Hence $\{\mathcal{W}_i\}_{i=1}^N \cup \{\mathcal{V}_i\}_{i=1}^{N_0-N}$ constitutes an A -tight fusion frame. \square

The number of m -dimensional subspaces added in Theorem 4.10 to force a fusion frame to become tight is in fact the smallest number that can be added in general. For this, let $\{\mathcal{W}_i\}_{i=1}^N$ be a fusion frame for \mathbb{R}^M with fusion frame operator S having eigenvalues $\{\lambda_j\}_{j=1}^M$. Suppose $\{\mathcal{V}_i\}_{i=1}^{N_1}$ is any family of m -dimensional subspaces with fusion frame operator S_1 , say, and so that the union of these two families is an A -tight fusion frame for \mathbb{R}^M . Thus

$$S + S_1 = AI,$$

which implies that the eigenvalues $\{\mu_j\}_{j=1}^M$ of S_1 satisfy

$$\mu_j = A - \lambda_j \quad \text{for all } j = 1, \dots, M,$$

and

$$\sum_{j=1}^M \mu_j = \sum_{j=1}^M (A - \lambda_j) = AM - Nm = N_1 m.$$

In particular,

$$AM = (N_1 - N)m = N_0 m.$$

Thus, we have examples to show that – in general – fusion frames with the above properties of S_1 cannot be constructed unless the hypotheses of Theorem 4.10 are satisfied. This shows that the smallest constant satisfying this theorem is in general the smallest number of subspaces we can add to obtain a tight fusion frame.

4.3. Extensions and Related Problems. Finally, we would like to discuss several extensions and related problems.

Weights. The handling of weights is particularly delicate. When turning a frame $\{f_i\}_{i=1}^N$ into the fusion frame $\{(\text{span}\{f_i\}, \|f_i\|)\}_{i=1}^N$ consisting of 1-dimensional subspaces and having the same (fusion) frame operator as well as the same (fusion) frame bounds [21, Prop. 2.14], we notice that the subspaces are generated by the frame vectors and the weights have to be chosen equal to the norms of the frame vectors. Thus choosing weights is in a sense comparable to choosing the lengths of frame vectors. However the design is more delicate due to the necessary compensation of the dimensions of the subspaces. Generalizing, for instance, Theorem 4.8 to weighted fusion frames requires careful handling and a thorough understanding of the interplay between subspace dimensions and weights. This is currently under investigation.

Chordal Distances. It was shown in [40] that maximal resilience of fusion frames to noise and erasures is closely related to the chordal distances between pairs of subspaces forming the

fusion frame. Hence it would be desirable to be able to control the set of chordal distances in construction procedures for fusion frames. The results in Section 3 already allow this control. However, for instance, for Theorem 4.8 this control is more difficult.

Sparsity. A fusion frame (also in the special case of a frame) with a desired fusion frame operator is often times designed to provide an optimal tool for data processing with specific performance metrics. The data processing ultimately needs to be performed with a DSP board and therefore it is desired to reduce the computational complexity (number of additions and multiplications required) as much as possible. This motivates the design of fusion frames, with fusion frame operators, which have sparsity properties. In other words, it is desired for the vectors of a frame as well as the generating vectors of the subspaces of a fusion frame to be sparse. The sparsity allows for fast vector-vector multiplications. In the construction presented in Section 4, this principle is deployed by only using sparse linear combinations of unit vectors. We conjecture that this construction enjoys maximal sparsity. However, we do not have a rigorous proof. In general, generating frames and fusion frames which allow for fast processing through additional inner structural properties such as sparsity is becoming more and more indispensable.

Equivalence Classes. Our results normally produce one fusion frame satisfying a desired property. However, from a scholarly point of view, it would be desirable to be able to generate each such fusion frame in the sense of the whole “equivalence class” of fusion frames satisfying a special property. This is beyond our reach at this point, since even the following apparently simple problem is still unsolved: Construct one Parseval frame in each equivalence class choosing unitary equivalence as the relation.

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