Hybrid Higher Order Numerical Methods in Electromagnetics

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Abstract — Hybrid higher order numerical methods for electromagnetic modeling are presented based on the method of moments in surface and volume integral equation formulations, finite element method, and diakoptic domain decomposition.

1 INTRODUCTION

Most of numerical methodologies for analysis of general electromagnetic structures in the frequency domain belong to one of the following three basic classes: (1) method of moments (MoM) in conjunction with the surface integral equation (SIE) approach [1], (2) MoM in the context of the volume integral equation (VIE) formulation [2], and (3) finite element method (FEM) [3]. From the numerical point of view, the three classes of methods have a lot in common. From the application point of view, each of the approaches has its advantages and deficiencies, and the choice of the “best” method depends on the particular problem that needs to be solved. However, the best way to exploit the full potential of any single approach is to combine it with other approaches.

This paper presents our on-going research efforts and some new advances in SIE, VIE, FEM, FEM-SIE, VIE-SIE, SIE-FEM-diakoptics, and SIE-VIE-diakoptics methods, where the diakoptics approach, a unique version of the domain decomposition method, is used to enhance the hybridizations. The diakoptics method is based on a linear combination of independent solutions of diakoptic subsystems, using explicit linear relations between coefficients in expansions of equivalent electric and magnetic surface currents on boundary surfaces of subsystems [4], [5]. The solutions are also enhanced by truly higher order geometrical and current/field numerical discretizations.

Two characteristic examples of higher order SIE-FEM-diakoptics and SIE-VIE-diakoptics modeling are presented to illustrate, validate, and evaluate individual numerical methods, formulations, hybridizations, and higher order solutions.

2 NUMERICAL BACKGROUND

In SIE techniques, metallic and dielectric surfaces of an electromagnetic structure (antenna or scatterer) under consideration are modeled using Lagrange-type generalized curved parametric quadrilaterals of arbitrary geometrical orders $K_u$ and $K_v$ ($K_u, K_v \geq 1$), shown in Fig. 1(a) and analytically described as [1]

$$r(u,v) = \sum_{k=0}^{K_u} \sum_{l=0}^{K_v} r_{kl} L^K_{kl} (u) L^K_l (v), \quad -1 \leq u, v \leq 1,$$

where $L^K_{kl}$ represent Lagrange interpolation polynomials given by

$$L^K_{kl}(u) = \prod_{i=k}^{K_u} \left( \frac{u - u_i}{u_k - u_i} \right),$$

with $u_i$ being the interpolation nodes along an interval $-1 \leq u \leq 1$ (note that $L^K_{kl}$ is unity for $u = u_k$ and zero at all other nodes), and similarly for $L^K_l (v)$. Electric and magnetic surface current density vectors, $\mathbf{J}$, and $\mathbf{M}$, over every generalized quadrilateral in the model are approximated by means of divergence-conforming hierarchical-type vector basis functions constructed from simple power functions ($P$) in parametric coordinates $u$ and $v$ [1],

$$\mathbf{J}_{su} = \frac{1}{S} \sum_{i=0}^{N_u} \sum_{j=0}^{N_v-1} a_{ij}(u) \hat{P}_i(u) P_j(v) \mathbf{a}_u,$$

and analogously for $\mathbf{M}_s$, where $\hat{P}$ denotes modified (divergence-conforming) power functions,

$$\hat{P}_i(u) = \begin{cases} 1 - u, & i = 0 \\ u + 1, & i = 1 \\ u^i - 1, & i \geq 2, \text{ even} \\ u^i - u, & i \geq 2, \text{ odd} \end{cases}, \quad P_j(v) = v^j,$$
$N_u$ and $N_v$ \((N_u, N_v \geq 1)\) are the orders of the polynomial current approximation, \(\{\alpha\}\) are unknown current-distribution coefficients, and \(\mathcal{J}\) is the Jacobian of the covariant transformation, found from unitary vectors \(a_u\) and \(a_v\), \(\mathcal{J} = a_u \times a_v\), \(a_u = \partial r/\partial u\), \(a_v = \partial r/\partial v\).

The building block for volumetric VIE and FEM modeling is a Lagrange-type interpolation generalized hexahedron, shown in Fig. 1(b), a volume (3-D) generalization of the quadrilateral patch in Fig. 1(a) [2], [3]. The equivalent electric displacement vector, \(D\), inside the VIE hexahedra and the electric and magnetic field vectors, \(E\) and \(H\), inside the FEM hexahedra are approximated by divergence- and curl-conforming hierarchical vector expansions obtained as generalizations of 2-D bases in (3)–(4) [2], [3].

Figure 1: (a) Generalized curved parametric quadrilateral patch for higher order SIE modeling. (b) Generalized curved parametric hexahedral volume element (with continuous spatial variation of medium parameters) for higher order VIE and FEM modeling.

Let us illustrate the SIE-FEM-diakoptics method in an example of a system composed of two subsystems, shown in Fig. 2, with one subsystem being analyzed by the SIE and the other by the FEM method. Also, let the total number of unknown coefficients for the approximation of currents \(J_s\) and \(M_s\) at the material discontinuities of the first subsystem be \(N_1\), that for the approximation of fields \(E\) and \(H\) in the second subsystem be \(N_2\), and the one for the approximation of the equivalent surface currents \(J_e\) and \(M_e\) at the diakoptic boundary be \(2D\) \((D\) coefficients for \(J_e\) and the same number for \(M_e\)). Each subsystem in Fig. 2, namely, the SIE subsystem \((i = 1)\) and the FEM subsystem \((i = 2)\), can be represented by a set of linear relations between column-matrices \(\mathbf{[e]}\) and \(\mathbf{[m]}\), filled with coefficients for the approximation of \(J_e\) and \(M_e\), cast in matrix form as [4]

\[
[\mathbf{J}_i] = [\mathbf{Y}_i][\mathbf{m}] + [\mathbf{J}_0], \quad i = 1, 2, \tag{5}
\]

where \(\mathbf{[Y]}\) is the diakoptic \(D \times D\) matrix of the subsystem and \(\mathbf{[J]}_0\) is the \(D \times 1\) column-matrix representing the excitation in the subsystem, as well as linear relations between coefficients \(\mathbf{[m]}\) and FEM coefficients of \(E\) and \(H\) for \(i = 2\),

\[
\begin{bmatrix}
\mathbf{e}_2 \\
\mathbf{h}_2
\end{bmatrix} = [\mathbf{C}]_2[\mathbf{m}] + \begin{bmatrix}
\mathbf{e}_2 \\
\mathbf{h}_2
\end{bmatrix}, \quad i = 1, 2, \tag{6}
\]

and similarly for relations between coefficients \(\mathbf{[m]}\) and SIE coefficients of \(J_s\) and \(M_s\) for \(i = 1\).
excitations in the $i$-th subsystem are turned off and the subsystem is excited with one unity-valued coefficient of $[\mathbf{m}_e]$, while all other coefficients of $[\mathbf{m}_e]$ are set to zero. By using SIE or FEM methods for the analyzed subsystem, we calculate the coefficients of $\mathbf{J}_s$, $\mathbf{M}_s$, and $\mathbf{J}_e$ in the SIE subsystem, and the FEM coefficients of $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{J}_e$ in the FEM subsystem. Therefore, the diakoptic method allows the analysis of a complex and large electromagnetic problem (which involves solving a large matrix equation) by solving smaller problems, which leads to a more efficient electromagnetic analysis.

In order to obtain the solution of the original electromagnetic problem, in Fig. 2, using the matrices that represent the two subsystems in (5)–(6), we relate the diakoptic coefficients of the equivalent electric and magnetic surface currents on the diakoptic boundaries between the individual subsystems as

\[ [\mathbf{e}_2] = [\mathbf{e}_1] \quad \text{and} \quad [\mathbf{m}_2] = [\mathbf{m}_1]. \tag{7} \]

Combining (5) and (7), we obtain the following diakoptic system of linear equations:

\[ ([\mathbf{Y}_1] - [\mathbf{Y}_2]) [\mathbf{e}_2] = -[\mathbf{J}_e]_0 + [\mathbf{J}_e]_0, \tag{8} \]

whose solution is $[\mathbf{m}_e]$. Diakoptic coefficients $[\mathbf{j}_e]$, $i = 1, 2$, are then computed from $[\mathbf{m}_e]$ using (5) for each subsystem, and coefficients $[\mathbf{e}_1]$ and $[\mathbf{h}_2]$ are obtained from (6), and similarly for $[\mathbf{j}_s]$ and $[\mathbf{m}_1]$. Once we have the coefficients for the subsystems, we can calculate the electromagnetic fields in both subsystems, as well as any other quantity of interest for the original electromagnetic structure in Fig. 2.

In the SIE-VIE-diakoptics method, coefficients $[\mathbf{e}]$ and $[\mathbf{h}]$ in (6) are replaced by coefficients $[\mathbf{d}]$ for the approximation of the equivalent electric displacement vector, $\mathbf{D}$, in the VIE subsystem [5].

### 3 NUMERICAL RESULTS AND DISCUSSION

As the first example, consider a $4 \times 4 \times 4$ array of cubical dielectric scatterers shown in Fig. 3 – analyzed by the SIE-FEM-diakoptics method. The dimensions are $a = s = 1\lambda_0$ ($\lambda_0$ being the free-space wavelength), and $\varepsilon_r = 2.25$ for the dielectric. The structure is modeled by 65 diakoptic subsystems (64 FEM domains and one open-region SIE domain). Each cubical diakoptic subsystem is modeled by a single (entire-domain) FEM element with orders $K_u = K_v = K_w = 1$ and $N_u = N_v = N_w = 3$, with a diakoptic cubical surface constructed from six SIE patches with $K_u = K_v = 1$ and $N_u = N_v = 3$. The number of FEM unknowns per subsystem is 144 and the total number of diakoptic unknowns (sum of unknowns on all diakoptic surfaces) is 13,824, so the size of the system of diakoptic equations is $D = 6,912$.

![Figure 3: 3-D array of cubical dielectric scatterers modelled by the higher order SIE-FEM-diakoptics method and by WIPL-D.](image)

Fig. 4 presents the normalized bistatic radar cross section (RCS) of the array in a characteristic cut for plane wave excitation of the system shown in Fig. 3. The diakoptic results are compared with the solution obtained by WIPL-D (MoM/SIE commercial software) [6], which serves as a reference (the total number of unknowns used for modeling in WIPL-D is 24,576), and we observe an excellent agreement of the two sets of results. When comparing the diakoptic approach to the pure MoM/SIE solution (by commercial software), the reduction in the number of unknowns in the final system of equations is by 3.55 times. Note that the higher order SIE-FEM-diakoptics method would allow modeling of inhomogeneous
and/or anisotropic scatterers in the array in Fig. 3 at essentially the same computational cost.

Next, we consider a $4 \times 4$ array of spherical scatterers shown in Fig. 5, where $a = \lambda_0/2.31$, $d = \lambda_0/4.29$, $\varepsilon_r = 4.0$, $\theta_{inc} = 90^\circ$, and $\phi_{inc} = 0^\circ$, and analyze it by the SIE-VIE-diakoptics method. Each diakoptic subsystem is modeled by a single (entire-domain) curved generalized hexahedral VIE element with $K_u = K_v = K_w = 2$ and $N_u = N_v = N_w = 4$, enclosed by a diakoptic surface composed of six curved generalized quadrilateral SIE patches with $K_u = K_v = 2$ and $N_u = N_v = 4$. The number of VIE unknowns per subsystem is 240 and the total number of diakoptic unknowns is 1,536, resulting in $D = 768$ diakoptic unknowns for the analysis; the computation time is 30 s on a modest PC computer and RAM memory consumption is 9 MB.

![Figure 5: Array of dielectric spherical scatterers modeled by higher order SIE-VIE-diakoptics, MoM/VIE, and MoM/SIE methods.](image)

![Figure 6: Normalized bistatic RCS in the $\phi = 90^\circ$ cut of the array of scatterers in Fig. 5.](image)

Fig. 6 shows the normalized bistatic RCS of the array in a characteristic plane. We compare the diakoptic results with the solution obtained by the higher order MoM/VIE method [2], which requires 3,840 unknowns, 420 s of computation time, and 236 MB of memory. An excellent agreement of the two sets of results is observed, with the computation time and memory usage being in favor of the diakoptic method — reduction by 26.2 times in computation time and by 4.7 times in memory consumption when compared to the MoM/VIE method. As additional validation, shown in Fig. 6 is an excellent agreement of the diakoptic solution with the RCS computed using the higher order MoM/SIE method [1].

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References


