Computation of 3-D Electromagnetic Cavity Resonances Using Hexahedral Vector Finite Elements with Hierarchical Polynomial Basis Functions

Milan M. Ilić* and Branislav M. Notaroš
University of Massachusetts Dartmouth, ECE Department
285 Old Westport Road, N. Dartmouth, MA 02747, www.bnotaros.umassd.edu

1. Introduction

Finite Element Method (FEM) solutions to 3-D electromagnetic field problems based on vector (edge) finite elements [1] have proved to be free of inherent shortcomings of traditionally used scalar (node-based) finite elements. Low-order vector elements have become a standard tool in modern computational electromagnetics (CEM) and are routinely used in a variety of applications. However, given great demands for fast CEM techniques capable of accurate and reliable real-time simulation and CAD on PC's and moderate-sized workstations, higher-order vector finite elements are, in the opinion of the authors of this paper, a necessity for FEM modeling today and in the future.

We propose a FEM technique based on using curl-conforming vector hierarchical polynomial basis functions in hexahedral finite elements and implement it in analysis of 3-D electromagnetic cavities. The 1-D version of the technique was presented in [2]. The field expansions automatically satisfy continuity boundary conditions for tangential fields on surfaces shared by adjacent hexahedrons in a FEM mesh. Hierarchical basis functions allow elements of different orders to be incorporated into a mesh. Polynomial degrees can be high, so electrically large finite elements can be used (large-domain FEM technique). This dramatically reduces the overall number of unknowns for a given problem.

Evaluation of electromagnetic resonances of closed cavities is crucial for understanding the operation of many devices, such as particle accelerators, microwave filters, microwave ovens, etc. In this paper, we compute the eigenvalues of simple rectangular cavities using the new hierarchical higher-order vector FEM elements and the Galerkin testing procedure. The results are compared with the analytical solutions and the numerical results obtained by different low-order (small-domain) FEM techniques. Excellent convergence properties of the new FEM basis functions are demonstrated, along with the computational superiority of the large-domain FEM technique over the corresponding small-domain solutions. We show that a rectangular cavity can be very accurately modeled by a single hexahedral finite element with 3-D polynomial basis functions in parametric coordinates using 6-8 unknowns per wavelength in each dimension. Note that this is actually an entire-domain FEM model.

2. Trilinear Hexahedral Elements with Hierarchical Polynomial Basis Functions

As a basic building block for geometrical modeling of 3-D structures, we use a trilinear hexahedron [2, 3]. This is a body determined solely by eight arbitrary points in space, which represent its vertices. It can be described analytically as

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$$r(u,v,w)=\frac{1}{8}\left[r_1(1-u)(1-v)(1-w)+r_2(u+1)(1-v)(1-w)+\ldots+r_8(u+1)(v+1)(w+1)\right]$$
$$=r_c+r_u u+r_v v+r_w w+r_{uv}uv+r_{uw}uw+r_{vw}vw+r_{uvw}uvw, \quad -1\le u,v,w\le 1 \tag{1}$$

where r_1, r_2, ..., r_8 are the position vectors of the hexahedron vertices. Fields inside every hexahedron are represented as

$$E=\sum_{i=0}^{n_u}\sum_{j=0}^{n_v}\sum_{k=0}^{n_w}\left[a_{uijk}\, f_{uijk}+a_{vijk}\, f_{vijk}+a_{wijk}\, f_{wijk}\right], \tag{2}$$

where f are curl-conforming hierarchical-type vector basis functions defined as

$$f_{uijk}(u,v,w)=u^i P_j(v)P_k(w)a'_u \quad , \qquad P_j(v)=\begin{cases}1-v, & j=0\\ v+1, & j=1\\ v^j-1, & j\ge 2, \text{even}\\ v^j-v, & j\ge 3, \text{odd}\end{cases} \tag{3}$$

with analogous expressions for f_{vijk} and f_{wijk}. Sum limits n_u, n_v, and n_w are determined by the adopted degrees of the polynomial approximation, and a_{uijk}, a_{vijk} and a_{wijk} are unknown field coefficients. The reciprocal unitary vector a'_u is defined as $a'_u=a_v\times a_w/J$ [4], and analogous cyclical expressions hold for a'_v and a'_w. J is the Jacobian of the covariant transformation, $J=(a_u\times a_v)\cdot a_w$, where $a_u=\mathrm{d}r/\mathrm{d}u$, $a_v=\mathrm{d}r/\mathrm{d}v$, and $a_w=\mathrm{d}r/\mathrm{d}w$, and r is given in Eq.(1). Note that similar (divergence-conforming) expansions in trilinear hexahedral elements are used in the large-domain volume integral equation (VIE) method of moments [3].

3. Computation of Cavity Resonances

In order to evaluate the efficiency, accuracy, and convergence of 3-D FEM basis functions in Eqs.(2)-(3), a Galerkin-type FEM procedure is implemented for the analysis of 3-D electromagnetic cavity resonances. The procedure is based on solving the electric field wave equation, $\nabla\times 1/\mu_r\,\nabla\times E-k_0^2\varepsilon_r E=0$. After invoking the divergence theorem and some vector identities, and upon enforcement of boundary conditions, the procedure yields a generalized eigenvalue problem:

$$[A]\{a\}=k_0^2[B]\{a\}, \tag{4}$$

where k_0^2 represents the eigenvalues of the system. Matrices $[A]$ and $[B]$ are given by

$$[A]=\begin{bmatrix}[UUC] & [UVC] & [UWC]\\ [VUC] & [VVC] & [VWC]\\ [WUC] & [WVC] & [WWC]\end{bmatrix} \quad \text{and} \quad [B]=\begin{bmatrix}[UUD] & [UVD] & [UWD]\\ [VUD] & [VVD] & [VWD]\\ [WUD] & [WVD] & [WWD]\end{bmatrix}, \tag{5}$$

where the elements of the corresponding submatrices have the form

$$UUC_{\hat{i}\hat{j}\hat{k}ijk}=\int_V\frac{1}{\mu_r}(\nabla\times f_{u\hat{i}\hat{j}\hat{k}})\cdot(\nabla\times f_{uijk})\,\mathrm{d}V, \quad UUD_{\hat{i}\hat{j}\hat{k}ijk}=\int_V\varepsilon_r f_{u\hat{i}\hat{j}\hat{k}}\cdot f_{uijk}\,\mathrm{d}V \tag{6}$$

with analogous expressions for the elements of other submatrices of $[A]$ and $[B]$. Vector $\{a\}$ is the column vector of the unknown coefficients. All integrals are defined over the volume of a hexahedron and are numerically integrated in the $u-v-w$ domain using the Gaussian integration formula at $(n_u+10)(n_v+10)(n_w+10)$ points.

As the first numerical example, consider a rectangular air-filled metallic cavity with dimensions 1 cm$\times$0.5 cm$\times$0.75 cm. Shown in Table 1 are the percentage errors of the resonant mode wave numbers, k_0, computed by the large-domain FEM and those obtained by small-domain FEM techniques using triangular prisms [5], bricks [6], and tetrahedrons [6], respectively, as basic elements. In the large-domain FEM approach, the cavity is modeled by a single trilinear hexahedral element (which in this case reduces to a brick) and 40 unknowns ($n_u=4$, $n_v=2$, $n_w=3$). It can be concluded from the table that, for the same level of accuracy, solutions obtained by means of the large-domain FEM require significantly less unknowns as compared to the solutions obtained by the other three methods (40 compared to 382, 270, and 260, respectively).

The next example is a cubical air-filled metallic cavity 0.5 cm on a side. A plot of the percentage error in calculating the first three degenerate eigenvalues against the number of unknowns is given in Fig.1(a), for the FEM models with small rectangular bricks [6], small tetrahedrons [6], and that with the cavity represented by a single large-domain (entire-domain) hexahedron. We observe great superiority of the large-domain FEM over small-domain FEM solutions in this case. For a better insight in the accuracy, convergence, and stability of the large-domain FEM in this example, Fig.1(b) shows the same large-domain FEM results in a logarithmic scale.

4. Conclusions

In [2], finite elements in the form of electrically large trilinear hexahedrons with higher-order polynomial field expansions are proposed and tested on a simple one-dimensional example. In this paper, the same approach is applied to a 3-D problem of computing electromagnetic cavity resonances. The eigenvalues of air-filled rectangular and cubical metallic cavities (for which the analytical solutions conveniently exist for comparison) were computed utilizing the proposed large-domain finite elements and the Galerkin testing procedure. The results are compared with the results obtained by different small-domain FEM techniques. The superiority of the proposed approach, in terms of the number of unknowns needed for certain accuracy, is observed. Our current and future work includes full numerical implementation of large-domain hexahedral finite elements for general electromagnetic problems and their hybridization with the large-domain surface integral equation (SIE) method.

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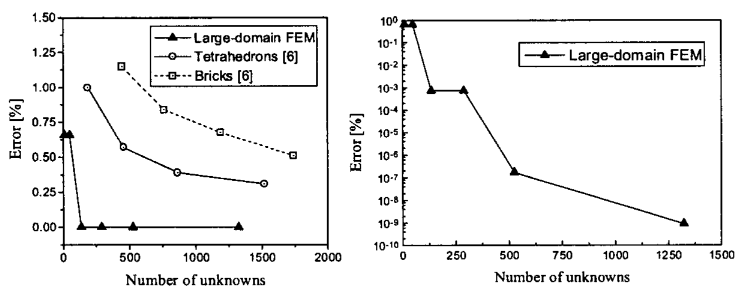
Table 1. Eigenvalue error comparison for a rectangular cavity (1 cm$\times$0.5 cm$\times$0.75 cm); the large-domain FEM and three small-domain FEM techniques.

Mode	k_0 [cm^{-1}] (Exact)	|Error| [%]			
		Prisms [5] 382 Unknowns	Bricks [6] 270 Unknowns	Tetrahedrons [6] 260 Unknowns	Large-domain FEM 40 Unknowns
TE_{101}	5.236	0.73	1.36	0.44	0.42
TM_{110}	7.025	2.32	2.23	0.70	0.53
TE_{011}	7.551	0.53	2.58	1.00	0.66
TE_{201}	7.551	0.64	3.13	0.56	2.38
TM_{111}	8.179	0.22	2.09	2.29	0.56
TE_{111}	8.179		2.09	0.70	0.56
TM_{210}	8.886		2.98	3.53	1.91
TE_{102}	8.947		5.38	1.70	2.76



(a) (b)

Fig.1. (a) Comparison of the large-domain FEM and two small-domain FEM techniques for a cubical metallic cavity 0.5 cm on a side. (b) Large-domain FEM results in a logarithmic scale.