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Short laser pulse parameters in a nonlinear medium: different approximations of the ray-pulse matrix

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Abstract

The purpose of this paper is to find an expression to describe the evolution of the temporal parameters of a short laser pulse crossing a Kerr nonlinear medium, suitable to be recast in a form consistent with the standard formulas for Gaussian beam transformations, via 2×2 ray pulse matrices. The goal is to find a satisfactory compromise between accuracy and simplicity. Expressions with different degree of approximation are analyzed and their validity and accuracy are discussed.

Keywords: Kerr lens mode locking; Pulse propagation; Ray-pulse matrices

1. Introduction

Self-phase modulation (SPM) terms a temporal modulation of the phase of an optical beam due to a nonlinear change of the refractive index. Self-actions of laser light like SPM and spatial self-focusing are optical phenomena that take place when an intense beam propagates in an intensity-dependent refraction index material [1]. The self-modulation effect is very important in experiments with picosecond and, even more noticeably, with femtosecond pulses. In fact, these nonlinear effects are at the basis of the operation of the usual sources of short and ultrashort pulses, and they are also a main issue in high-energy ultrafast experiments.

The aim of this work is to estimate the error made when the temporal parameters of a laser pulse travelling through a nonlinear material with positive dispersion are calculated with the simple 2×2 matrix formalism. This has immediate applications, such as improving the capacity of modeling self-mode locked lasers [2–6] (especially those in which the active media are not the nonlinear element [7–9]), and describing the short pulse beam evolution through optical systems, when the terms higher than the second order in the phase can be assumed to be not relevant. The matrix formalism provides, in addition, a simple way to evaluate the stability of the self-mode locked lasers solutions [10,11].

This paper can be viewed as complementary to Ref. [12], where the nonlinear matrix for self-focusing was obtained. Here, we deal with the related (but not identical) temporal counterpart. Although a complete temporal equivalence of the self-focusing ma-

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trix of Ref. [12] cannot be obtained, different approximations are studied and compared.

The formulation of pulse propagation in terms of 2×2 temporal matrices [13] allows the use of the theory of Gaussian transformations [14,15] on systems containing nonlinear Kerr elements, where the refractive index has a quadratic dependence with intensity. For a Gaussian pulse of temporal width τ and a chirp parameter $\partial^2\phi/\partial t^2 \equiv S$ in a material with refractive index n, the complex temporal pulse parameter p is defined as:

$$\frac{1}{p} = \frac{1}{k}S - \frac{\mathrm{i}n\lambda}{\pi\tau^2}, \tag{1}$$

where $k = 2n_0\pi/\lambda$ and n_0 is the linear refractive index. After crossing a system modeled with a 2×2 matrix defined by

$$M_T = \begin{pmatrix} K & I \\ J & L \end{pmatrix}, \tag{2}$$

the output pulse results with a parameter p_{out} given in terms of the input parameter p_{in} by [15]:

$$\frac{1}{p_{\mathrm{out}}} = \frac{\lambda J + L\dfrac{1}{p_{\mathrm{in}}}}{K + \dfrac{I}{\lambda}\dfrac{1}{p_{\mathrm{in}}}}. \tag{3}$$

2. Solution of the parabolic equation with the addition of a nonlinear phase modulation

The electric field of a linearly polarized beam propagating along the z axis can be written as:

$$E(z,t) = \frac{1}{2}\{u\exp[\mathrm{i}(\omega t - kz)] + \mathrm{c.c.}\}, \tag{4}$$

where ω is the angular frequency. The temporal pulse envelope of Eq. (4) in a dispersive medium with nonlinear phase modulation is described, within the parabolic approximation, by [16,17]:

$$\frac{\partial u}{\partial z} + \mathrm{i}\frac{\beta''}{2}\frac{\partial^2 u}{\partial t^2} - \mathrm{i}b|u|^2 u = 0, \tag{5}$$

which is the parabolic equation with the addition of a self-phase modulation nonlinear term. The factor β'' is the second derivative of the propagation constant (named group velocity dispersion, GVD), b is the self-phase modulation coefficient (see below) and t is the displaced time coordinate whose origin $t=0$ is centered at the time of arrival of the pulse at each plane z.

The analogy between the paraxial wave equation and the dispersive wave equation has been already pointed out [14,16,18]. In the same way, a formal equivalence exists between the nonlinear dispersive wave Eq. (5) and the nonlinear spatial equation, that takes into account the self-focusing of the laser beam due to the Kerr effect [12]. In that case, the second derivative with respect to t is replaced by the transverse laplacian acting on the x–y coordinates orthogonal to the z axis. The essential difference with the spatial case studied in Ref. [12] arises on the dimensionality of the problem. In the spatial case, the manifold transverse to the propagation axis is two-dimensional (the plane where the spot-size is defined), while in the temporal case it is one-dimensional.

Assuming that the temporal profile of the solution can be modeled with a Gaussian function, the complex envelope (slowly varying) field can be written as:

$$u(z,t) = \sqrt{\frac{U}{\tau}}\exp\left[\frac{\mathrm{i}kt^2}{2}\frac{1}{p} + \mathrm{i}\varphi\right], \tag{6}$$

where U is the energy of the pulse, τ the pulse width, φ is the linear phase shift and p is the complex pulse parameter, Eq. (1). The pulse chirp S, the pulse width τ and the phase shift φ are functions of the distance on the axis, z. In order to obtain an analytical solution of (5), we approximate the nonlinear term $|u|^2$ by a least squares weighted parabolic fit [12]:

$$|u|^2 = \frac{U}{\tau}\exp\left(-\frac{2t^2}{\tau^2}\right) \cong \frac{5\sqrt{2}}{8}\frac{U}{\tau}\left(1 - \frac{4t^2}{5\tau^2}\right). \tag{7}$$

The resulting equation is a parabolic equation with a ‘source’:

$$\frac{\partial u}{\partial z} + \mathrm{i}\frac{\beta''}{2}\frac{\partial^2 u}{\partial t^2} = \mathrm{i}\frac{5\sqrt{2}}{8}\frac{Ub}{\tau}\left(1 - \frac{4t^2}{5\tau^2}\right)u. \tag{8}$$

Inserting the Gaussian pulse ansatz, the nonlinear equation reduces to an equation involving a constant multiplier of (6) and a t^2 multiplier. If we set them equal to zero individually, we obtain two complex equations, which can also be split in their real and imaginary parts. This leads to four ordinary differential equations but two of them differ in a constant multiplier, so that we are left with:

$$-\frac{1}{\tau}\frac{\partial\tau}{\partial z}+\beta'' S=0 \tag{9a}$$

$$\frac{\partial S}{\partial z}+S^2\beta''-\frac{\sqrt{2}\,Ub}{\tau^3}-\frac{4\beta''}{\tau^4}=0 \tag{9b}$$

$$\frac{\partial\varphi}{\partial z}-\frac{\beta''}{\tau^2}-\frac{5\sqrt{2}}{8}\frac{Ub}{\tau}=0\,. \tag{9c}$$

From (9a) we obtain the first derivative of the chirp with respect to z:

$$\frac{\partial S}{\partial z}=\frac{1}{\beta''\tau}\frac{\partial^2\tau}{\partial z^2}-\frac{1}{\beta''\tau^2}\left(\frac{\partial\tau}{\partial z}\right)^2. \tag{10}$$

combining (9b) and (10) the equation for the evolution of the pulse width in a dispersive medium with a nonlinear phase shift can be written as:

$$\frac{\partial^2\tau}{\partial z^2}=\frac{4\beta''^2}{\tau^3}+\frac{\sqrt{2}\,Ub\beta''}{\tau^2}\,. \tag{11}$$

This equation has an analytical (inverted) solution, given by

$$z=\frac{+(-)\sqrt{C\tau^2-4\beta''^2-2\sqrt{2}\,Ub\beta''\tau}-\tau_0^2S_0\,\beta''}{C}+(-)\frac{\sqrt{2}\,Ub\beta''}{C^{\frac{3}{2}}}\cdot\ln\left[\frac{\sqrt{C}\left(\tau-\frac{\sqrt{2}\,Ub\beta''}{C}\right)+\sqrt{C\tau^2-4\beta''^2-2\sqrt{2}\,Ub\beta''\tau}}{\sqrt{C}\left(\tau_0-\frac{\sqrt{2}\,Ub\beta''}{C}\right)+(-)\tau_0^2S_0\,\beta''}\right], \tag{12}$$

where C is:

$$C=\frac{4\beta''^2}{\tau_0^2}+\tau_0^2S_0^2\,\beta''^2+\frac{2\sqrt{2}\,Ub\beta''}{\tau_0}\,, \tag{13}$$

where τ_0 and S_0 are the pulse width and the chirp at $z=0$. The expression (12) cannot be inverted analytically, but τ as a function of z can be obtained numerically. The chirp is obtained as a function of t, using the relation (9a) and the expression (12) as:

$$S=+(-)\frac{\sqrt{C\tau^2-4\beta''^2-2\sqrt{2}\,Ub\beta''\tau}}{\tau^2\beta''}\,. \tag{14}$$

In the absence of SPM ($b=0$) Eqs. (12) and (14) reduce to the expressions for the propagation of a pulse in a medium with GVD per unit length β''. If we put $\beta''=0$, from Eqs. (9a), (9b) and (9c) we obtain $\tau=\tau_0$; $S=\sqrt{2}Ubz/\tau_0^3+S_0$. This is the evolution of a pulse through a medium which has an equivalent matrix defined by $K=L=1$, $I=0$, $J=\sqrt{2}Ubz/(2\pi\tau_0^3)$, that is, a ‘temporal lens’ in the notation of Ref. [14]. In order to set J to agree (in this limit of $\beta''=0$) with the self-phase modulation matrix of Ref. [15], the nonlinear coefficient b is defined as:

$$b\equiv\frac{16n_2}{\lambda\sigma^2\sqrt{\pi}}\,. \tag{15}$$

The nonlinear index of the material is n_2 and σ is the beam spot-size (1/e radius of the field). The obtained solution cannot be put in a form consistent with Eq. (3), that is, there is no pulse ray matrix that describes the evolution of the pulse parameters ruled by Eqs. (12)–(14). Nevertheless, this solution is useful to evaluate the accuracy of the predictions of approximate expressions in the form of 2×2 matrices, which is done in the next section.

3. Approximated matrices

The simplest matrices for a material with GVD and SPM are the two commuted products of the matrices describing each of these effects separately. If the input beam propagates first through a nonlinear medium and then through a dispersive medium, a matrix is obtained and, using the bilinear transformation (3) the pulse width at the end of the material is written as:

$$\tau_{SG}^2 = \tau_0^2\left[\left(1+\frac{\sqrt{2}\,Ub\beta'' z^2}{\tau_0^3}+\beta'' S_0 z\right)^2 + \left(\frac{2\beta'' z}{\tau_0^2}\right)^2\right], \quad (16)$$

If the media are encountered in the inverse order, the pulse width is:

$$\tau_{GS}^2 = \tau_0^2\left[(1+\beta'' S_0 z)^2 + \left(\frac{2\beta'' z}{\tau_0^2}\right)^2\right]. \quad (17)$$

Note that (17) does not show any effect of the nonlinearity. This is easy to understand following the analogy between the temporal and the spatial parameters: (17) is equivalent to the evolution of the spot-size of a beam propagating through a distance and then crossing a (divergent) lens. The (de)focusing effect cannot be perceived *immediately after* the lens. Some propagation (here, some dispersive medium) is necessary.

None of the two descriptions above appear adequate for modeling a GVD–SPM element, because both effects occur simultaneously and continuously as the beam crosses the material. To obtain a more accurate matrix description of the phenomenon, we can model the medium as an m-term product of the two-matrices product, each elementary matrix being the propagation through a distance z/m in the GVD–SPM medium. The closed form of the m-term product matrix coefficients can be obtained, applying the unitary transformation that reduces the original matrix to a diagonal form, performing the m-term product on the diagonal representation of the matrix and then applying the inverse transformation. Each coefficient is now a sequence on m, and their convergent expressions in the limit $m=\infty$ can be calculated. A matrix M_H whose elements are hyperbolic circular functions, as in a 'lenslike' medium, is obtained. This is analog to a (spatial) beam propagating through a region of negative quadratic variation of the refraction index [19]. The matrix M_H is symmetrical (i.e., $K=L$), which means that the same result is obtained regardless which one of the two elementary matrices starts the product, which is a desired property for the solution. The matrix M_H is not, however, very practical for performing calculations, especially in the case of modeling self-mode locked lasers. But, at first order in the nonlinear coefficient, M_H can be approximated as:

$$M_H \cong M_L = \begin{pmatrix} 1+\dfrac{\sqrt{2}\,Ub\beta'' z^2}{2\tau_0^3} & 2\pi\beta'' z \\ \dfrac{\sqrt{2}\,Ubz}{2\pi\tau_0^3} & 1+\dfrac{\sqrt{2}\,Ub\beta'' z^2}{2\tau_0^3} \end{pmatrix}. \quad (18)$$

Note that this approximation is like a 'symmetric mixture' of the simple commuted products that lead to expressions (16) and (17).

In order to work within the validity range of Eq. (18), two facts must be kept in mind: first, the nonlinear material should be no longer than the beam confocal parameter, in order to avoid another nonlinear effect generated on the spatial variation of the pulse. Second, the last approximation from the hyperbolic functions matrix that leads to Eq. (18) is done assuming the condition:

$$\mu z \equiv \sqrt{\frac{\sqrt{2}\,Ub\beta''}{\tau_0^3}}\, z \ll 1. \quad (19)$$

This is the practical limit of the model. However, the neglected terms are of order $(\mu z)^3/6$ or smaller, which means that, for example, using this matrix to describe a 3 mm long intracavity Ti:Sapphire rod, with 70 fs pulses (input duration) and 70 nJ intracavity power focused on a 60 μm waist, the relative error of the matrix coefficients is 10%, compared with the 'exact' hyperbolic functions matrix. In the same conditions but with a 30 fs input pulse, the error is near 50%.

Now, we compare the evolution of the temporal parameters given by the different expressions developed above, namely: Eqs. (12)–(14) (numerically inverted), and Eqs. (16)–(18). We choose two different situations of practical interest: a beam propagating through a 10 mm thick glass rod inside a short-pulsed (4 ps) mode locked Nd:YAG laser cavity, and

Table 1
Parameters used for the two illustrative examples

Parameter	Nd:YAG, intracavity	Ti:sapphire, external
U (nJ)	120	5
σ (mm)	0.03	0.02
λ (mm)	1.064×10^{-3}	0.82×10^{-3}
Confocal parameter (mm)	4.8	2.7
τ_0 (ps)	4	0.065
S_0 (ps^{-2})	0	0

a 65 fs Ti:sapphire laser pulse focused on a 5 mm thick glass rod. The Nd:YAG example is chosen because most of the designs of self-mode locked neodymium-ion lasers need an intracavity element to supply enough nonlinearity to sustain the pulsed regime. The Ti:sapphire case is useful as a model for cavity design, pulse reshaping outside the cavity and z-scan experiments. The dispersion constant of the glass material (SF57 glass) is $\beta'' = 1.46\times10^{-4}$ ps^2/mm and the nonlinear coefficient is $n_2 = 0.41$ mm^2 ps/J [7]. Other parameters are summarized in Table 1.

The results for the first example are shown in Fig. 1. The 4 ps pulse (energy: 120 nJ) starts unchirped and begins to spread. Due to the (relatively) long pulse, the nonlinear phase modulation and the dispersion alter the pulse parameters only slightly. The maximum pulse width difference between the distinct models after travelling through 10 mm of heavy glass is less than 10^{-3} ps in a 4 ps pulse. The evolution of the pulse width described with the expression (17) does not take into account the self-phase modulation, and the pulse broadening is only due to the dispersion. The pulse evolution obtained with the commuted matrix product (16) overestimates the nonlinear effect, while the first order approximation matrix (18) shows an excellent agreement with the (more accurate) solution obtained from the inversion of (12). These last two curves are overlapped, showing a perfect fit for the considered example.

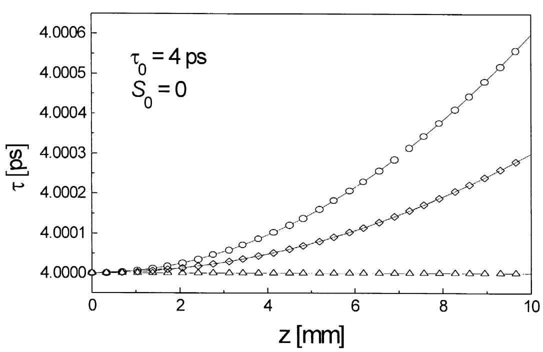

Fig. 1. Evolution of the 1064 nm pulse width on 10 mm of glass in the distinct approaches. The incident pulse duration is 4 ps and the initial chirp parameter is $S_0 = 0$. —: solution of the differential equation, (numerical inversion of Eq. (12)); -○-: GVD followed by SPM; -△-: SPM followed by GVD; -◇-: first order approximation (Eq. (18)) matrix.

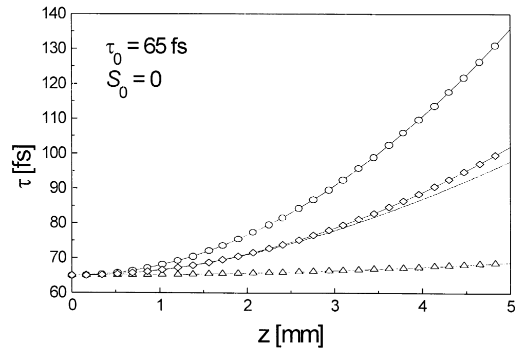

Fig. 2. Evolution of the 820 nm pulse width on 5 mm of glass. The incident pulse duration is 65 fs, transform limited. The curves are indicated as in Fig. 1.

The chirp parameter is a less sensitive magnitude, and consequently all the approximations correctly fit the approximated solution (all the curves, not shown, are superimposed). The chirp parameter varies almost linearly with z and at the end of the rod it is

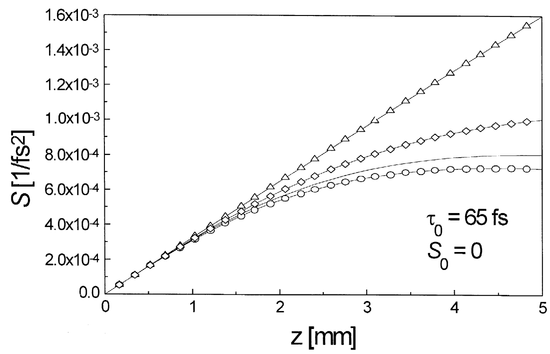

Fig. 3. Evolution of the pulse chirp parameter, the same conditions as in Fig. 2.

0.1 ps^{-2}; this means that the pulse comes out from the glass with a normalized time-bandwidth product $\sqrt{1+(S^2\tau^4)/4} \approx 1.3$.

Figs. 2 and 3 show the pulse width and chirp for the case of the focused Ti:sapphire laser pulse. The initial condition for this example is $\tau_0 = 65$ fs (transform limited) and the pulse energy is 5 nJ. In spite of this low pulse energy, the nonlinear factor is more than 10^4 times larger than in the previous case, thanks to the τ_0^{-3} dependence of the nonlinear term. For fs pulses, the combined effect of the SPM and dispersion generates a significant pulse broadening. Again, the simple products between an 'only GVD' and an 'only SPM' matrices give an unsatisfactory description of the process, specially in the pulse width evolution. As in the previous example, one of them only takes into account the dispersion and the other overestimates the self-modulation. The chirp parameter in this case is not linear, and the first order approximation (18) shows again a good correspondence with the solution obtained from the inversion of (12).

The first order expansion (18) fits the solution obtained solving the master equation in the parabolic approximation (the numerical inversion of (12)), but one may question the accuracy of this last solution. Hence, as an additional check, we obtain the numerical solution of the master Eq. (5) directly, with a Runge–Kutta Fehlberg (RKF45) algorithm [20] (using 512 points for the temporal mesh). In this way we check the error made in the parabolic approximation at Eq. (7). For the Ti:sapphire example, we have fit the field envelope of the RKF45 numerical solution with a Gaussian field and obtained the equivalent pulse width and chirp parameter. The numerical inversion of Eq. (12) agrees satisfactorily with the results of this calculation, even for peak powers as high as 3.6 GW/cm^2.

4. Summary

An accurate model of the effects of nonlinear phase modulation and dispersion by using the matrix formalism is of practical interest. We have introduced a solution of the nonlinear equation (the numerical inversion of Eq. (12)), and compared it with different descriptions of the problem, compatible with the matrix representation. The first order (in the nonlinear coefficient) approximated matrix (18) shows an accurate description of the combined GVD–SPM effects, at least for pulses not much shorter than 100 fs. It appears to be the best compromise between the simplicity of a matrix description, together with simple expressions for the matrix elements, and numerical accuracy. The self-shortening matrix of Ref. [12] (for the spatial part) and the matrix (18) (for the temporal part) can be composed in a 4×4 matrices model for a more accurate and complete, but still simple, description of the beam and pulse variations in a complex optical system including a nonlinear medium. A nonlinear self-consistent Poincaré map of a resonant cavity for design or analysis of self-mode locked lasers can be immediately calculated in this way, with improved accuracy. This will allow not only the calculation of the operation values of the laser, but also a trivial evaluation of their stability [10,11].

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References

[1] S.A. Akhmanov, R.V. Khokhlov, A.P. Sukhorukov, in: F.T. Arecchi, E.O. Schulz-Dubois (Eds.), Laser Handbook, Chap. E3, North-Holland, Amsterdam, 1972.

[2] K. Liu, C.J. Flood, D.R. Walker, H.M. van Driel, Kerr lens mode locking of a diode-pumped Nd:YAG laser, Opt. Lett. 17 (1992) 1361.

[3] U. Keller, G.W. 'tHooft, W.H. Knox, J.E. Cunningham, Femtosecond pulses from a continuously self-starting passively mode locked Ti:sapphire laser, Opt. Lett. 16 (1991) 1022.

[4] F. Krausz, Ch. Spielmann, T. Brabec, E. Wintner, A.J. Schmidt, Generation of 33-fs optical pulses from a solid-state laser, Opt. Lett. 17 (1992) 204.

[5] T. Miura, K. Kobayashi, Z. Zhang, K. Torizuka, F. Kannari, Stable-mode locking operation in a Cr:forsterite laser with a five-mirror cavity, Opt. Lett. 24 (1999) 554.

[6] I.T. Sorokina, E. Sorokin, E. Wintner, A. Cassanho, H.P. Jenssen, R. Szipöcs, 14-fs pulse generation in Kerr lens mode locked prismless Cr:LiSGaF and Cr:LiSAF lasers: observation of pulse self-frequency shift, Opt. Lett. 22 (1997) 1716.
[7] B. Henrich, R. Beigang, Self-starting Kerr lens mode locking of a Nd:YAG-laser, Opt. Comm. 135 (1997) 300.
[8] G.P.A. Malcolm, A.I. Ferguson, Self-mode locking of a diode-pumped Nd:YLF laser, Opt. Lett. 16 (1991) 1967.
[9] U. Keller, T.H. Chiu, J.F. Ferguson, Self-starting femtosecond mode locked Nd:glass laser that uses intracavity saturable absorbers, Opt. Lett. 18 (1993) 1077.
[10] A. Hnilo, Self-mode locking Ti:sapphire laser description with an iterative map, J. Opt. Soc. Am. B 12 (1995) 718.
[11] M. Marioni, A. Hnilo, Self-starting of self-mode locking Ti:Sapphire lasers. Description with a Poincaré map, Opt. Commun. 147 (1998) 89.
[12] V. Magni, G. Cerullo, S. De Silvestri, *ABCD* matrix analysis of propagation of Gaussian beams through Kerr media, Opt. Comm. 96 (1993) 348.
[13] A.G. Kostenbauder, Ray-pulse matrices: a rational treatment for dispersive optical systems, IEEE J. Quantum Electronics 26 (1990) 1148.
[14] S.P. Dijaili, A. Dienes, J.S. Smith, *ABCD* matrices for dispersive pulse propagation, IEEE J. Quantum Electronics 26 (1990) 1158.
[15] O.E. Martinez, J.L.A. Chilla, Self-mode locking of Ti:sapphire lasers: a matrix formalism, Opt. Lett. 17 (1992) 1210.
[16] A.E. Siegman, Lasers, Chap. 9, University Science Books, Mill Valley, CA, 1986.
[17] H.A. Haus, J.G. Fujimoto, E.P. Ippen, Structures for additive pulse mode locking, J. Opt. Soc. Am. B 8 (1991) 2068.
[18] B.H. Kolner, M. Nazarathy, Temporal imaging with a time lens, Opt. Lett. 14 (1989) 630.
[19] A. Yariv, Quantum Electronics, Chap. 6, J. Wiley and Sons, 1975.
[20] E. Fehlberg, Low-order Classical Runge–Kutta Formulas with Stepsize Control, NASA-TR-R-315, 1969.