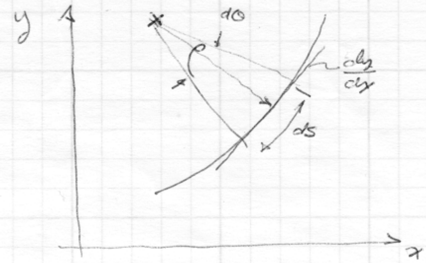
RADIUS OF CURVATURE

$$y = f(x)$$

$$\frac{dy}{dx} = f'(x)$$

$$\frac{d^2y}{dx^2} = f''(x)$$



Cartesian

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

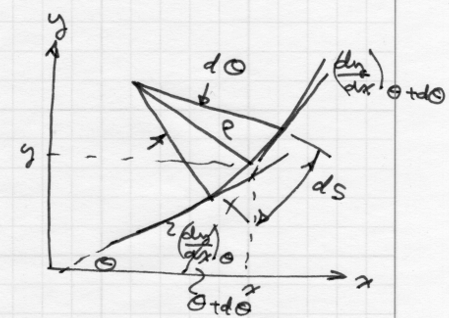
~~Polar or~~ Cylindrical

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

note in {Pure cylindrical motion} $\frac{dr}{d\theta} = 0$ $\frac{d^2r}{d\theta^2} = 0$ so

$$\rho = \frac{\left[r^2 + 0\right]^{3/2}}{r^2 + 2(0)^2 - r(0)} = \frac{r^{2\,3/2}}{r^2} = \frac{r^3}{r^2} = r$$

Derivation (cartesian)



(A)

$$\rho\, d\theta = ds$$

$$\therefore \rho = \frac{ds}{d\theta}$$

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$d\theta = \theta_2 - \theta_1 = \tan^{-1}\left(\frac{dy}{dx}\right)_2 - \tan^{-1}\left(\frac{dy}{dx}\right)_1$$

Taylor series expansion

$$\tan^{-1}\left(\frac{dy}{dx}\right)_2 = \tan^{-1}\left(\frac{dy}{dx}\right)_1 + \frac{d\left(\tan^{-1}\left(\frac{dy}{dx}\right)\right)}{ds} ds + \cdots$$

$$d\theta = \tan^{-1}\left(\frac{dy}{dx}\right)_1 + \frac{d\left(\tan^{-1}\left(\frac{dy}{dx}\right)\right)}{ds} ds + \cdots - \tan^{-1}\left(\frac{dy}{dx}\right)_1$$

$$\rho = \frac{ds}{\frac{d\left(\tan^{-1}\left(\frac{dy}{dx}\right)\right)}{ds} ds} = \frac{1}{\frac{d\left(\tan^{-1}\left(\frac{dy}{dx}\right)\right)}{dx}\left(\frac{dx}{ds}\right)}$$

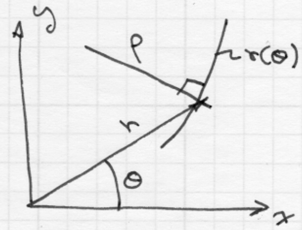
but $\frac{dx}{ds} = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$ (see A above)

$$\frac{d \tan^{-1} u}{dx} = \frac{1}{1+u^2}\frac{du}{dx} \text{ or}$$

$$\frac{d \tan^{-1}\left(\frac{dy}{dx}\right)}{dx} = \frac{1}{1+\left(\frac{dy}{dx}\right)^2}\frac{d^2y}{dx^2}$$

so

$$\boxed{\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left(\frac{d^2y}{dx^2}\right)}}$$



$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\frac{dx}{d\theta} = -r\sin\theta + \cos\theta\frac{dr}{d\theta}.$$

$$\frac{dy}{d\theta} = +\cos\theta\, r + \sin\theta\frac{dr}{d\theta}$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{+r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{+r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta}\right)^2$$

$$= \left[r^2\sin^2\theta - 2r\frac{dr}{d\theta}\sin\theta\cos\theta + \left(\frac{dr}{d\theta}\right)^2\cos^2\theta + r^2\cos^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta\right] \Big/ \left[r^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + \left(\frac{dr}{d\theta}\right)^2\cos^2\theta\right]$$

$$= \frac{\left[r^2\overbrace{(\sin^2\theta + \cos^2\theta)}^{1} + \left(\frac{dr}{d\theta}\right)^2\overbrace{(\sin^2\theta + \cos^2\theta)}^{1}\right]}{\left(-r\sin\theta + \frac{dr}{d\theta}\cos\theta\right)^2}$$

$$= \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}{\left[-r\sin\theta + \frac{dr}{d\theta}\cos\theta\right]^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$\frac{d\left(\frac{dy}{dx}\right)}{d\theta} = \frac{d}{d\theta}\left[\frac{+r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta}\right]$$

$$= \frac{\left(+\frac{dr}{d\theta}\cos\theta - r\sin\theta + \frac{d^2r}{d\theta^2}\sin\theta + \frac{dr}{d\theta}\cos\theta\right)\left(-r\sin\theta + \frac{dr}{d\theta}\cos\theta\right)}{\left(-r\sin\theta + \frac{dr}{d\theta}\cos\theta\right)^2}$$

$$- \frac{\left[+r\cos\theta + \frac{dr}{d\theta}\sin\theta\right]}{\left(-r\sin\theta + \frac{dr}{d\theta}\cos\theta\right)^2}\left(-\frac{dr}{d\theta}\sin\theta - r\cos\theta + \frac{d^2r}{d\theta^2}\cos\theta - \frac{dr}{d\theta}\sin\theta\right)$$

$$= \Big\{ \Big[-r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta - r\frac{d^2r}{d\theta^2}\sin^2\theta - r\frac{dr}{d\theta}\cos\theta\sin\theta$$

$$+ \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - r\frac{dr}{d\theta}\sin\theta\cos\theta + \frac{dr}{d\theta}\frac{d^2r}{d\theta^2}\sin\theta\cos\theta +$$

$$\left(\frac{dr}{d\theta}\right)^2\cos^2\theta \Big] -$$

$$\Big[-r\frac{dr}{d\theta}\cos\theta\sin\theta - r^2\cos^2\theta + r\frac{d^2r}{d\theta^2}\cos^2\theta - r\frac{dr}{d\theta}\cos\theta\sin\theta$$

$$- \left(\frac{dr}{d\theta}\right)^2\sin^2\theta - r\frac{dr}{d\theta}\sin\theta\cos\theta + \frac{dr}{d\theta}\frac{d^2r}{d\theta^2}\sin\theta\cos\theta$$

$$- \left(\frac{dr}{d\theta}\right)^2\sin^2\theta \Big] \Big\} \Big/ \left[-r\sin\theta + \frac{dr}{d\theta}\cos\theta \right]^2$$

$$= \frac{r^2(\sin^2\theta+\cos^2\theta) - r\frac{d^2r}{d\theta^2}(\sin^2\theta+\cos^2\theta) + 2\left(\frac{dr}{d\theta}\right)^2(\cos^2\theta+\sin^2\theta)}{\left[-r\sin\theta + \frac{dr}{d\theta}\cos\theta \right]^2}$$

$\therefore$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2 \right]^{3/2} \Big/ \left[-r\sin\theta + \frac{dr}{d\theta}\cos\theta \right]^3}{\dfrac{\left[r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2} \right] \Big/ \left[-r\sin\theta + \frac{dr}{d\theta}\cos\theta \right]^2}{\left[-r\sin\theta + \frac{dr}{d\theta}\cos\theta \right]}}$$

$$\boxed{\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2 \right]^{3/2}}{\left[r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2} \right]}}$$