Printed in U.S.A.

75-FE-27

an ASME publication

A Semi-Empirical Transition Criterion for Laminar Wall Boundary Layers

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A semi-empirical formulation which provides a general and sufficiently satisfactory method for estimating the incipience of laminar-to-turbulent transition of a variety of wall boundary layer flows is presented. It is shown that a laminar flow becomes unstable to disturbances, l_e, when $Sn > N\phi(l_e)$, and trips to a turbulent flow when $Sn > N\psi(l_e)$, where Sn is a Stability number, the ratio of the driving force on the fluid to the fluid resistance to motion; N is a numerical constant whose value depends on whether the flow boundaries are rigid-rigid, rigid-free or free-free; and $\phi(l_e)$ and $\psi(l_e)$ are the effects of the disturbances, l_e, which can be modeled semi-empirically in terms of the free stream turbulence intensity.

Contributed by the Fluids Engineering Division of The American Society of Mechanical Engineers for presentation at the Joint Fluids Engineering and Lubrication Conference, Minneapolis, Minn., May 5-7, 1975. Manuscript received at ASME Headquarters January 17, 1975.

Copies will be available until February 1, 1976.

Discussion of this paper should be delivered to the Technical Editor, Journal of Fluids Engineering, Box 69, Hanover, N. H. U.S.A. 03755.

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ABSTRACT

A semi-empirical formulation which provides a general and sufficiently satisfactory method for estimating the incipience of laminar-to-turbulent transition of a variety of wall boundary layer flows is presented. It is shown that a laminar flow becomes unstable to disturbances, I_e, when Sn $>_N \phi(I_e)$, and trips to a turbulent flow when Sn $>_N \Psi(I_e)$, where Sn is a Stability number, the ratio of the driving force on the fluid to the fluid resistance to motion; N is a numerical constant whose value depends on whether the flow boundaries are rigid-rigid, rigid-free or free-free; and $\phi(I_e)$ and $\Psi(I_e)$ are the effects of the disturbances, I_e, which can be modeled semi-empirically in terms of the free stream turbulence intensity.

NOMENCLATURE

Ax; Ay	Transpiration Number $(U_1\Delta/\nu)(v_o/U_1)$; $(v_o\Delta/\nu)^2$
C, (C*)	Wall Curvature Parameter, Δ/r_o, (δ^*/r_o)
c_p	Specific Heat at Constant Pressure
G	Görtler Number, $Re_{\delta*}(\delta^*/r_o)^{\frac{1}{2}}$
g	Gravitational Constant
H	Velocity Profile Shape Factor, δ^*/θ
I	Local Total Disturbance Intensity
i, j, k	Indices
k*	Coefficient of Thermal Conductivity
L	Characteristic Length Scale
Pr	Prandtl Number, $\mu c_p/k^*$
q	$t \times 10^3/(1+v^*)$
r_o	Wall Radius of Curvature
Ra	Rayleigh Number, $gL^3c_p(T_o-T_1)/(\nu k^* T_1)$
Ra_M	Apparent Rayleigh Number Due to Compressibility
Re	Reynolds Number, U_1L/ν
Sn	Section Stability Number
t	$10^{2.34\tanh(10/H - 4)}$
T_1, T_o	Free Stream Temperature, Wall Temperature
Ta	Taylor Number $(2Re_L^2L/r_o)$
U_1, u	Free Stream Velocity, Local Flow Velocity in x-direction
v	Local Flow Velocity in the y-direction
v_o	Wall Transpiration Velocity (Positive for Injection)
v*	v_o/U_1
x, y, z	Coordinates
Δ	Modified Boundary Layer Thickness
δ	Boundary Layer Physical Thickness, $y_u = 0.995U_1$
δ^*	Boundary Layer Displacement Thickness
θ	Boundary Layer Momentum Thickness
η	Dimensionless Height from Wall, y/Δ
ν	Kinematic Viscosity
Φ, ϕ, Ψ, ψ	Functions as Defined in Text

Subscripts and Superscripts

$(\ )_{ci}$	Value at Point of Neutral Stability
$(\ )'$	Fluctuating Quantity
$(\ )_o$	Initial Value, or Value at the Wall
$(\ )_{ct}$	Value at Point of Incipient Transition
$(\tilde{\ })$	Non-Dimensional Quantity
$(\ )_x$	In the x-direction
$(\ )_y$	In the y-direction

INTRODUCTION

In reference (1)[1] it was noted that conceptual fluid particles in a fluid execute motions determined by the constraining force field on the fluid, the force of primary importance in determining the stability of the flow being that resistive force associated with the bonding of the fluid particles. This resistive force was related to a proposed dynamic fluid property of cohesiveness; clearly, this concept of cohesiveness is fundamental to the generally accepted concept of viscosity in fluid flow, and derives from a much broader philosophical theory of universal continuity. It was further suggested that there must be a critical minimum fluid cohesiveness (corresponding to a critical force field) necessary to maintain a laminar flow. If the fluid cohesiveness falls below this critical value, flow perturbations may amplify; that is, the flow becomes unstable. As the constraint of the fluid cohesive property is further overwhelmed, the motions of the fluid particles become progressively more random; the onset of turbulence, which serves to arrest the tendency for the fluid cohesiveness to completely disappear. The intense fluctuational motions of the fluid particles create a new force field, the so-called Reynolds shear forces, proportional to the intensity of the fluctuations. This new force field is resistive and builds up the fluid cohesiveness, thus

[1] Underlined numbers in parentheses designate references at end of paper.

restoring the flow system to a more stable configuration. Many natural systems behave in this manner; when they become dynamically unstable in one configuration, they tend to a new and more stable configuration. The influence of the fluid cohesiveness on flow instability and transition is conceptualized in the schematic drawing, Figure (1).

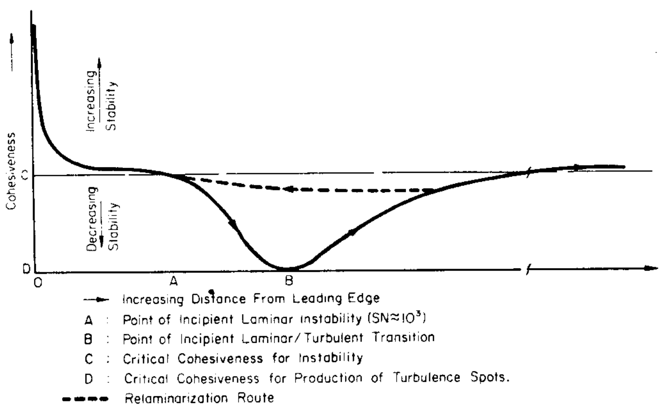

Fig. 1 Conceptual Influence of Fluid Cohesiveness on Flow Stability

Furthermore, reference (1) suggested that a laminar flow force field may be described by a Stability number, Sn, and the position variable, η; and that laminar instability in a wall boundary layer becomes incipient at the flow position for which $Sn = Sn_{ci} \simeq 10^3\phi(I_e)$, where I_e was defined as the equivalent free stream disturbance intensity which alone would produce the same effect on a flow as a combination of the actual free stream and wall disturbances. Reference (2) suggested the following empirical formulation for the equivalent free stream disturbance intensity: $I_e = I_1 + (30I_1 + 0.006)\exp\{-(2 + 0.5 \times 10^4 I_1)\}$. The function $\phi(I_e)$ is discussed later. The Stability number is essentially the ratio of the total driving force on the fluid (such as thermal, magnetic, centrifugal, inertial, etc.) to the total fluid resistance to motion (which is basically viscous). The formulation for the Stability number for a two-dimensional curved wall boundary layer with heat and mass transfer at the wall was given as:

$$Sn = \{Sn_x^2 + a_1 Sn_y\}^{\frac{1}{2}} \tag{1}$$

where:

$$Sn_x = \{Re_\Delta(1+v^*) + Ra_x\}; \quad Sn_y = \{Ra_y + Ay + Ta\};$$

$$\Delta = \delta^*/t; \quad t = 10^{2.34\tanh(10/H - 4)}; \quad a_1 = 11 \times 10^6/(1 + \overset{*}{Re}_\Delta)$$

and $\overset{*}{Re}_\Delta$ is the critical Reynolds number for flow instability or flow transition for the inertially driven flat plate flow. The length scale used in all the engineering numbers is Δ, as defined above. Sn_x is essentially the ratio of the driving forces in the x-direction to the fluid viscous forces in the same direction; Sn_y is the equivalent ratio in the y-direction. Thus, Sn_x and Sn_y can be additively expanded to include any other forces such as electrical and magnetic forces which have not been included in the present considerations. It will be shown later, for instance, that for compressible flows, $Sn_y = \{Ra_y + Ay + Ta + Ra_M\}$, where Ra_M is an apparent Rayleigh number due to the impressed thermal boundary layer. Except in the case of vertical or steeply-inclined wall convective flows, the Rayleigh number, Ra_x, in the x-direction is usually negligibly small; for generality, however, Ra_x will be appropriately retained in equation (1) and subsequent equations.

This paper attempts to obtain a formulation for the prediction of the incipience of laminar-to-turbulent transition in wall boundary layers, a problem of great practical interest. The approach adopted is an extension of concepts and results discussed in references (1) and (2) and summarized above and in the Appendix. A definite and general mathematical formulation of a transition criterion for laminar wall boundary layers which yields very encouraging predictions is obtained.

THE TRANSITION CRITERION

On the basis of the general discussion, above, it appears that flow transition from a laminar to a turbulent regime would be incipient when the magnitude of the total disturbance kinetic energy, $\tilde{K}$, attains a critical value, $\tilde{K}_{ct}$; this is considerably supported by experimental results, such as of Elder (3), Klebanoff, et al (4), Kovasnay, et al (5), and Tani and Komoda (6), which indicate that turbulence spots first begin to form at a distance from the wall corresponding to about $y/\delta = 0.3$, if the local turbulence intensity, $u^* > 18 \pm 2.5\%$, where u^* is the r.m.s. fluctuational velocity, non-dimensionalized with U_1.

In reference (2), the essence of which is summarized in the Appendix, an approximate solution for the maximum amplification factor in an unstable laminar boundary layer was obtained as:

$$(\tilde{K}(x)/\tilde{K}_o) = \exp(\xi)/\{1 + \omega^{-1}\Phi(I_e)(1 - \omega)\exp(\xi)\} \tag{2}$$

where:

$$\xi = 0.003(Sn - Sn_{ci}); \quad \Phi(I_e) = (0.002\tanh(120I_e);$$

$\tilde{K}$ = Total disturbance kinetic energy, (K/U_1^2); $\omega = Sn_{ci}/Sn$; and $\tilde{K}_o$ is the value of the disturbance kinetic energy that initially enters the boundary layer. $\tilde{K}_o$ was assumed in reference (2) to be given generally by $\tilde{K}_o \propto \Phi(I_e) \quad \phi_2(\eta)$. In flow situations where disturbances exist in the subcritical region even up to the point of neutral stability, $\tilde{K}_o = \tilde{K}_{ci}$, the value at the point of neutral stability. In supercritical cases, where any previous disturbance is completely damped prior to neutral stability and no further disturbances arise until downstream of the critical point, $\tilde{K}_o$ corresponds to the value of the input disturbance kinetic energy.

At incipient transition in the critical region of the boundary layer, $(\tilde{K}(x)/\tilde{K}_o)$ becomes $(\tilde{K}_{ct}/\tilde{K}_o)$, which is readily seen to be directly proportional to $1/\tanh(120I_e)$; thus equation (2) can be rewritten as:

$$\xi_{ct} + \ell n\,\{1 - 0.00067\xi_{ct}\} \propto -\ell n\,\{\tanh(120I_e)\} \tag{3}$$

The term $\ell n\,\{1 - 0.00067\xi\}$ approximates to,(-0.00067ξ), so that equation (3) reduces to the following transition criterion:

$$Sn_{ct} \simeq Sn_{ci} - A^*\ell n\,\{\tanh(120I_e)\} \tag{4}$$

where A^* is a numerical constant. Substituting for

Sn_{ci} (which, from reference (1) is equal to $N\phi(I_e)$) one obtains equation (4) in the form:

$$Sn_{ct} \simeq N\{\phi(I_e) - B\ell n\{\tanh(120I_e)\}\} \equiv N\Psi(I_e) \quad (5)$$

where: $\Psi(I_e) = \phi(I_e) - B\ell n\{\tanh(120I_e)\}$, and B is the numerical constant A^*/N, which is estimated to be of magnitude 2.6, by comparison with experimental data, if $\phi(I_e) \simeq 1.2\exp(-5I_e)$; Figure (2) shows the curve fit.

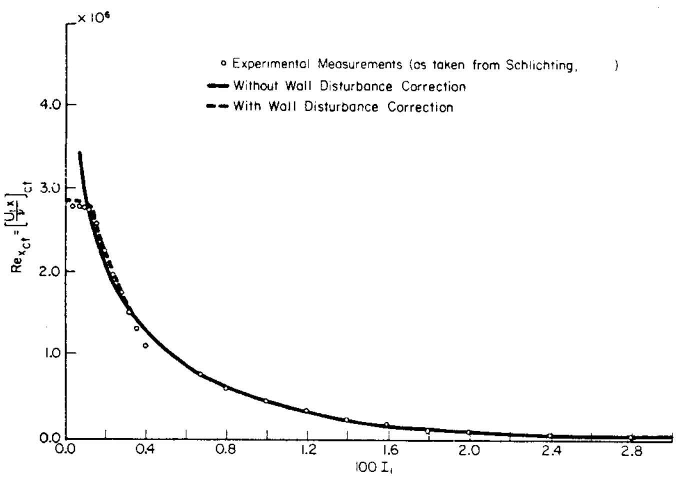

Fig. 2 Effect of Free Stream Turbulence Intensity on Incipient Transition Reynolds Number

Equation (5) is valid for all I_e, and indicates that the influence of I_e on incipient transition reduces to $\phi(I_e)$, if $I_e > 0.03$; in other words, if $I_e > 0.03$, the regions of instability and transition are practically coincident. This has been observed experimentally, also. In summary, it is suggested that laminar-to-turbulent wall boundary layer transition is incipient when:

$$Sn > N\Psi(I_e) \quad (6)$$

where: $\Psi(I_e) = 1.2\exp(-5I_e) - 2.6\ell n\{\tanh(120I_e)\}$, and $N = 10^3$ for wall boundary layers. The transition criterion, (6), is semi-empirical, being a direct result of the approximate analytical solution, summarized in the Appendix, for the maximum growth characteristics of total disturbance kinetic energy; the empiricism in (6) stems from the formulations for I_e and $\phi(I_e)$.

APPLICATIONS OF THE STABILITY AND TRANSITION CRITERIA

The stability criterion was given in reference (1) as $Sn_{ci} \simeq N\phi(I_e)$, and the transition criterion has been suggested above to be $Sn_{ct} \simeq N\Psi(I_e)$, where $N = 10^3$ for wall boundary layers and ϕ, Ψ are as previously defined. Substituting for the Stability number in each of the above criteria one obtains that:

(a) at incipient laminar flow instability:

$$Sn_x^2 + (a_1)_{ci} Sn_y \simeq 10^6 \phi^2 \quad (7)$$

(b) and at incipient laminar-to-turbulent flow transition:

$$Sn_x^2 + (a_1)_{ct} Sn_y \simeq 10^6 \Psi^2 \quad (8)$$

where the terms are as previously defined. It is easily seen that $(a_1)_{ci} = 11 \times 10^6/(1 + 10^3\phi)$ and $(a_1)_{ct} = 11 \times 10^6/(1 + 10^3\Psi)$.

From equations (7) and (8) the critical value of any appropriate engineering number for the stability or transition of a given laminar wall boundary layer flow can be readily computed. An engineering number which is usually of great interest in flow considerations is the critical Reynolds number, Re_{δ^*}, based on some appropriate length scale (the boundary layer displacement thickness, in this case). By direct substitution for Sn_x, Sn_y and a_1 in equations (7) and (8), it can be shown that the critical Reynolds number, Re_{δ^*}, for incipient laminar flow instability or for incipient laminar-to-turbulent flow transition satisfies the following quadratic equation:

$$\alpha_1 Re_{\delta^*}^2 + \alpha_2 Re_{\delta^*} + \alpha_3 = 0 \quad (9)$$

where:

$$\alpha_1 = 1 + 11 \times 10^6\{2(\delta^*/r_o)/t + v^{*2}\}/\{(1 + v^*)^2 (1 + 10^3\varepsilon)\}$$

$$\alpha_2 = 2tRa_x/(1 + v^*)$$

$$\alpha_3 = t^2Ra_x^2/(1 + v^*)^2 + 11q^2Ra_y/(1 + 10^3\varepsilon) - q^2\varepsilon^2$$

$$q = t \times 10^3/(1 + v^*)$$

$$t = 10^{2.34\tanh(10/H - 4)}; \text{ and}$$

$$\varepsilon = \phi, \text{ for instability, } \varepsilon = \Psi, \text{ for transition.}$$

Equation (9) has two real or imaginary solutions; the smaller positive solution is, of course, the one of interest. Imaginary solutions will be seen to have significant implications. To estimate the critical Reynolds number for laminar flow instability one merely substitutes Re_{δ^*ci} for Re_{δ^*} in equation (9), with $\varepsilon = \phi(I_e)$ in α_j ($j = 1, 2, 3$); for incipient transition, $Re_{\delta^*} = Re_{\delta^*ct}$ and $\varepsilon = \Psi(I_e)$ in α_j ($j = 1, 2, 3$). The well known solution of equation (9) is:

$$Re_{\delta^*} = \{-\alpha_2 \pm (\alpha_2^2 - 4\alpha_1\alpha_3)^{\frac{1}{2}}\}/2\alpha_1 \quad (10)$$

In order to estimate the critical Reynolds number for laminar flow instability or for laminar-to-turbulent flow transition one thus requires some or all of the following input, described continuously (or discretely) in the appropriate coordinate direction:

(a) the flow boundary geometry (that is, wall curvature, etc.),
(b) the flow parameters: displacement thickness, δ^*; velocity profile shape factor, H: Prandtl number, Pr, or temperature profile shape factor,
(c) the free stream disturbance intensity, I_1, and
(d) the mass and/or heat transfer at the wall (that is, $v^* = v_o/U_1$, and $(T_o-T_1)/T_1$).

These input data are usually ideally provided by conventional computational schemes for laminar wall boundary layers. The predictions given by the above criteria in some frequently encountered wall boundary layers will now be considered.

Curved Wall Flows

By inspecting the coefficient, α_1, in equation (9), which is the only term that contains the direct influence of the wall curvature, (δ^*/r_o), it becomes clear that if the absolute value of (δ^*/r_o), is significantly less than $\{(1 + 10^3\varepsilon)/22\}t \times 10^{-6}$, the wall curvature has no effect on the flow instability or transition. In other words, a curved wall flow is practically a flat plate flow, if: $|\delta^*/r_o| << \{(1 + 10^3\varepsilon)/22\}t \times 10^{-6}$. Figure (3) shows a plot of equation (10) for various wall curvatures compared against some experimental flow transition data.

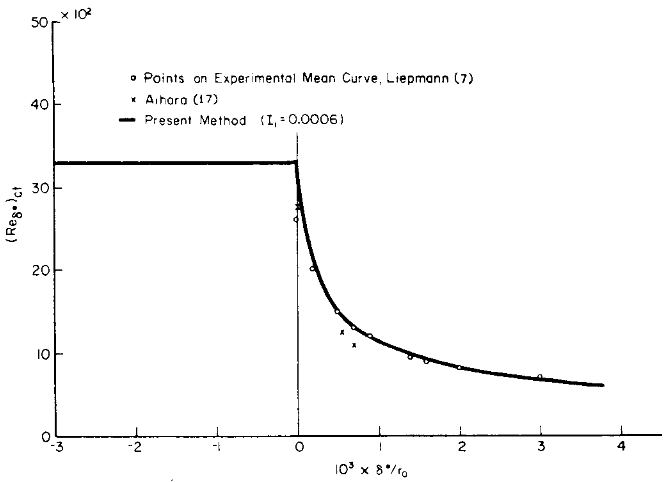

Fig. 3 Effect of Streamwise Wall Curvature on Flow Transition

For convex curvature (δ^*/r_o) is negative and if the effect of wall curvature is appreciable, α_1 would be negative, resulting in imaginary solutions to equation (9). If one interprets the incipience of imaginary solutions in this case to imply the incipience of laminar flow separation with turbulent reattachment, then the critical Reynolds number for flow instability or transition in a convex wall flow with appreciable wall curvature would be practically the same as in flat plate flows.

Equations (7) and (8) can be appropriately rewritten to emphasize the critical Görtler or the Taylor number, as may be desired.

Flow with Mass Transfer at the Wall

By inspection of the coefficients α_1, α_2, and α_3 of equation (9), it can readily be inferred that the effect of wall transpiration (suction or injection) must be very similar to that of pressure gradient on a boundary layer flow; it modifies the mean velocity profile shape factor, H. Suction, like negative pressure gradient, decreases the shape factor, which is stabilizing, whereas injection, like positive pressure gradient, increases the shape factor, which is destabilizing. For two values of free stream turbulence intensity, Figure (4) shows the resulting effect of transpiration on transition Reynolds number, while Figure (5) shows the influence of pressure gradient. The curves have identical form.

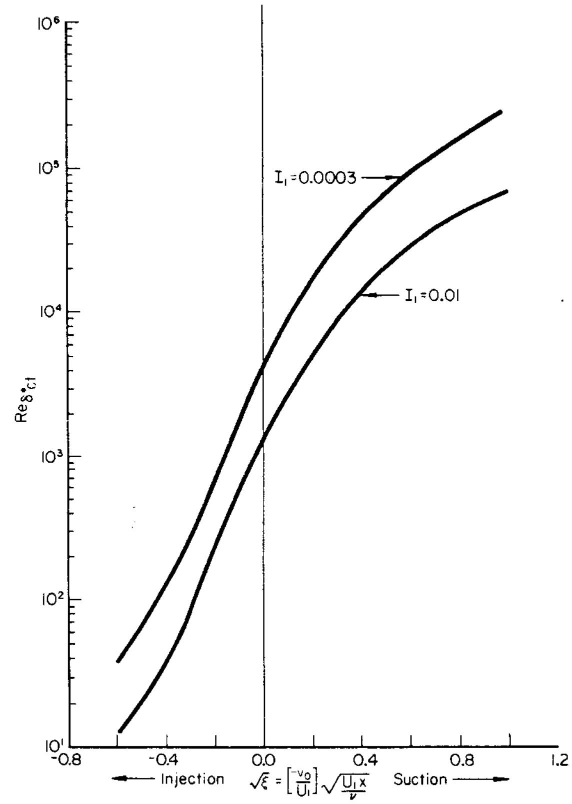

Fig. 4 Effect of Wall Transpiration on Flow Transition

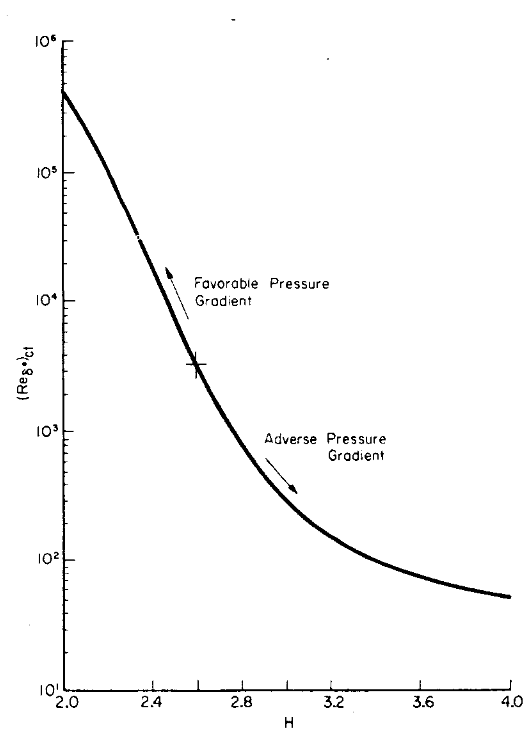

Fig. 5 Effect of Pressure Gradient on Flow Transition

Flow with Heat Transfer at the Wall

The critical Reynolds numbers for flows with heat transfer at the wall can be calculated from equation (10). However, it is rather obvious from inspection of the coefficients of equation (9) that the influence of Reynolds number on thermal instability and transition is of secondary importance, unless the free stream disturbance intensity is large. The appropriate engineering number of interest for this class of flows is the Rayleigh number whose critical values for instability and transition are given from equations (7) and (8), respectively, as follows:

$$Ra_{\delta(\text{thermal})} \simeq \{\phi^2 \times 10^{4.8} - hRe_{\delta*}^2\}\{t/(HPr^{1/3})\}^3 \quad (11)$$

(instability)

$$Ra_{\delta(\text{thermal})} \simeq 10 \times \{\Psi^2 \times 10^{4.3} - hRe_{\delta*}^2\} \{t/(HPr^{1/3})\}^3 \quad (12)$$

(transition)

where $h \simeq 0.02/t^2$. The conventional assumption that $\delta(\text{thermal}) \simeq \{1/(1.026Pr^{1/3})\}\ \delta(\text{momentum})$ has been made in order to obtain equations (11) and (12). The factor, $(H\ Pr^{1/3})$, which appears in the above equations is interpreted as the temperature profile shape factor; thus, as anticipated, the temperature and velocity profiles are identical, if the Prandtl number is unity. For the case of no mean flow, $(H\ Pr^{1/3})$ can either be calculated directly or simulated by assuming a hypothetical Blasius velocity profile ($H \simeq 2.59$) with a suitably chosen Prandtl number to yield the desired temperature profile. Equations (11) and (12) indicate that for natural convective flows, if H is arbitrarily assumed to be 2.59 and $Pr \simeq 2$ the critical Rayleigh number for cellular instability (with $I_1 = 0.001$) is about 278, and for incipient transition to turbulence is about 2.7×10^3.

Compressible Flows

Although the above stability and transition criteria were obtained from the analyses of incompressible flows, they could be applied directly to compressible flows, if one recognizes that the major effect of compressibility is to establish a thermal boundary layer within the momentum layer. Therefore, for compressible flows an apparent Rayleigh number effect must be considered simultaneously with the Reynolds number and other effects. The apparent Rayleigh number due to compressibility, written to emphasize the Mach number rather than the thermal effect, is given as: $Ra_M \equiv -Pr^{1.5}M^2\ (\gamma-1)\ (g\Delta^3/\nu^2)/2$. It can readily be shown that $(g\Delta^3/\nu^2)$ is a function of only the mean velocity profile shape factor, H, and can be approximated by: $2/t^2$. Thus, the critical Reynolds numbers for laminar instability and laminar-to-turbulent transition in a compressible wall boundary layer are given respectively as:

$$Re_{\delta*_{ci}} = \{\phi^2(I_e) - 0.027\ (Ra_y + Ra_M + Ta + Ay)\}^{1/2}q \quad (13)$$

$$Re_{\delta*_{ci}} = \{\Psi^2(I_e) - 0.00194\ (Ra_y + Ra_M + Ta + Ay)\}^{1/2}q \quad (14)$$

In hypersonic flows (M>4), dissociation of the fluid may occur, so that for validity of equations (13) and (14), the Mach number influence on the functions $\phi(I_e)$, $\Psi(I_e)$, and on the shape factor, H, and the kinematic viscosity, ν, must be known. The experimental results of Van Driest and Boison (8) show that the influence of the free stream turbulence intensity decreases as the Mach number increases; that is, $\phi(I_e)$ and $\Psi(I_e)$ tend to unity as M gets large. The dependence of the shape factor, H, on Mach number for laminar flows is discussed by Wilson (9); the value of H lies parabolically between 2.6 and 5 in the Mach number range, $0<M<4$, and increases more rapidly as M increases. Thus, equations (13) and (14) would indicate decreasing critical Reynolds number up to a value of M of about 4, and increasing critical Reynolds number independent of the mean velocity profile shape factor for M>4. This trend agrees with experimental observations.

CONCLUDING REMARKS

The results of this work and of references (1) and (2) are both quantitatively and qualitatively satisfactory, and inspire some confidence in the advocated physical model for laminar flow instability and transition. The suggestion in reference (1) that the force field of a laminar flow is adequately described by the Stability number, Sn, and the local spatial position, η, appears to be very much in order. With slight modifications, these results can very easily be incorporated in existing computer codes for flow calculations to provide continuous computation of general flows from laminar through transitional to turbulent regimes.

ACKNOWLEDGMENTS

Research support by NSF Grant GK-23989 from the Engineering Mechanics Division of the National Science Foundation is gratefully acknowledged, as is also the computer time and clerical help from Systems Research & Development Associates, Inc. of Boston, Massachusetts.

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APPENDIX

Maximum Amplification of Disturbances in Laminar Wall Boundary Layers

The invariant modeling technique of Donaldson (10) can be used to obtain equations for the disturbance kinetic energies in a boundary layer, in a closed form. An equation for the total disturbance kinetic energy is then obtained by summing these equations for the x, y, z - component. Neglecting the vertical variation of kinematic viscosity, the total perturbation kinetic energy equation for time-independent incompressible flow simplifies to the following:

$$u\partial K/\partial x + v\partial K/\partial y = \nu\, \partial^2 K/\partial y^2 - 2\nu K/y^2 + 2\sigma\partial u/\partial y \quad (15)$$

where:
$K = \langle u'^2\rangle + \langle v'^2\rangle + \langle w'^2\rangle$ and $\sigma = \langle -u'v'\rangle$

Making the transformations: $y = \Delta\eta$; $K = K_{ci}Q$ (where K_{ci} is the value of K at the point of neutral stability); $\sigma = U_1^2\,\tilde{\sigma}$; and $v = U_1\tilde{v}$, the dimensionless form of equation (15) becomes:

$$Q'' + a_1Q' + a_2Q = a_3Q_x + a_4 \quad (16)$$

where:

$Q_x = \partial Y/\partial\ (x/\Delta)$, and

$a_1 = (2K'_{ci}/K_{ci} - \tilde{v}U_1\Delta/\nu)$

$a_2 = (K''_{ci}/K'_{ci} - 2/\eta^2 - \tilde{v}K'_{ci}U_1\Delta/(\nu K_{ci}))$

$a_3 = \tilde{u}U_1\Delta/\nu$

$a_4 = -2(U_1\Delta/\nu)\tilde{\sigma}(\partial\tilde{u}/\partial\eta)/\tilde{K}_{ci}$

From the results of J. T. Stuart (11) it is apparent that a disturbance distorts the mean velocity profile, $u/U_1 = \phi_1(\eta,Sn)$ by introducing a perturbation term, $(b\tilde{\sigma}Sn)$, so that the mean velocity profile of a disturbed laminar flow is $u/U_1 = \phi_1(\eta,Sn) + b\tilde{\sigma}Sn$.

Since in a laminar wall boundary layer, the transverse and streamwise mean velocities are related as follows: $v/u = \phi_3\ (\eta)\ Re^{-\frac{1}{2}}$, one may generally represent the transverse mean velocity as follows: $\tilde{v} = \phi_3(\eta)\ \ \{\phi_1/Sn + b\tilde{\sigma}\}$.

Without loss of generality one could assume that the total disturbance kinetic energy at the point of neutral stability has a distribution across the boundary layer given by the curve, $\tilde{K}_{ci} \propto \Phi(I_e)\phi_2(\eta)$, where I_e is the intensity of the free stream and wall disturbances, and $\tilde{K}_{ci} = K_{ci}/U_1^2$; and that the cross-correlation $\tilde{\sigma}$ is related to the total disturbance kinetic energy as follows: $\tilde{\sigma} = \phi_7(\eta)K + \phi_8(\eta)K' + \phi_9(\eta)K'' + \ldots$.

Applying the above relations and making an order of magnitude analysis based on experimentally observed magnitudes of the different variables of equation (16) in laminar and transitional wall boundary layers one obtains the following simplified equation:

$$Q'' + F_2Q'^2 + F_1Q' + F_o = 0 \quad (17)$$

where:

$Q \equiv Q(\eta,Sn,I_e)$, and

$F_o \simeq -G_1(\eta)Q_x$

$F_1 \simeq -bG_2(\eta)\Phi\ (I_e)SnQ$

$F_2 \simeq bG_3(\eta)\Phi\ (I_e)Sn$

Equation (17) is the classical polynomial class of non-linear differential equations of the second order which describes systems in which "predatory energy" is taken interactively from a basic "prey-supply". Similar equations had been obtained by Volterra (12), for the problem of prey and predator and also by Lord Rayleigh (13), in his consideration on sound motion.

A Special Solution of the Disturbance Equation

Under the transformation: $Q'(\eta,Sn,I_e) = 1/U(Q)$, equation (17) reduces to the Abel's equation of the first type:

$$U'(Q) = F_oU^3(Q) = F_1U^2(Q) + F_2U(Q) \quad (18)$$

Equation (18) is readily solved by the method discussed in Kamke (14); the solution obtained is the equation:

$$Q' = \sum_{i=1}^{\infty} \alpha_i Q^i \quad (19)$$

where: $\alpha_1 \simeq G_4(\eta)Q_x/Q$, $\alpha_2 \simeq -bG_5(\eta)\ \Phi\ (I_e)SnQ_x/Q$, and subsequent terms are of magnitude always much smaller than $\alpha_1 Q$ and $\alpha_2 Q^2$. Interestingly, J. T. Stuart (11), obtained by a different method an equation which is very similar to equation (19), for the streamwise component of disturbance. Since $(U_1\Delta/\nu) \propto Sn$, it is readily shown that $(x/\Delta) \propto Sn$. Thus, if (x/Δ) is measured from the point of incipient flow instability, $(x/\Delta) \propto (Sn-Sn_{ci})$. Representing (x/Δ) by ξ $(\equiv Sn(1-\omega))$, where $\omega = Sn_{ci}/Sn$, equation (19) can be rewritten as:

$$(\partial Q/\partial \xi) \quad \{1 - m(\xi+c)Q\} \simeq G_6(\eta) \quad (\partial Q/\partial \eta) \qquad (20)$$

where: $m = b\Phi(I_e)$, and $c = Sn_{ci}$. An approximate solution of equation (20) is:

$$Q \simeq \exp(A) \;\; / \;\; \{1 + B\exp(A)\} \qquad (21)$$

such that: $A = h\xi$; $B = {}^{m}\xi G_7(\eta)$.

Usually one is interested in the maximum amplification characteristics in the unstable laminar flow, that is, in the amplification characteristics at the critical layer. The critical layer of a laminar wall boundary layer is for most practical purposes a constant η-layer. For the critical layer, therefore, $G_7(\eta)$ is a constant. At the critical layer, also, $Q = Q(\max)$.

Close to the point of neutral stability, (that is, $\xi \to 0$), $Q(\max) \to \exp(A)$, and from experimental results such as those of Schubauer and Skramstad (15), one readily estimates that $h \simeq 0.003$. Furthermore, using the invariant models of Donaldson (10), the amplification factors at transition can be calculated in the manner discussed in reference (16), for different values of free stream disturbance intensity, I_e. The results, Figure (6), indicate that approximately: $mG_7(\eta) \simeq (0.002/Sn_{ci})\tanh\{120I_e\}$. Thus, a special solution of the disturbance equation at the critical layer is:

$$Q(\max) \equiv (K/K_{ci})_{\max} \simeq \exp(\gamma)/\{1 + \beta\exp(\gamma)\} \qquad (22)$$

where:

$$\gamma \simeq 0.003Sn(1-\omega), \text{ and } \beta \simeq 0.002\tanh(120I_e)\{(1-\omega)/\omega\}$$

Curves of equation (22) shown in Figure (7) for different values of I_e are qualitatively and quantitatively realistic.

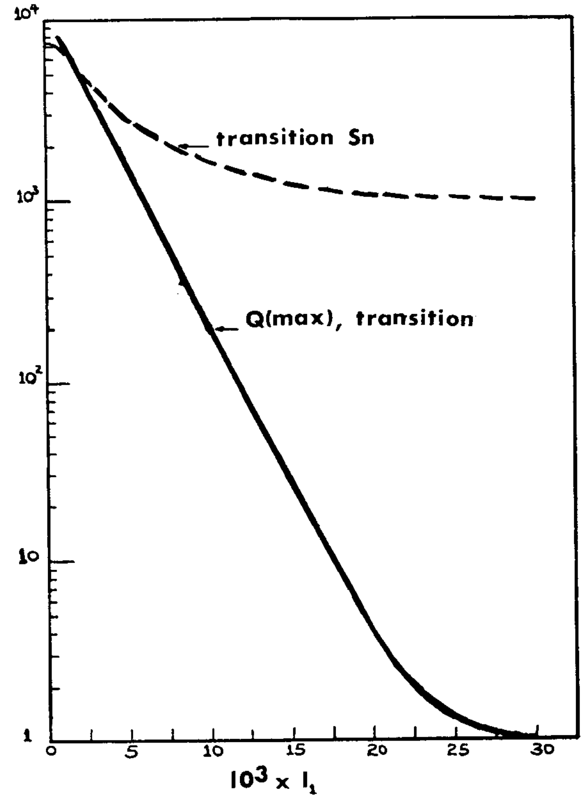

Fig. 6 Disturbance Amplitude Effect on Flow Transition

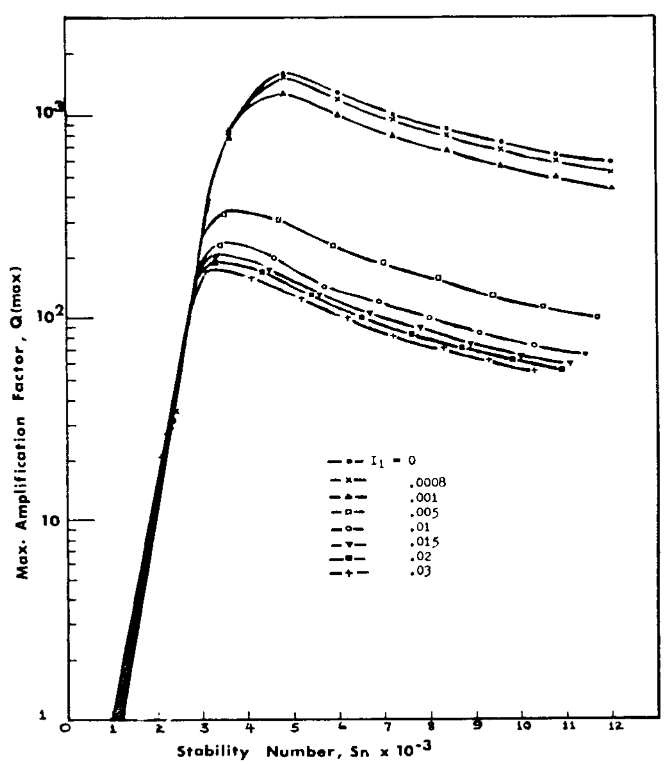

Fig. 7 Spatial Variation of Max. Amplification Factor