TAYLOR-GOERTLER VORTICES IN LAMINAR WALL JETS ALONG CURVED SURFACES

by

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This paper presents some results of a linear stability analysis on the formation of Taylor-Goertler Vortices in laminar wall jets along curved surfaces. The classical instability of a laminar boundary layer along a concave curved wall which results in a system of counter rotating streamwise oriented vortices is fairly well documented [1]. The parameter governing the stability of such a flow is the Goertler Number defined as $R_d \sqrt{\delta/R}$ where R_d is the Reynold's Number based on a characteristic length δ of the boundary layer, and R is the radius of curvature of the wall. For such a flow to be unstable, two conditions must be satisfied: the wall must possess concave curvature; and the Goertler Number of the flow should exceed some critical value. The instability is a direct consequence of an unstable distribution of angular momentum throughout the boundary layer, which results in an imbalance between the centrifugal and pressure forces. The wall jet however, may display this centrifugal instability on convex as well as concave curved walls due to the nature of the longitudinal component of it's mean velocity profile.

Following convention, we define the region between the wall and the maximum in the velocity profile as the inner flow, while the region beyond is designated the outer flow [fig. (1)]. When the flow is along a concave surface, the inner flow is unstable and we designate this as the Type I instability. Flow along a convex surface results in an instability of the outer flow and this we name the Type II instability.

Theory

Following Smith [2], a linear perturbation analysis yields the following set of four coupled linear total differential equations:

$$\frac{d^2u}{d\eta^2} - \left[V_o\ R_d + K\right] \frac{du}{d\eta} + \left[B^2 - A^2 - R_d\left(B\,U_o - \frac{dV_o}{d\eta}\right) + K\eta B\left(2B - U_oR_d\right)\right]u - \left[R_d\frac{dU_o}{d\eta} + K\left(2B - U_oR_d\right)\right]v = \frac{B}{2}\,C\left(1 + K\eta\right) \qquad (1)$$

$$\frac{d^2v}{d\eta^2} - \left[V_oR_d + K\right]\frac{dv}{d\eta} + \left[B^2 - A^2 + 2K\eta B^2 - BR_dU_o\left(1 + K\eta\right) - R_d\frac{dV_o}{d\eta}\right]v - \left[2K\left(R_dU_o - B\right)\right]u = \frac{1}{2}\frac{dC}{d\eta} \qquad (2)$$

$$\frac{d^2w}{d\eta^2} - \left[V_oR_d + K\right]\frac{dw}{d\eta} + \left[B^2 - A^2 + 2KB^2\eta - BR_dU_o\left(1 + K\eta\right)\right]w = -\frac{A}{2}C \qquad (3)$$

$$Bu + KBu\eta + \frac{dv}{d\eta} + Aw - Kv = 0 \qquad (4)$$

where u, v, w, are the dimensionless fluctuations of velocity with respect to η in the x, y, z directions respectively and C is the dimensionless pressure fluctuation. η is the dimensionless vertical (y) coordinate. U_o and V_o are the dimensionless streamwise and vertical velocities respectively, of the primary flow. A, B and K are the dimensionless wavenumber, amplification parameter and curvature respectively, concave curvature being denoted by a positive K. The scaling length and velocity used are those proposed by Glauert [3] in a similarity analysis of the laminar wall jet. Consequently, this permits direct use of his similarity solutions for the mean velocities U_o and V_o in the stability calculations. The Reynold's number then becomes $\left(\frac{U_s x}{\nu} \right)^{1/4}$ where U_s is a velocity related to the outer momentum flux, ν the kinematic viscosity and x the distance downstream from a virtual origin. The boundary conditions applicable are

$$\eta = 0, \qquad u = v = w = 0$$

$$\eta \to \infty, \qquad u, v, w, \to \infty$$

Additionally, the continuity equation (4) for the fluctuations yields the auxiliary boundary condition

$$\eta = 0, \qquad \frac{dv}{d\eta} = 0$$

The equations are homogenous with homogeneous boundary conditions and therefore constitute an eigenvalue problem. The information sought is the variation of K with A, all other parameters remaining fixed. A multiple shooting method using a Hamming Predictor-Corrector integration formula was used in the numerical solution procedure. In order to restrict the range of integration to a finite interval, asymptotic solutions valid in the far field region were derived and matched to the numerical integration at $\eta = 10$. Details of the method are described in Conte [4] and in Kahawita [5].

Results

The results are presented in terms of the Goertler Number defined here as $R_d \sqrt{K}$. They were found to be independent of Reynold's Number, a result also found by Smith [2]. The neutral stability curves (B = 0) for the Type I and Type II instabilities are shown in fig. (2). The critical values of Goertler Number and wavenumber for the Type I instability are considerably lower than those of Type II. This is most probably due to the action of the vertical velocity component of the primary flow in limiting the size of the Type II disturbances.

Previous work [6] has demonstrated the importance of the vertical velocity component of the primary flow in stability calculations. In the Type I instability, the unstable inner flow is bounded below by the wall and above by stable fluid. However, throughout the unstable layer, the vertical velocity component of the primary flow is directed outwards, thus encouraging free-stream penetration even though it changes sign later on in the outer flow. The Type II fluctuations which originate in the outer region, are restricted above by the vertical velocity component of the primary flow directed inwards, and below by, again the vertical velocity component directed upwards and the solid wall. An examination of the eigenfunctions in fig. (3) demonstrates this restricted extent of penetration by the Type II fluctuations.

Concluding Remarks

The work reported herein could also find application in turbulent wall jets. Tani [7] has demonstrated the utility of stability calculations of Taylor-Goertler vortices applied to turbulent boundary layers.

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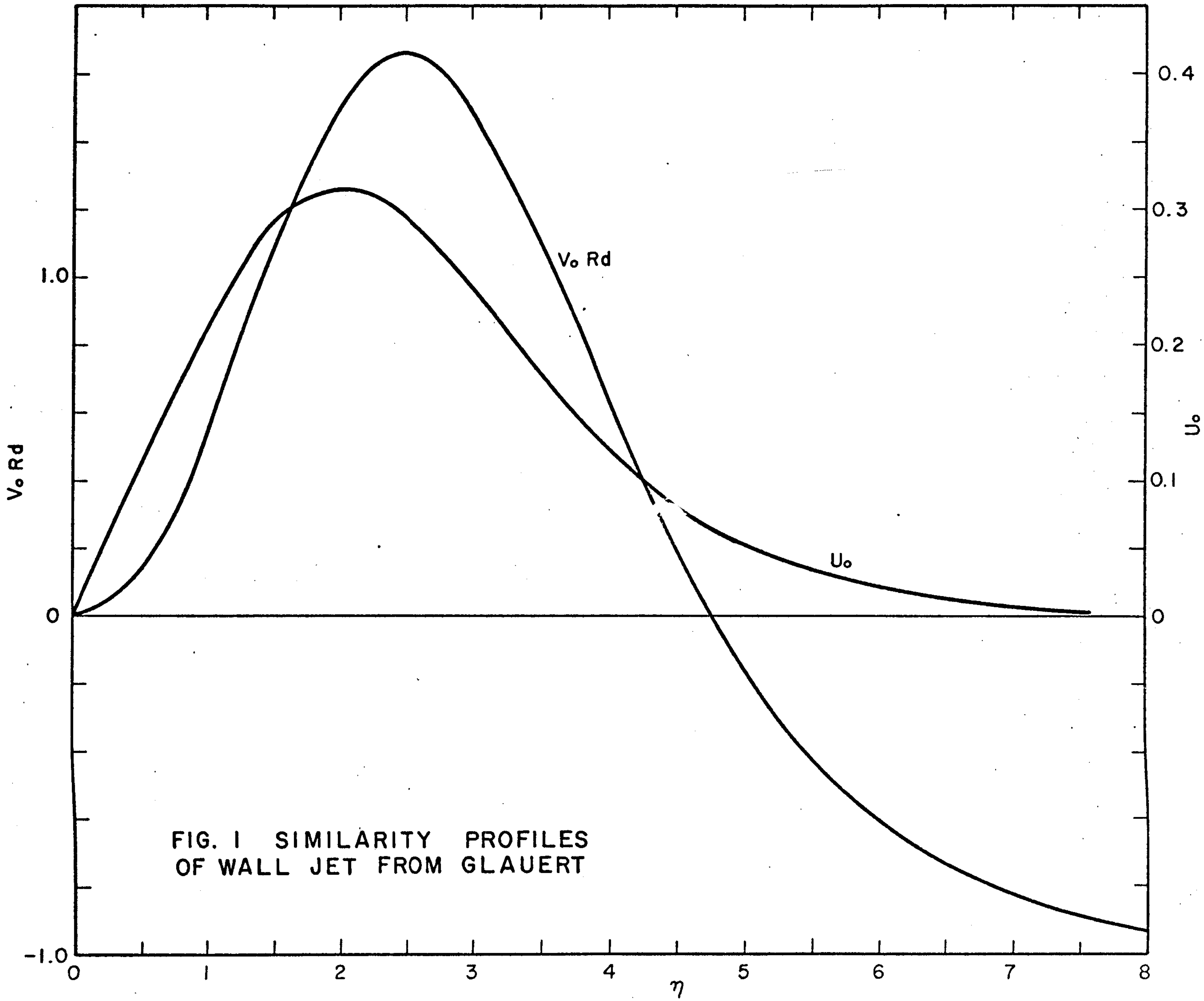

FIG. I SIMILARITY PROFILES OF WALL JET FROM GLAUERT

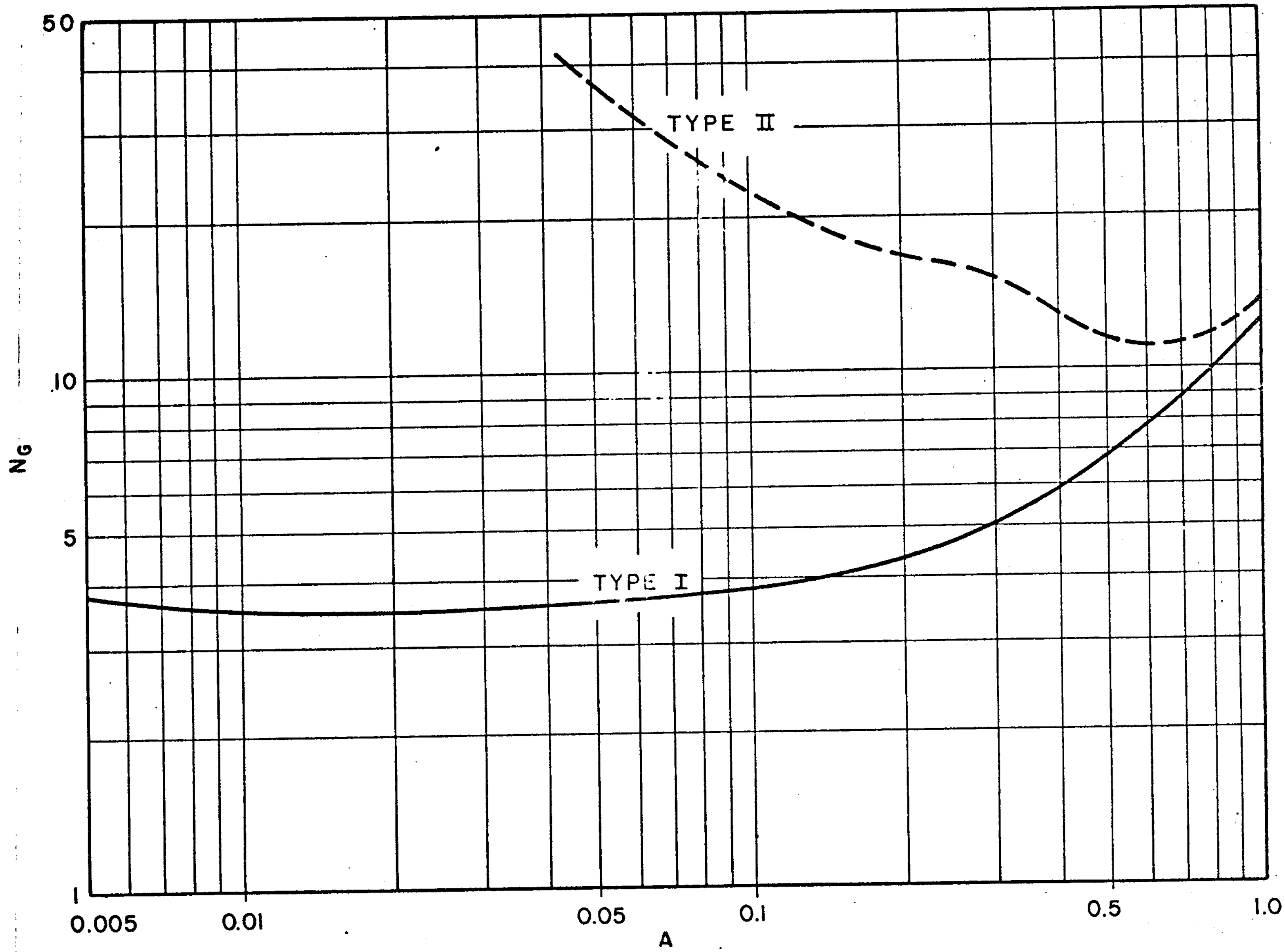

FIG. 2 NEUTRAL STABILITY CURVES

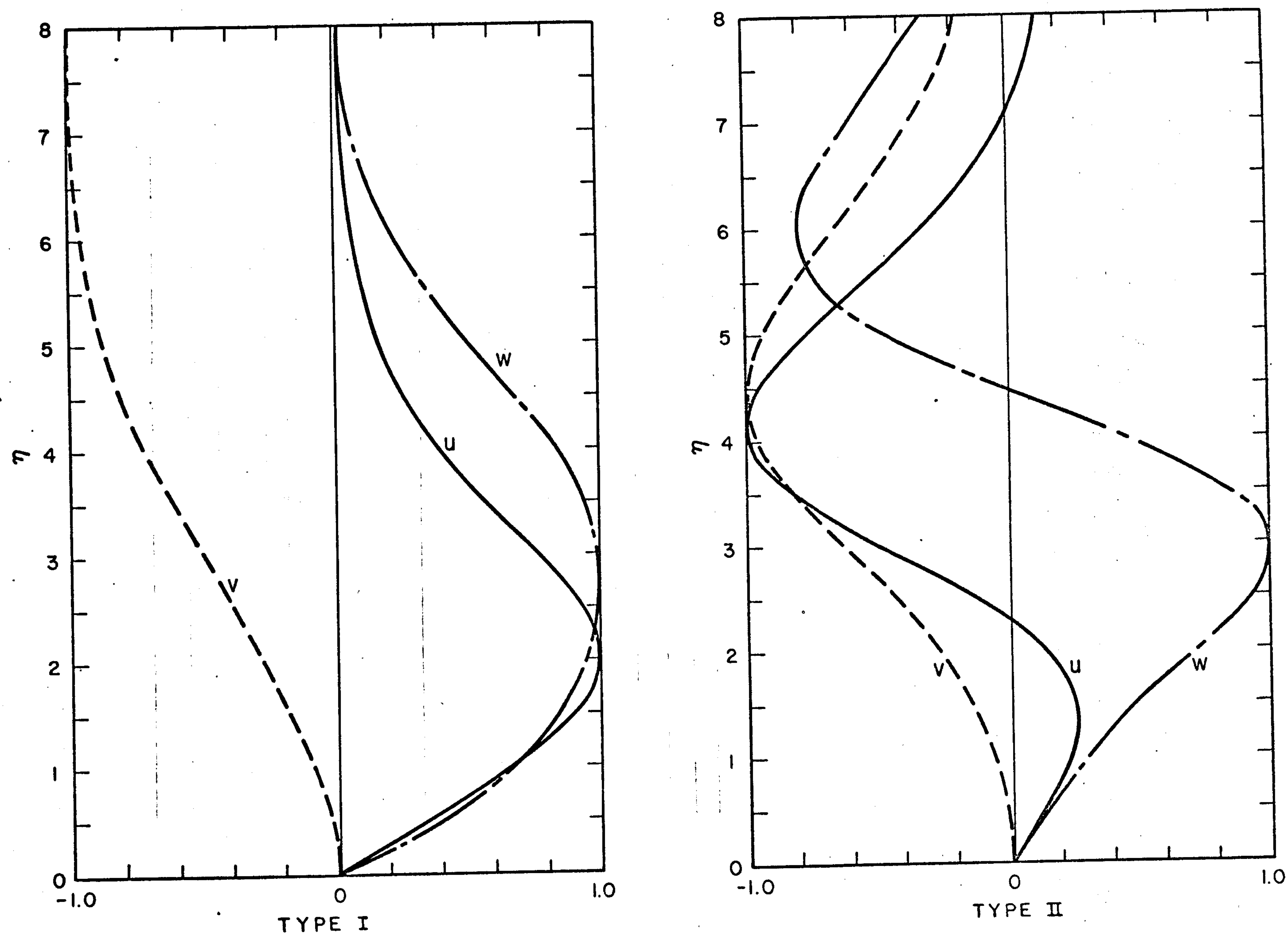

FIG. 3 EIGENFUNCTIONS AT CRITICAL CONDITIONS