TAYLOR-GOERTLER VORTICES IN LAMINAR WALL JETS
ALONG CURVED SURFACES

by

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This paper presents some results of a linear stability analysis on the formation of Taylor-Görtler Vortices in laminar wall jets along curved surfaces. The classical instability of a laminar boundary layer along a concave curved wall which results in a system of counter rotating streamwise oriented vortices is fairly well documented [1]. The parameter governing the stability of such a flow is the Görtler Number defined as \( R_d \frac{\delta}{\sqrt{R}} \) where \( R_d \) is the Reynolds' Number based on a characteristic length \( \delta \) of the boundary layer, and \( R \) is the radius of curvature of the wall. For such a flow to be unstable, two conditions must be satisfied: the wall must possess concave curvature; and the Görtler Number of the flow should exceed some critical value. The instability is a direct consequence of an unstable distribution of angular momentum throughout the boundary layer, which results in an imbalance between the centrifugal and pressure forces. The wall jet however, may display this centrifugal instability on convex as well as concave curved walls due to the nature of the longitudinal component of its mean velocity profile.

Following convention, we define the region between the wall and the maximum in the velocity profile as the inner flow, while the region beyond is designated the outer flow [fig. (1)]. When the flow is along a concave surface, the inner flow is unstable and we designate this as the Type I instability. Flow along a convex surface results in an instability of the outer flow and this we name the Type II instability.
Theory

Following Smith [2], a linear perturbation analysis yields the following set of four coupled linear total differential equations:

\[
\frac{d^2 u}{d \eta^2} - \left( V_o R_d + K \right) \frac{d u}{d \eta} + \left[ U_o^2 - A^2 - R_d \left( B U_o - \frac{d V_o}{d \eta} \right) \right] u = \frac{B}{2} C (1 + K \eta) \tag{1}
\]

\[
\frac{d^2 v}{d \eta^2} - \left( V_o R_d + K \right) \frac{d v}{d \eta} + \left[ U_o^2 - A^2 + 2 K \eta - 2 R_d U_o \left( 1 + K \eta \right) - R_d \frac{d V_o}{d \eta} \right] v = - \frac{1}{12} \frac{d C}{d \eta} \tag{2}
\]

\[
\frac{d^2 w}{d \eta^2} - \left( V_o R_d + K \right) \frac{d w}{d \eta} + \left[ U_o^2 - A^2 + 2 K B^2 \eta - B R_d U_o \left( 1 + K \eta \right) \right] w = - \frac{A}{2} C \tag{3}
\]

\[
B u + K B u \eta + \frac{d v}{d \eta} + A w - K v = 0 \tag{4}
\]

where \( u, v, w \), are the dimensionless fluctuations of velocity with respect to \( \eta \) in the \( x, y, z \) directions respectively and \( C \) is the dimensionless pressure fluctuation. \( \eta \) is the dimensionless vertical \( (y) \) coordinate. \( U_o \) and \( V_o \) are the dimensionless streamwise and vertical velocities respectively, of the primary flow. \( A, B \) and \( K \) are the dimensionless wavenumber, amplification parameter and
curvature respectively, concave curvature being denoted by a positive $K$. The scaling length and velocity used are those proposed by Glauert [3] in a similarity analysis of the laminar wall jet. Consequently, this permits direct use of his similarity solutions for the mean velocities $U_0$ and $V_0$ in the stability calculations. The Reynolds's number then becomes $\left( \frac{U_g x}{v} \right)^{1/4}$ where $U_g$ is a velocity related to the outer momentum flux, $v$ the kinematic viscosity and $x$ the distance downstream from a virtual origin. The boundary conditions applicable are

$$\eta = 0, \quad u = v = w = 0$$
$$\eta \to \infty, \quad u, v, w \to \infty$$

Additionally, the continuity equation (4) for the fluctuations yields the auxiliary boundary condition

$$\eta = 0, \quad \frac{dv}{d\eta} = 0$$

The equations are homogenous with homogeneous boundary conditions and therefore constitute an eigenvalue problem. The information sought is the variation of $K$ with $A$, all other parameters remaining fixed. A multiple shooting method using a Hamming Predictor-Corrector integration formula was used in the numerical solution procedure. In order to restrict the range of integration to a finite interval, asymptotic solutions valid in the far field region were derived and matched to the numerical integration at $\eta = 10$. Details of the method are described in Conte [4] and in Kahawita [5].
Results

The results are presented in terms of the Coertler Number defined here as $R_d \sqrt{K}$. They were found to be independent of Reynold's Number, a result also found by Smith [2]. The neutral stability curves ($B = 0$) for the Type I and Type II instabilities are shown in fig. (2). The critical values of Coertler Number and wavenumber for the Type I instability are considerably lower than those of Type II. This is most probably due to the action of the vertical velocity component of the primary flow in limiting the size of the Type II disturbances.

Previous work [6] has demonstrated the importance of the vertical velocity component of the primary flow in stability calculations. In the Type I instability, the unstable inner flow is bounded below by the wall and above by stable fluid. However, throughout the unstable layer, the vertical velocity component of the primary flow is directed outwards, thus encouraging free-stream penetration even though it changes sign later on in the outer flow. The Type II fluctuations which originate in the outer region, are restricted above by the vertical velocity component of the primary flow directed inwards, and below by, again the vertical velocity component directed upwards and the solid wall. An examination of the eigenfunctions in fig. (3) demonstrates this restricted extent of penetration by the Type II fluctuations.
Concluding Remarks

The work reported herein could also find application in turbulent wall jets. Tani [7] has demonstrated the utility of stability calculations of Taylor-Görtler vortices applied to turbulent boundary layers.

References


FIG. 3 EIGENFUNCTIONS AT CRITICAL CONDITIONS