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A Force Field Theory: Part I. Laminar Flow Instability

JOSHUA C. ANYIWO*
Colorado State University, Fort Collins, Colo.

A force field theory is stated, namely, that: "The dynamic characteristics of a body in motion are completely determined by the forces (external and internal) acting on the body, either instantaneously or over a period in time or space." This theory facilitates the definition of a generalized Stability Number in terms of the Engineering Numbers that characteristically describe flows. It is shown that, as defined in this paper, the Stability Number and the position parameter are the necessary and sufficient parameters for describing the dynamic characteristics of flows, in general. Any flow becomes unstable to applied disturbances if the local Stability Number is greater than $N\phi(I_e)$, where N is a numerical constant and $\phi(I_e)$ is a function of the wall and freestream disturbance intensity. The value of N depends on whether the boundaries of the flow are rigid-rigid, rigid-free or free-free. $N = 10^3$ for rigid-free boundaries. The great simplicity and accuracy of this method makes it by far the most practicable for estimating flow instability.

Nomenclature

$A, B, \ldots Z$; $a, b, \ldots z$ = numerical constants
$Ax; Ay$ = Transpiration number, $(U_1\Delta/\nu)(r_0/U_1)$; $(r_0\Delta/\nu)^2$
C = Wall curvature parameter, Δ/r_0
F = forcefield
G = Görtler number, $Re_\theta(\theta/r_0)^{1/2}$
H = Velocity Profile Shape Factor, δ^*/θ
I = local total disturbance intensity
i, j, k = indices
Mini (.) = minimum value of (.)
Pr = Prandtl number
r_0 = wall radius of curvature
Ra = Rayleigh number
Re = Reynolds number
Sn = section stability number
Ta = Taylor number
U_1 = freestream velocity
u = local streamwise velocity

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* Research Scientist. Associate Member AIAA.

v = local transverse velocity
w = local lateral velocity
x, y, z = coordinates
Δ = modified boundary-layer thickness
δ = boundary-layer physical thickness, $y_u = 0.955U_1$
δ^* = boundary-layer displacement thickness
θ = boundary-layer momentum thickness
λ = local disturbance scale length
η = dimensionless height from wall, y/Δ
ν = kinematic viscosity
ξ = $10^{\{3+2.34 \tanh(10/H-4)\}}$
Ω = forcefield angle
ϕ Φ, ψ Ψ = functions as defined in text
χ = pressure gradient function, $\{(10/H)-4\}$
$(\ )_{ci}$ = value at point of neutral stability
$(\ )'$ = fluctuating quantity
$(\ )_0$ = value at the wall

Introduction

MOST analytical studies of fluid flow confine themselves to finding exact or approximate solutions to simplified forms of the Navier-Stokes equations. This approach has so far been very difficult and, in terms of practical useable results, very inefficient. Without doubt, the analytical solution of the governing equations of general fluid flow is very desirable. However, in the light of the inadequacy of current mathematical tools for tackling this problem, one must begin to think of more tractable alternatives which yield satisfactory and useable results. This paper pursues such an alternative, using the force field theory.

The basic Navier-Stokes equations were derived from Newton's fundamental principle relating the external force on, and the acceleration of, a body in motion. This fundamental principle implies the force field theory namely that: "The dynamic characteristics of a body in motion are completely determined by the forces (external and internal) acting on the body, either instantaneously or over a period in time or space." Thus, if one can describe the instantaneous or average force field prevailing on a body, one can readily determine the complete instantaneous or average characteristics of the motion of that body. It would seem that a limitation to the force field approach would lie in the difficulty to completely describe a force field. However, the net force on a body can always be decomposed into a driving force on the body and a resistive force due to the body. The ratio of the driving to the resistive force provides a dimensionless descriptive quantity for the local force field on the body. Thus, if one can locally or sectionally describe the above ratio, one has locally or sectionally defined, quantitatively, the prevailing force field.

It appears clear, that explicitly, the force field theory emphasizes the "tenacity" or "cohesiveness" of a fluid in motion. A high "cohesiveness" means that conceptual fluid particles are strongly bonded together, so that external perturbations on the fluid are strongly resisted. This situation will be taken to correspond to a large force field. This rather odd definition of the force field is adopted in this paper in order to emphasize that, for the stability of a flow, the force of primary interest is that due to the fluid reaction to external impressed forces. If this resistive force is large, the flow is stable to perturbations. A zero force field corresponds then to the situation of a critical "cohesiveness" for which the fluid could resonate to certain types of disturbances present in it. Since a negative "cohesiveness" is physically meaningless, a fluid must readjust to increase its "cohesiveness" whenever the property tends to disappear. This is precisely what happens in the transition from laminar to turbulent flow. The fluid undergoes a phase change with respect to its cohesive property. The same situation is observed on a more severe scale in the phase change from liquids to gaseous state or vice versa.

Unlike the approach of seeking analytical solutions to the Navier-Stokes equations, the force field theory permits a philosophical pursuance of the general flow problem, using physical analogies and the abundant experimental results available. This paper discusses the general force field theory and its application to the simplification of the laminar flow stability problem. Generalized stability criteria are derived in an amazingly simple manner and yield rather excellent results when applied even to the most general type of flow problem.

Force Field Concept of a Stability Number

The force field prevailing on a fluid may be described by the local forces as follows.

Local Force Field =
ψ_1 (Local Driving Force, Local Resistive Force)

If one focuses attention on planes perpendicular to the streamwise direction, then in dimensionless form, for a two-dimensional flow:

Force Field =
ψ_2 (Section Driving Force/Section Resistive Force, η)

where $\eta = y/\Delta$ and Δ is a suitable scaling length. Since the force field theory implies that the stability characteristics of a flow are determined by the prevailing force field, one may appropriately define the dimensionless force quantity in the above expression as the flow Stability number. In general, one may then write:

Stability number =
(Net Driving Force/Net Resistive Force) = Sn

Hence, Force Field = $\psi_2[Sn, \eta]$. Thus, the necessary and sufficient parameters with which one may describe the general dynamic characteristics of any flow are the section flow Stability number Sn and the dimensionless position η as defined above.

Stability Number for Particular Flows

Consider the two simple one-dimensional flow problems sketched in Figs. (1a, 1b). The first problem is a one-dimensional flow along a horizontal plane, Fig. (1a). For this case, the prevailing force field is determined by the inertial and resistive (primary viscous) forces. Thus, the descriptive force quantity, Sn, is the ratio of the inertial to the viscous force, conventionally called the Reynolds number $U_r\Delta/\nu_r$. U_r and ν_r are respectively the reference velocity and kinematic viscosity and Δ is a suitable scaling length. If, in addition, heat- and mass-transfer occur at the boundary of the flow sketched in Fig. (1a), the net driving force may be distinguished as the algebraic sum of an inertial, and a buoyancy force and a force associated with the momentum imbalance due to mass transfer, in the streamwise direction. It can easily be verified that the ratio of the streamwise component of the force associated with the momentum imbalance due to transpiration or aspiration, to the streamwise viscous force is given by $Ax = (U_r\Delta/\nu_r)(v_0/U_r)$, where v_0 is positive for transpiration. The ratio of the streamwise buoyancy force to viscous force will simply be called the x-Rayleigh number, Ra_x. Thus, the Stability number for the horizontal one-dimensional flow with heat and mass transfer at the boundary is given by

$$Sn_x = [Re_\Delta(1 + v_0/U_r) + Ra_x] \quad (1)$$

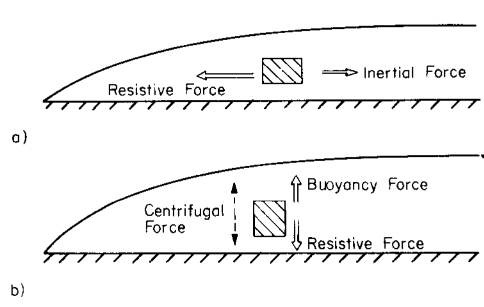

Fig. 1 Schema of boundary-layer sections.

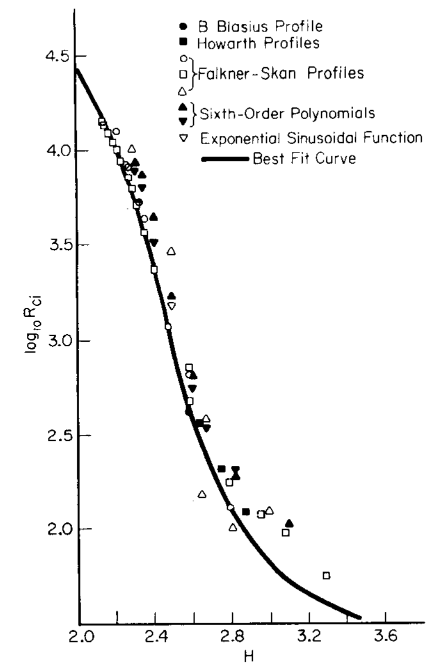

Fig. 2 Functional relations between $(U_1\delta^*/\nu)_{ci}$ and the shape factor, H.

The second flow problem considers the force influence in the vertical direction only. If there is no heat or mass transfer at the boundary, but the boundary is curved, the descriptive force quantity in the vertical direction would be the ratio of the centrifugal force to the vertical viscous force. That is, the Taylor number, Ta $(\equiv 2CRe_\Delta^2)$, where C $(\equiv \Delta/r_0)$ and r_0 is the wall radius of curvature which is positive for concave curvature. If heat- and mass-transfer occur at the boundary, then the net driving force in the vertical direction is the algebraic sum of a centrifugal and a buoyancy force and the vertical component of the force associated with the momentum imbalance due to mass transfer at the boundary. Again it may easily be verified that the ratio of the vertical component of the force associated with the momentum imbalance due to transpiration or aspiration, to the vertical viscous force is given by $Ay = (v_0\Delta/\nu_r)^2$. The ratio of the vertical buoyancy force to the vertical viscous force is the y-Rayleigh number, Ra_y. Thus, the Stability number for the vertical one-dimensional flow with heat and mass transfer at the boundary is given by

$$Sn_y = [Ra_y + Ta + Ay] \tag{2}$$

A Generalized Stability Number for Two-Dimensional Flows

In the more general two-dimensional flows, the force field becomes slightly more complicated to describe. However, the net driving force components, Fx and Fy, may be distinguished for the streamwise and vertical directions, respectively. Associated with these driving forces are net resistive forces (primarily viscous), Vx and Vy, due to the fluid. One may therefore want to write that the appropriate Stability number is given by

$$Sn = [(Fx^2 + Fy^2)/(Vx^2 + Vy^2)]^{1/2} \tag{3}$$

which is the ratio of the vector sum of the driving forces to the vector sum of the resistive forces. Equation (3) is, however, incomplete, as it omits the qualitative effect of the resultant force direction. Suppose that the resultant force makes an angle Ω with the streamwise direction, where Ω is positive if Fy is in the positive y direction. If the streamwise direction is defined, the influence of the sign of the angle Ω is contained in the vertical forces $\bar{F}y$ and $\bar{V}y$. A new force ratio, $\bar{F}y^2/(\bar{V}y^2 + \bar{V}x^2)$, is the more appropriate force quantity to use for the vertical direction because in the physical flows, the instability modes are very sensitive to the direction of Fy, and it should not suffice to define this direction separately from Fy. Thus,

$$Sn = [Fx^2/Vx^2(1+\bar{V}y^2/Vx^2) + \bar{F}y^2/\bar{V}y^2(1+Vx^2/\bar{V}y^2)]^{1/2} \tag{4}$$

where $\bar{F}y$ and $\bar{V}y$ are, respectively, the vertical driving and resistive forces redefined to include the qualitative influence of the sign of Ω

$$Fx^2/[Vx^2(1+\bar{V}y^2/Vx^2)] \equiv Sn_x^2/[1+0(Re_\Delta^{-2})] \doteq Sn_x^2$$
$$\bar{F}y^2/[\bar{V}y^2(1+Vx^2/\bar{V}y^2)] \equiv \text{Function}\,\{Sn_y/[1+0(Re_\Delta)],\ \text{sign}\,\Omega\} \tag{5}$$

A function of two variables can usually be expressed in powers of the one variable with functional coefficients of the other variable. Thus

$$\bar{F}y^2/[\bar{V}y^2(1+Vx^2/\bar{V}y^2)] \equiv a_1 Sn_y + a_2 Sn_y^2 + a_3 Sn_y^3 + \cdots \tag{6}$$

where a_i is a function of sign Ω divided by a quantity of order of magnitude Re_Δ^i, and $i = 1, 2, 3 \ldots$. It is clear that, for $i > 1$, $a_i \to 0$, rapidly, for noncreeping flows. Thus, only the first coefficient a_1 of Eq. (6) may be appreciably different from zero if Re_Δ^i is not small. Hence, one may define a generalized Stability number for two-dimensional flows, away from the flow entrance region, as follows:

$$Sn = [Sn_x^2 + a_1 Sn_y]^{1/2} \tag{7}$$

where Sn_x and Sn_y are as previously defined and $a_1 \approx \text{const}/Re_\Delta$. The above result is similar in form to one obtained by D. D. Joseph[1] using a variational method for plane Couette flows heated from below.

A Generalized Stability Criterion for Boundary Layers

Equation (7) indicates that for a simple two-dimensional flow with no heat- or mass-transfer at the boundary the generalized Stability number reduces to the Reynolds number, $(U_r\Delta/\nu_r)$. Conventionally one uses the freestream velocity as U_r and a boundary-layer thickness as the scaling length Δ. One soon finds, however, that with such a Reynolds number the flow characteristics depend on the mean velocity profile shape factor as well as on the Reynolds number. According to the force field theory, the flow should depend only on the local Reynolds number. It will thus be desirable to define a Reynolds number which implicitly contains the influence of the velocity profile shape factor.

Let the freestream velocity, U_1, be retained as the reference velocity. Figure 2 shows a plot of the critical Reynolds number against the mean velocity profile shape factor for a fairly wide class of flows as taken from Rosenhead.[2] A best fit curve for these data may easily be obtained if the shape factor parameter, $(10/H-4)$, is used. With this parameter one obtains from Fig. 2 that

$$\log_{10}(U_1\delta^*/\nu)_{ci} \doteq 3 + 2.34\tanh(\chi) \tag{8}$$

where $\chi = (10/H-4)$ and the subscript ci denotes values at the point of neutral stability. Equation (8) may be rewritten as: $(U_1\delta^*/\nu)10^{-2.34\tanh(\chi)} \doteq 10^3$ at the point of neutral stability. Thus, if a new length scale, $\Delta \equiv \delta^*[10^{-2.34\tanh(\chi)}]$ is defined, a universal constant value, 10^3, is obtained for the critical Reynolds number $U_1\Delta/\nu$, at the point of incipient laminar instability. For a general two-dimensional flow, then

$$Sn_{ci} = [Sn_x^2 + a_1 Sn_y]_{ci}^{1/2} \doteq 10^3 \tag{9}$$

with $\Delta \equiv \delta^*[10^{-2.34\tanh(10/H-4)}]$.

An attempt will now be made to formalize this universal stability criterion. Consider a two-dimensional laminar flow in which, somehow, a perturbation has entered such that the velocity components can be represented by a mean portion and a fluctuating portion, whose mean is zero. That is

$$u = \bar{u} + u', \quad v = \bar{v} + v', \quad w = 0 + w' \tag{10}$$

Consider, further, a control volume whose dimensions are proportional to the dimensions of the local disturbance length scale, λ. If the local amplitudes of the disturbance in the control volume are, in the relevant Cartesian coordinate system u', v', and w' which may be finite or infinitesimal, the stability problem reduces to finding the conditions under which these amplitudes will be sustained and amplified, or damped out.

The characteristic local fluctuational velocity scale within the control volume is related to the rms velocity,

$$J \equiv [(\langle u'^2 \rangle + \langle v'^2 \rangle + \langle w'^2 \rangle)/3]^{1/2} \qquad (11)$$

Also, the characteristic period which specifies the order of magnitude of the time required for the occurrence of a fluctuation, is related to λ/J. The energy of the fluctuation in the control volume mentioned above is derived from the main flow, through some mechanism, and manifests itself as primarily kinetic energy. When the fluctuation occurs, its energy, per unit mass, is proportional in order of magnitude to J^2. The fluctuational energy per unit time and mass is therefore of order of magnitude $J^2/(\lambda/J)$. Thus, when a fluctuation occurs, the amount of energy which goes over from the main flow to the fluctuation in the control volume, per unit time and mass is equal in order of magnitude to J^3/λ. The same result may be obtained from the primary production of turbulence term, $\overline{u_i'u_j'}(\partial \bar{u}_i/\partial x_j)$, if Prandtl's mixing length argument is used to show that $(\partial \bar{u}_i/\partial x_j)$ is given in order of magnitude by the ratio J/λ. The previous derivation is, however, more general than this latter alternative. The fluctuational energy dissipation in an incompressible flow with velocity fluctuations is given generally, per unit time and mass, in Cartesian tensor notation by:

$$\varepsilon = \nu\langle(\partial u_i'/\partial x_j + \partial u_j'/\partial x_i)(\partial u_i'/\partial x_j)\rangle \qquad (12)$$

If it is assumed that the local velocity gradients of the fluctuations, $\partial u_i'/\partial x_j$, etc, are given by the ratio, J/λ, of the local scale values, then the energy loss by fluctuations in the control volume, per unit time and mass, is equal in order of magnitude to $\nu J^2/\lambda^2$. If all other perturbations on the energy of the main flow in the control volume are small compared to those mentioned previously, then the fluctuation in the fluid will be sustained only if the production quantity is greater in magnitude than the dissipation quantity. Growth of the fluctuation implies, in the conventional sense, instability of the flow. Thus, the flow in the control volume is unstable if

$$J^3/\lambda > \nu J^2/\lambda^2 \qquad (13)$$

If vertical distances are nondimensionalized with the characteristic length Δ and all velocities nondimensionalized with U_1, the instability criterion (13) may be reduced to the following form:

$$U_1\Delta/\nu > 1/(I\lambda/\Delta) \qquad (14)$$

where I is the local disturbance intensity defined as $I = J/U_1$. The inequality (14) indicates that for the fluctuations in a flow to be sustained, the appropriate dimensionless force quantity for the flow must be greater than some function of the local disturbance intensity, I, and the local dimensionless disturbance length scale, (λ/Δ). Thus, in general,

$$Sn < \psi(I, \lambda/\Delta) \qquad (15)$$

for flow stability.

At the critical layer, it is quite legitimate to assume that λ/Δ is approximately constant. This quantity, λ/Δ, contains the influence of the over-all mean flowfield. Further, the local disturbance intensity, I, depends on the freestream and wall disturbances, as well as on the local force field. Thus one may write that, for general flow stability,

$$Sn < N\Phi(I_1, I_0) \qquad (16)$$

where I_1 = freestream disturbance intensity, I_0 = intensity of disturbances characteristic of the wall, and N = numerical constant, whose value depends on whether one has rigid-rigid, rigid-free or free-free boundaries.

The aforementioned relation is in agreement with the result obtained earlier in this section. The deduction was made from analyses on flows of various mean velocity profile shapes that, if Δ is used as the characteristic length, a constant value is obtained for the critical stability number, for laminar flow stability. Since the best fit curve used to evaluate the constant matched the flat plate experimental result, $Re_{\delta *_{ci}} = 420$ for Blasius profiles, it may be rightly assumed that the flows analyzed simulated physical flows with freestream disturbance intensities in the

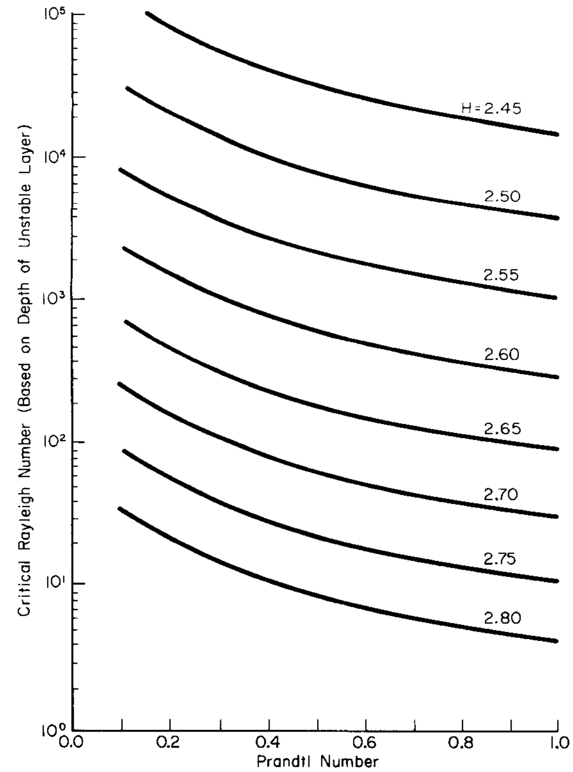

Fig. 3 Influence of Prandtl number on critical Rayleigh number for laminar flow instability.

neighborhood of $I_1 = 0.01$; this value being the average value for the experiments that yielded the result, $Re_{\delta *_{ci}} = 420$ for Blasius profiles. The effect of wall disturbances is rather difficult to define. Experiments such as that of Schubauer and Skramstad[3] indicate, however, that for aerodynamically smooth walls, wall disturbance effects are very small, compared to $I_1 = 0.01$. Thus, the relation (16) may be rewritten as follows

$$Sn < 10^3\phi(I_1) \qquad (17)$$

for laminar flow stability. That is, $N = 10^3$ for rigid-free boundaries. From relation (14), $\phi(I_1)$ must be a decreasing function of I_1, and from (17), $\phi(I_1)$ must be of value close to unity for I_1 in the neighborhood of 0.01. The proper formulation for $\phi(I_1)$ is given in Pt. II (Ref. 4) of this series. An approximate estimate was given in Ref. (5) as $\phi(I_1) \approx 1/(1+280I_1^2)$. Relation (17) may be recast in the following more conventional form, using the formulation for Sn:

$$Re_{\delta *} \leqq \{\phi^2(I_1) - b(Ra_y + Ta + Ay)\}^{1/2}\xi/(1 + v_0/U_1) \qquad (18)$$

for laminar flow stability, where b is a numerical constant which corresponds to the value of $a_1/10^6$ at the point of neutral stability, and $\xi = 10^{\{3+2.34 \tanh(\chi)\}}$. The constant b may be fixed from the result for critical Reynolds number for asymptotic suction profiles. For such profiles, the critical Reynolds number, $U_1\delta^*/\nu$, for laminar flow stability has been calculated to be between 4×10^4 and 7×10^4, notably by C. C. Lin,[6] Iglisch,[7] and Ulrich.[8] Equation (18) indicates that for a flat plate with suction and no heat transfer at the plate, the critical Reynolds number is given by

$$Re_{\delta *_{ci}} \doteq (1 - bAy)\xi \qquad (19)$$

for moderate freestream disturbance intensity, $I_1 \approx 0.007$. The asymptotic suction profile corresponds to the case $Ay = 1$, $H = 2$ and $(-v_0/U_1)^2(U_1 x/\nu) = \infty$. Using a value for $Re_{\delta *_{ci}}$ between $(4 \text{ and } 7) \times 10^4$ in Eq. (19), one obtains that the constant b has value in the neighborhood of 0.027.

The following generalized stability criteria may now be stated. For laminar flow stability

$$Sn < 10^3\phi(I_1) \qquad (20)$$

or

$$Re_{\delta*} < \{\phi^2(I_1) - 0.027(Ra_y + Ta + Ay)\}^{1/2}\xi/(1 + v_0/U_1) \quad (21)$$

Results Using the Generalized Stability Criteria

The generalized stability criteria have been stated in the above section. The following principal results have been obtained, using the criteria, for various types of flows.

Flows with Heat Transfer at the Wall (Stratified Flows)

If the influence of the freestream disturbance intensity is negligible, then for a flat plate flow with heat transfer at the boundary the stability criterion (20) reduces to the following:

$$Re_\Delta{}^2 + 2.7 \times 10^4 Ra_\Delta < \phi^2(I_1) \times 10^6 \quad (22)$$

for flow stability. If the Rayleigh number is based on the depth of the major unstable layer, the above relation becomes approximately

$$Ra < [\phi^2(I_1)/(H^3 Pr)] \times 10^{\{4.8 + 7.02 \tanh(10/H - 4)\}} \quad (23)$$

for flow stability, where Pr is the Prandtl number. In obtaining the inequality (23), it has been assumed that

$$\delta(\text{thermal}) \doteq [1/(1.026 Pr^{1/3})]\delta(\text{momentum}) \quad (24)$$

after the manner of Eckert and Drake.[9]

For a Blasius velocity profile with a Prandtl number of unity, one obtains that the critical Rayleigh number for laminar flow stability is of the order of 300. Figure 3 shows a chart of the Prandtl number effect on the critical Rayleigh number, for various mean velocity profile shape factors.

No experimental data appear to be available for this type of flow with which to compare the present result. Some numerical results, nevertheless, have been obtained by other workers, for the related case of penetrative cellular perturbations in a horizontal layer of fluid composed of a layer of unstable density gradient above which is a layer of stable density gradient. For the classical rigid-free boundary solution corresponding to the limiting case of infinite stability on top of the unstable layer, Chandrasekhar[10] gives a critical Rayleigh number based on the depth of the layer of about 1100. In reality, of course, the convective cells do penetrate the stable layer. Rintel[11] and Stix,[12] have shown that for small stability on top of the unstable layer, the critical Rayleigh number is much smaller than that obtained by not taking penetration into account. For various degrees of such penetrative cellular perturbations close to the limiting case of infinite penetration, Stix[12] and Rintel,[11] obtained critical Rayleigh numbers based on the depth of the layer of 225 and 172, respectively. Although these results were obtained for no mean flow, they indicate that the present estimate of 300 for a Blasius mean flow with unit Prandtl number is very good.

The case of no mean flow may be simulated in the present work by assuming a Blasius mean flow and choosing a suitable Prandtl number to obtain the desired temperature profile. This approximation is possible because the influence of Reynolds number on thermal instability is of secondary importance at moderate flow speeds.

If one defines the Richardson number, Ri, as the ratio of the buoyancy force to the inertial force, it is easy to show that:

$$Ri_L \equiv -BRa_L/Re_L{}^2 \quad (25)$$

where L is a characteristics scale length and B is a numerical constant. The stability criterion (20) can be rewritten, using (25) in the following form:

$$Ri_L > 0.42 \times 10^{-4} A(1 - \phi^2(I_1) \times 10^6/Re_\Delta{}^2) \quad (26)$$

for stability, where A is a numerical constant. As $Re_\Delta \to \infty$, the critical Richardson number attains a constant value given by $0.42 \times 10^{-4} A$. Schlichting[13] investigated the stability of flows with density stratification with the aid of Tollmien's theory. The calculation was based on the assumption of a Blasius profile for the case of penetrative cellular perturbations in a horizontal layer of fluid composed of a lower layer of unstable

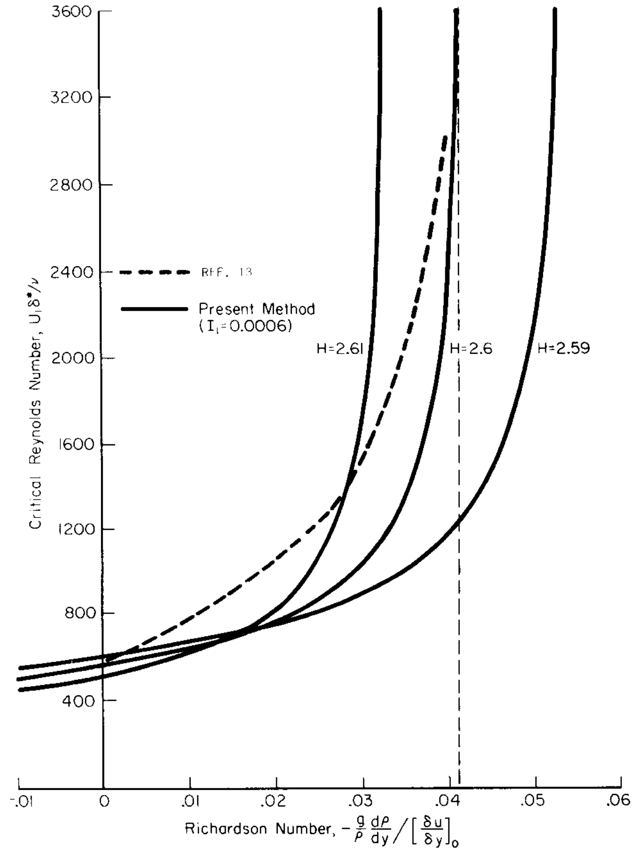

Fig. 4 Critical Reynolds number for stratified flows.

density gradient above a flat plate with a density gradient in the boundary layer and a constant density outside it. Schlichting[13] obtained that the critical Reynolds number, $Re_{\delta*}$, increased rapidly as the Richardson number increased, changing from $Re_{\delta*_{ci}} = 575$ for $Ri = 0$ to $Re_{\delta*_{ci}} = \infty$ for $Ri = 1/24$. The value for $Ri = 0$ is rather too high compared to the accepted value of 420. If, however, it is assumed that Schlichting's asymptotic result $Re_{\delta*_{ci}} = \infty$ for $Ri = 1/24$, is correct for wall bounded flows, and if the characteristic scale length L is chosen as the depth of the unstable layer, the inequality (26) becomes

$$Ri_{\delta(\text{thermal})} > (8.81/H^3 Pr)10^{7.02 \tanh(\chi)}[1 - \phi^2(I_1) \times 10^6/Re_\Delta{}^2] \quad (27)$$

or $Re_{\delta*} < \phi(I_1)\xi/[1 - 0.114 H^3 Pr Ri_{\delta*} \times 10^{-7.02 \tanh(\chi)}]^{1/2}$, for laminar flow stability. Figure 4 shows a plot of this result for various velocity profile shape factors, in comparison to Schlichting's[13] results for the Blasius profile. The present results show the proper trend, at least.

Flows over Curved Walls

For flows over a curved boundary, the present method indicates that

$$Re_\Delta{}^2 + 5.4 \times 10^4 Re_\Delta{}^2(\Delta/r_0) < \phi^2(I_1) \times 10^6 \quad (28)$$

for laminar stability, where r_0 is the wall radius of curvature which is positive for concave curvature. If Δ/r_0 is of the order of 10^{-4} or greater, the influence of wall curvature becomes appreciable. Otherwise, the flow behaves much like a flat plate flow. For the case where the influence of wall curvature is not negligible, the stability criterion (28) may be written as follows

$$Re_\Delta{}^2(\Delta/r_0) < 18.52\phi^2(I_1) \quad (29)$$

for laminar stability.

In terms of the boundary-layer momentum thickness (29) becomes

$$Re_\theta{}^2(\theta/r_0) < (18.52\phi^2(I_1)/H^3)10^{7.02 \tanh(\chi)} \quad (30)$$

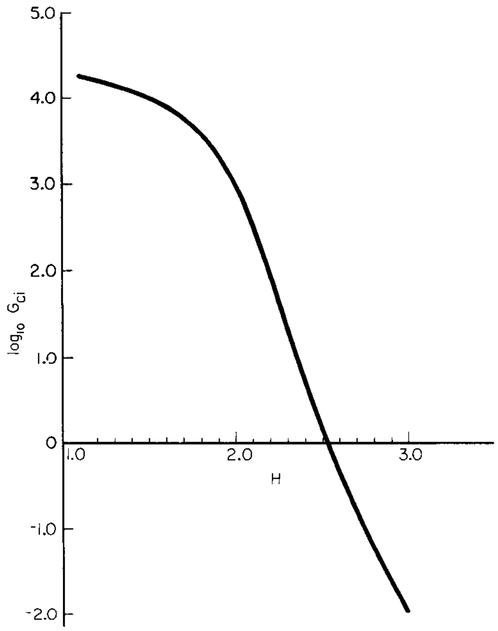

Fig. 5 Influence of shape factor on critical Görtler number.

or $Re_\theta(\theta/r_0)^{1/2} < (4.3\phi(I_1)/H^{1.5})10^{3.51\tanh(z)}$, for laminar flow stability.

For a Blasius profile, $H = 2.6$, at moderate freestream turbulence intensity, the above criterion yields that the critical Görtler number, $Re_\theta(\theta/r_0)^{1/2}$, for laminar stability is 0.30 for all Re_θ. For $H = 2.59$, the critical Görtler number is 0.34.

These estimates compare very well with the values of 0.34 and 0.58 given respectively by A. M. O. Smith[14] and Görtler.[15] Figure 5 shows a plot of the critical Görtler number for various velocity profile shape factors, and Fig. 6 shows the influence of streamwise wall curvature on the critical Reynolds number for stability.

Flat Plate Flows with Mass Transfer at the Wall

For this class of flows the stability criterion (21) reduces to the following:

$$Re_{\delta*_{ci}} < [\phi^2(I_1) - 0.027Ay]^{1/2}\xi/(1 + v_0/U_1) \qquad (31)$$

for laminar stability. The flows studied by C. C. Lin[6] and T. S. Chen et al.,[16] have been recomputed using the force field method. The comparison is shown in Fig. 7. The present estimates

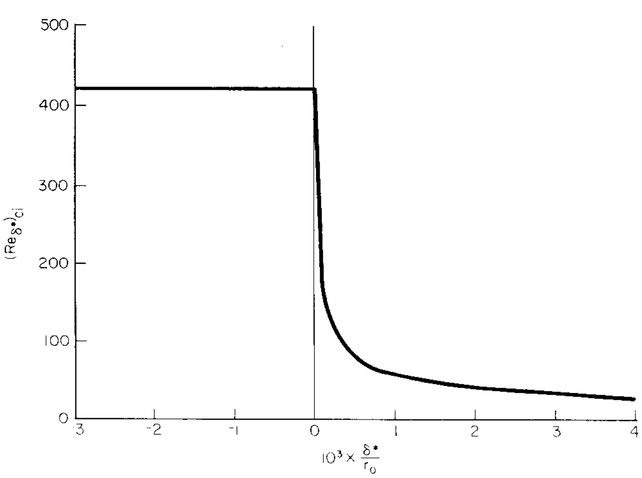

Fig. 6 Effect of streamwise wall curvature on flow stability.

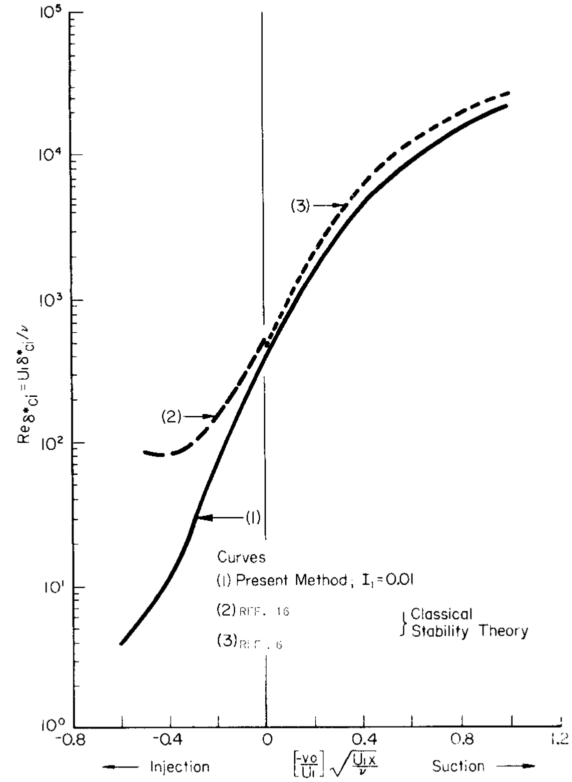

Fig. 7 Effect of wall transpiration on critical Reynolds number.

are very good. It is clear from the relation (31), that wall transpiration affects the flow much like pressure gradient. The major influence is to alter the mean velocity profile shape factor. Wall suction decreases the shape factor, which is stabilizing, while wall injection increases the shape factor, which is destabilizing.

Concluding Remarks

This work has been based on a force field hypothesis, which emphasizes a dynamic fluid property called the "cohesiveness" or fluid "tenacity." This fluid property defines the ability of the fluid to resist perturbations, and appears to be the primary factor in the determination of flow characteristics. All the preliminary results derived on the basis of the force field hypothesis show no significant deviation from reality. In fact, the incompleteness of some previous boundary-layer stability theories becomes very obvious in view of the force field concept. Boundary-layer stability phenomena and, indeed, general natural stability phenomena seem to follow exactly according to this force field idea. A general force field theory is therefore stated as follows: "Particles in any system in nature will tend to execute independent behavior or motion in accordance with their separate internal force fields except in as constrained by the prevalent external force field. It requires a steady force field above a certain critical magnitude to establish "order" among the particles. The magnitude of this critical force field is proportional to the average internal force field of the particles."

For the special case of two-dimensional wall boundary layers, the following conclusions are drawn, on the basis of the force field theory:

1) The general characteristics of the wall boundary layer may be described completely, solely by the local "cohesive" fluid property which is a function of position and section Stability number, only. The section Stability number for a laminar flow is given by the following relation:

$$Sn \doteq [Re_\Delta^2(1 + v_0/U_1)^2 + a_1(Ra_y + Ta + Ay)]^{1/2}$$

where $a_1 \approx 1.1 \times 10^7/Re_\Delta$.

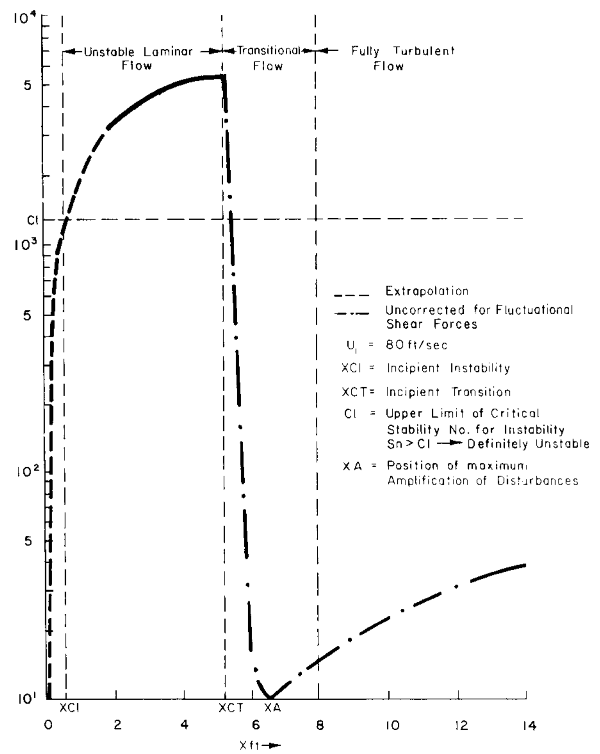

Fig. 8 Distribution of the Stability number in a flat plate boundary layer.

2) The position of incipient laminar instability is given approximately by $Sn_{ci} \doteq \phi(I_1) \times 10^3$ for a length scale $\Delta \equiv \delta^* 10^{-2.34 \tanh(10/H-4)}$. This corresponds to

$$Re_{\delta^*_{ci}} \doteq \{\phi^2(I_1) - 0.027(Ra_y + Ta + Ay)\}^{1/2}\xi/(1 + v_0/U_1)$$

In all the arguments in the text, it has been tacitly assumed that the resistive force is the conventional viscous force. Viscous forces are inherent forces in fluids which manifest themselves whenever there is shear, that is, a velocity differential among the fluid particles. The viscous forces conventionally considered are those due to streamwise and transverse mean shear. Viscous forces due to fluctuational shear have not been considered in this text. Thus, the formulations for the dimensionless force quantities given previously are valid only for laminar flows where the fluctuational shear forces are negligible. For turbulent flows, one must reformulate Sn_x and Sn_y to include the fluctuational shear forces. Figure 8 shows the distribution of the Stability number, as defined previously, in a flat plate boundary layer.

All the preliminary results obtained using the deductions from the force field theory are excellent predictions of real flow characteristics. The simplicity of these results suggest more fervent pursuance of the methods of the force field theory.

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