EFFECTIVE-VISCOSITY MODEL FOR TURBULENT WALL BOUNDARY LAYERS $^{\Delta}$

by

R. N. Meroney *

J. C. Anyiwo * *

Fluid Dynamics and Diffusion Laboratory
College of Engineering
Colorado State University
Fort Collins, Colorado

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* Associate Professor, College of Engineering
Colorado State University
Fort Collins, Colorado

** Graduate Research Assistant
Colorado State University
(Formerly Cambridge University, U. K.)

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ABSTRACT

Existing effective viscosity models which have been very valuable in the mean field closure method for turbulent boundary layer computation have shown certain undesirable limitations for certain realistic but general boundary layer flows. The more general flows usually involve non-negligible considerations of pressure gradients and such wall conditions as roughness, curvature and aspiration or transpiration in varying degrees of importance. The effects of these external and wall influences have, unfortunately, been underplayed by most existing effective viscosity models.

The present model of the effective viscosity is developed for a general flow and has shown remarkable agreement with experimentation, without being any more complex than existing models.

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1. INTRODUCTION

The use of an effective viscosity assumption provides closure for the turbulent boundary layer equations of motion for enhanced mathematical treatment. Boussinesq introduced and assumed a linear relation, of the same form as the Newtonian Law of friction, between the Reynolds shear stress and the mean velocity gradient. The coefficient in this linear relation has since become known as eddy-viscosity. Although the concept of the eddy-viscosity, and worse still, a constant eddy-viscosity, is physically unsound, it nevertheless allows reliable and useful predictions to be made for most types of flows. One is thus compelled to ignore the lack of physical rigor.

An effective viscosity may therefore be defined as the sum of the kinematic molecular viscosity, and the eddy viscosity.

i.e. $$\nu_e = \nu + \varepsilon \qquad 1.1$$

If, further, it is assumed that the effect of wall roughness is very similar to the effect of the molecular viscosity on the flow, so that an apparent kinematic viscosity , ν_a , may be defined, then:

$$\nu_e = \nu_a + \varepsilon \qquad 1.2$$

The form of equation (1.2) may then be used to reduce the turbulent boundary layer momentum equations to a form similar to the laminar case, which is closed and soluble by some mathematical techniques. The form of the eddy-viscosity is difficult to establish especially near the wall.

Many models of the eddy-viscosity have been introduced for the wall region of the boundary layer. Reichardt[5], for instance, assumed the following:

$$\frac{\varepsilon}{\nu} = \kappa\,(y_+ - \delta_1^+ \tanh \frac{y^+}{\delta_\ell^+}) \qquad 1.3$$

on the basis that in the boundary layer, the eddy-viscosity, ε , increases with y^{+3} as $y^{+} \to 0$, and changes monotonically into a linear function of y^{+} as the region beyond the overlap region is approached, Deissler[6] tried to take account of a turbulence diminishing toward the wall and suggested the relation:

$$\frac{\varepsilon}{\nu} = au^{+}y^{+} \; [\; 1 - \exp\; (\; - au^{+}y^{+}\;)\;] \qquad 1.4$$

Where, a , is a numerical constant. This relation gives an eddy viscosity which is proportional to y^{+4} in the inner wall region. Van Driest[7], on the basis of a modified Prandtl's mixing length, suggested the following form

$$\varepsilon = \kappa^2 y^2 \; [\; 1 - \exp\; (\; - y^{+}/A\;)\;]^2 \; \left| \frac{\partial u}{\partial y} \right| \qquad 1.5$$

where $A = 26$ for the zero pressure gradient flow along a smooth wall. This relation also approaches the wall as y^{+4}. The more familiar recent additions to the eddy-viscosity models for the inner wall region are the models due to G. Mellor and A.M.O. Smith and their colleagues. Mellor et al.[3,14] argued that in the wall region, the eddy-viscosity is a universal function of y, $\frac{\partial u}{\partial y}$, and the molecular viscosity, ν , which can be represented by the functional curve

$$\frac{\varepsilon}{\nu} = \chi^4/(\chi^3 + \bar{A}^3) \qquad 1.6$$

where $\chi = \frac{\kappa y}{\nu}\sqrt{\frac{\tau}{\rho}}$, and $\bar{A}$ is a constant (= 6.9). The form of equation (1.6) is as yet the most satisfactory form of the eddy-viscosity introduced for the wall region, and except in severe cases, it is only very slightly affected by pressure gradients and wall conditions.

Smith et al.,[1,2] modified the Van-Driest model in the inner wall region obtaining an analytical relation for the general variation of the turbulence function A.

$$\varepsilon = \kappa^2 y^2 \left[1 - \exp\left(-y^+/A_+\right) \right]^2 \left| \frac{\partial u}{\partial y} \right| \qquad 1.7$$

$$\text{where} \quad A_+ = 26 \left\{ -\frac{FU_1^{+3}}{V_o^+} \left[\exp\left(11 \cdot 8 V_o^+\right) - 1 \cdot \right] + \exp\left(11 \cdot 8 V_o^+\right) \right\}^{-1/2}$$

In the outer region of the boundary layer, not many changes have been made to Clauser's original assumption of a constant eddy viscosity, ($\frac{\varepsilon}{U_1 \delta} = 0.018$). Mellor et al.,[14] assumed a constant value of $\frac{\varepsilon}{U_1 \delta^*} = 0.016$, which they said could be modified by the curvature of the wall in the following manner:

$$\frac{\varepsilon}{U_1 \delta^*} = 0.016 \left[1 - A^* \frac{\delta^*}{r_o} \right] \qquad 1.8$$

where A^* is a numerical constant.

Smith et al.,[2] modified their outer region constant eddy viscosity with an intermittency factor and thus obtained:

$$\frac{\varepsilon}{U_1 \delta^*} = 0.0168 \left[1 + 5.5 \eta^6 \right]^{-1} \qquad 1.9$$

Most other researchers in this field have simply chosen a constant in the neighborhood of 0.018 which satisfied their particular kind of flow.

In the overlap region of the turbulent boundary layer, Smith et al.[2] applied the constraint of continuity of the eddy-viscosity from the wall outward. Their wall eddy viscosity function however approximates a linear function of y^+ in this region. Mellor et al.[14] assumed that the eddy viscosity in the overlap region must be some universal function of y , $\frac{\partial u}{\partial y}$, and $U_1\delta^*$ (chosen as a suitable scaling parameter), and obtained the following linear function:

$$\frac{\varepsilon}{U_1\delta^*} = \chi / R_{\delta^*} \qquad 1.10$$

Earlier, Rotta[8] had assumed that the mixing length at $y = 0$ is not necessarily zero (especially in rough-wall flows) but has some finite value ℓ_o . Hence the applicable mixing length in the transition region should be $\ell = \ell_o + \kappa y$.

$$\text{i.e. } \frac{\varepsilon}{U_1\delta^*} = (\ell_o + \kappa y)^2 \left| \frac{\partial u}{\partial y} \right| / U_1\delta^* \qquad 1.11$$

Most of the models introduced here have been obtained essentially on the basis of wide experience with turbulent boundary layers, leading to intuitive guesses that satisfy some boundary constraints and approximate turbulence characteristics on a large scale. In fact, all the wall eddy-viscosity models, including the one presented herein, are curve fits scaled on suitable wall region parameters. Fortunately the wall region boundary layer is virtually insensitive to external influences, so that curves for the eddy-viscosity obtained in that region give the illusion of a universal character.

In the overlap region of the turbulent boundary layer, the dimensionless eddy-viscosity $(\varepsilon/U_1\delta^*)$ appears to be a linear function of y^+ as assumed by all the models. The influence of external and wall conditions on the eddy-viscosity in this region is only quantitative, linearity still being maintained, as the overlap region is virtually a constant shear stress region.

All the models except that of Smith et al.,[1,2] assumed a constant value for the non-dimensional eddy-viscosity in the outer region of the boundary layer. Clearly the eddy-viscosity is not a constant in this region but decreases far from the wall across the zone of intermittency. When corrected for intermittency, a sizeable portion of the outer boundary layer shows a fairly constant value for $\varepsilon/U_1\delta^*$. Hence the assumption of a constant outer region eddy viscosity will not be expected to have a large influence on predictions of velocity profiles except in the outermost region where, anyway, the vertical gradient, $\partial u/\partial y$, of the velocity is usually very small. In predicting shear stress distributions; in diffusion problems and generally in problems involving large gradients of mean quantities as in the atmosphere, the use of a constant outer eddy viscosity should be noticeably erroneous. Moreover, all experimental data indicate that the outer region eddy viscosity is not only intermittency controlled but is also quantitatively dependent on streamwise pressure gradients and wall condition. The reader is referred to the data of Moffat _et al._,[11] and Fraser[10]. This is obvious also from an examination of the boundary layer momentum equations for the outer region. Only in the simple case of fully developed channel and pipe flows can a constant eddy-viscosity be compatible in the outer boundary layer.

2. THE EFFECTIVE VISCOSITY IN WALL BOUNDARY LAYERS

Having reviewed the developments in boundary layer eddy viscosity modelling, we will now try to establish a satisfactory functional form for the effective viscosity in the wall boundary layer. The effective viscosity was defined as:

$$\nu_e = \nu_a + \varepsilon \tag{2.1}$$

where, ν_a , is the apparent kinematic molecular viscosity due to wall roughness.

It is clear that what we are trying to do by defining, ν_a , is to increase the effective viscosity close to the rough wall. In that case, we can alternatively effect all such changes via the eddy viscosity. If the roughness geometry is such that an equivalent Nikuradse[12] sand roughness height, k_s , can be meaningfully defined (i.e., flow effects and turbulence mechanism are preserved, under the transformation) then we may define an augmented dimensionless vertical distance.

$$\chi = \frac{\kappa}{\nu}(y + k_s)\sqrt{\tau/\rho} \tag{2.2}$$

Certainly, if we cannot mathematically model the roughness geometry by some transformation which preserves turbulence mechanism, we cannot hope to obtain a mathematical model of the roughness influence on an effective viscosity. Perry and Joubert[13] discuss an alternative approach to modelling the apparent viscosity.

In the wall region of the boundary layer, perhaps the best way to look at the problem of the eddy-viscosity form is a method used by Meroney[15]. Meroney obtained an expression for the Reynolds shear stress distribution near the wall, by a Taylor series expansion of the total flow velocity. If such an expansion is repeated for the wall boundary layer as defined in the present study, one finds that:

$$\overline{u'v'} = \left[4\nu u_4 - \left(\frac{v_o}{\nu}\right)^2 \cdot \frac{1}{3!}\left(v_o u_1 - U_1 \frac{dU_1}{dx}\right)\right] y^3 + [5\nu u_5 - v_o u_4] y^4 + \ldots \qquad 2.3$$

where $u_i = \frac{1}{i!} \frac{\partial^i u}{\partial y^i} \Big|_{y=o}$

From (2.3) it is seen that:

$$\varepsilon = \left[\frac{1}{6}\left(\frac{v_o}{\nu}\right)^2 \left(v_o - \frac{U_1}{u_1}\frac{dU_1}{dx}\right) - 4\nu \frac{u_4}{u_1}\right] y^3 + \left[v_o \frac{u_4}{u_1} - 5\nu \frac{u_5}{u_1}\right] y^4 + \ldots \qquad 2.4$$

It will now be assumed that at the wall where the Taylor series expansion is valid y^5 is very small and may be dropped from equation (2.4), or in non-dimensional form,

$$\frac{\varepsilon}{U_1 \delta^*} = \phi_w = \left[\frac{v_o^{+2}}{6\kappa^3}\left(v_o^+ - FU_1^{+3}\right) - 4\, Z\right] \frac{\chi^3}{R_{\delta^*}} + \frac{v_o^+ Z}{\kappa R_{\delta^*}} \chi^4 \qquad 2.5$$

where

$$Z = \frac{\nu^4}{U_\tau^5 \kappa^3} \left(\frac{\partial^4 u}{\partial y^4}\right)_{y=o} \quad \text{and} \quad F = \frac{\nu}{U_1^2} \frac{dU_1}{dx} \qquad 2.6$$

From equation (2.5) it becomes clear that the wall conditions have a direct influence on the form of the eddy-viscosity in the wall region. In the absence of wall effects, the eddy-viscosity at the wall is proportional to y^{+3}, but is otherwise proportional to y^{+n}, $(3<n<4)$. In the transition region of the turbulent boundary layer, the eddy viscosity is directly proportional to χ and may be written as

$$\frac{\varepsilon}{U_1 \delta^*} = \phi_t = \frac{g}{R_{\delta^*}} \chi \qquad 2.7$$

where, g, is a simple functional constant for any particular flow through which pressure gradients and wall conditions quantitatively affect ϕ_t.

Except for the functions, Z and g in equations (2.6) and (2.7), we have now obtained the functional relations for the eddy viscosity at the wall and in the overlap region of the turbulent boundary layer. Since the eddy viscosity function must be continuous from the wall outward, we will now fit an empirical curve to the two relations represented by equations (2.5) and (2.7). On the basis of the power law behavior observed in equation (2.5) and (2.7) we postulate a curve of the form

$$\phi_{w\tau} = \left[\frac{\lambda_1 \chi^4}{(A^3 + \chi^3)} + \frac{\lambda_2 \chi^3}{(A^2 + \chi^2)} \right] \cdot \frac{1}{R_{\delta *}} \qquad 2.8$$

where

$$\lambda_1 = v_o^+ \left[g - A^2 v_o^{+2} \; (v_o^+ - FU_1^{+3}) / \kappa \right] / (v_o^+ - 4\kappa/A) \qquad 2.9$$

$$\lambda_2 = \left[A^2 v_o^{+2} \; (v_o^+ - FU_1^{+3}) - \frac{(4g\kappa/A - 4Av_o^{+2} \; (v_o^+ - FU_1^{+3}))}{(v_o^+ - 4\kappa/A)} \right]$$

For the curve of equation (2.8), the function, g , varies with y by about 0.5 about the value unity, such that we can empirically approximate g by

$$g = 1 + 0.031 \times 10^{-3} \; (v_o^+ \; \chi^3) \qquad 2.10$$

For this value of g , the value of A which consistently gives the best fit for a number of experimental results is A = 7.2. Herring and Mellon used a value of A = 6.9 in the absence of additional corrections on g . With these estimates, the wall region effective viscosity for the general turbulent wall boundary layer is satisfactorily described by

$$\frac{\partial}{\partial y}\left[\frac{u}{(1-cy)}\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left(V\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{cuV}{1-cy}\right) - \frac{\partial}{\partial x}\left(\frac{cu^2}{1-cy}\right)$$
$$= \frac{-1}{\rho}\frac{\partial p}{\partial x}\cdot\frac{\partial}{\partial y}\left(\frac{1}{1-cy}\right) + \frac{\partial^2}{\partial y^2}\left[\nu\frac{\partial u}{\partial y} - \overline{u'v'}\right] + \frac{\partial}{\partial y}\left(\frac{c\overline{u'v'}}{1-cy}\right) \qquad 2.13$$

where $c(=1/r_o)$ is the wall curvature, and is positive for concave curvature. In the absence of pressure gradients and wall effects, the outer region eddy viscosity is adequately represented by

$$\frac{\varepsilon}{U_1\delta^*} = 0.016 \text{ x An Intermittency factor.}$$

In that case, if we assumed a suitable power-law velocity profile, Eq. (2.13) can be analyzed further to yield the following outer region effective viscosity profile (See Appendix for analytical details):

$$\phi_o = \frac{\nu_e}{U_1\delta^*} = 0.016\ \gamma\left(1+\eta\frac{a\delta^*}{r_o}\right)\left[1+\left(b\frac{V_o}{U_1} - dFR_{\delta^*} + m\frac{\delta^*}{r_o} - h\delta^*\frac{d}{d_x}(1/r_o)\right)f(\eta)\right] \qquad 2.15$$

where γ = an intermittency factor $(=1/(1+5\cdot5\eta^6))$ and a , b , d , m and h are numerical constants.

For the cases of interest, the streamwise gradient of curvature is usually very small compared to the remaining terms in Eq. (2.15). Hence we finally obtain by comparison with experimental data of Fraser[10], that:

$$\phi_o = 0.016\ \gamma\left(1+a\eta\frac{\delta^*}{r_o}\right)\left[1+\left(2\cdot8\frac{V_o}{U_1} - FR_{\delta^*} + m\frac{\delta^*}{r_o}\right)f(\eta)\right] \qquad 2.16$$

where $f(\eta) \simeq 500\sinh(\eta)$, and $a \simeq 12\cdot1$.

$$\nu_e = \frac{1}{R_{\delta*}} \left\{ \frac{\lambda_1 \chi^4}{[7\cdot 2^3 + \chi^3]} + \frac{\lambda_2 \chi^3}{[7\cdot 2^2 + \chi^2]} + 1 \right\} \qquad 2.11$$

where

$$\lambda_1 = \left[g - 126 v_o^{+2} \, (v_o^+ - FU_1^{+3}) \right] v_o^+ / (v_o^+ - 0.23)$$

$$\lambda_2 = \left[v_o^{+2} \, (126 v_o^+ - 15\cdot 6) \, (v_o^+ - FU_1^{+3}) - 0.23 g \right] / (v_o^+ - 0.23) \qquad 2.12$$

With reference to the data of Fraser[10], the eddy viscosity profile in the outer region of the turbulent boundary layer shows a distinct trend with the streamwise distance, x , which, however, may be diminished by non-dimensionalizing the eddy diffusity, ε , with $U_1 \delta^*$. A slight trend with x is still noticeable probably because the displacement thickness, δ^* , calculated in the conventional way, does not adequately represent the growth influence of the turbulent boundary layer especially when wall effects and external conditions on the flow are severe. Moreover, the eddy viscosity profile shows definite dependence on the external and wall conditions, i.e., pressure gradients, transpiration, etc., on the flow. Examination of the momentum equation of fluid motion shows that the effects of transpiration, etc., on the eddy viscosity are local, not history-oriented as one would naturally anticipate. The experimental results of Fraser[10] also confirm this.

In the orthogonal system of parallel curves Fig. (6), the vorticity transport equation for two-dimensional fluid motion can be reduced to the following approximate form, for the outer boundary layer:

Interestingly, Eq. (2.16) reduces to a form previously suggested by Mellor *et al.*,[3] (except for an additional δ^*/r_o factor), in the absence of pressure gradient and wall transpiration. The constant, m , has not been evaluated in the present work because adequate curved wall data were not available to the present authors. In the continuing research on analytical boundary layer predictions, the results of a mean turbulent closure method will be regressed to compute the constant, m , and to modify the present model for cases of thermally stratified flows.

3. COMPARISON WITH EXPERIMENTAL RESULTS

Prior to the formulation of the present effective viscosity model, one of the most satisfactory effective viscosity models in turbulent wall boundary layer computation appears to be that due to Herring and Mellor[14]. The forms of the Herring-Mellor model and the present model are compared against the experimental measurements of Moffat et al.,[11] in Fig. 1. The boundary layer studied by Moffat et al., is due to a highly accelerated flow with wall transpiration. It seems obvious that the Herring-Mellor model is most satisfactory only for simple flat-plate flows without mass transfer at the wall. When pressure gradients and wall transpiration may be considered, the Herring-Mellor model under-estimates the viscosity of the flow especially in the inner wall region and the outermost region of the turbulent boundary layer. The present model provides an accurate description of the inner wall region since it is tailored to obey the governing differential equations at the wall, but suffers a negligible deviation in the lower overlap region. The matching point between the inner wall and the overlap solutions fluctuates with the type of flow and is not exactly fixed by the present curve fit. However, apart from this small deviation in the overlap region of the boundary layer, the present model shows a much more satisfactory fit to the experimental results of very general types of flows. Further illustration of the superiority of the present effective viscosity model over previous models is obtained from the direct use of the models in boundary layer computations. The computer program of Herring and Mellor[3] was used for all computations, with modifications only in the effective viscosity models.

The example used is the imcompressible boundary layer flow studied by Moses[16], which was used at the Stanford Symposium on the "Computation of Turbulent Boundary Layers." The first portion of the flow is in an adverse pressure gradient. Subsequently, the pressure gradient is removed and the layer relaxes to conditions of nearly zero pressure gradient. Comparisons with Moses' data are shown in Figs. (2) and (3), where theoretical skin friction, shape factor and mean velocity profiles are plotted against the data points. While the present method gives an almost exact fit to the experimental data points in all cases, the model of Herring and Mellor shows definite deviations (up to 15%) especially in the skin friction distribution and in the outer region of the mean velocity distribution. Figure (4) shows most clearly the difference between the two models. The Herring-Mellor model shows marked deviations in the wall and outermost boundary layer regions.

The present effective viscosity model has also been used for the compressible turbulent boundary layer of Moore and Harkness[17], and has shown very good agreement with experiment, Fig. 7.

4. CONCLUDING REMARKS

We may now conclude that if a scalar eddy viscosity is assumed to be a valid concept in wall turbulent boundary layer theory, then such an eddy viscosity has the following characteristics:

It is a continuous function of y which approaches the wall surface as a power function, y^{+n}, of the vertical distance y from the wall. If the external and the wall conditions on the boundary layer flow are unimportant $n \to 3$, but $n \to 4$ as the effects of the flow environment become important.

Away from the wall the eddy viscosity function is a linear function of y, soon reaching a maximum and then gradually decreasing to zero very far from the wall. Even in the region away from the wall, the eddy viscosity function is at least quantitatively dependent on the conditions at the wall and in the free stream.

For moderate roughness, it appears that when an equivalent Nikuradse sand roughness height can be meaningfully defined, the influence of roughness can adequately be modelled as an apparent y - shift effect.

Finally, a mean velocity field closure method which predicts the turbulent boundary layer downstream of a known station should yield very close predictions, as the iterative scheme of such a method usually corrects slight errors in the effective viscosity assumption. Moreover, the dimensionless effective viscosity appears to be a function of the local boundary layer parameters and is history-oriented only through such parameters.

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(i) Terms in $(\frac{\delta^*}{r_o})^2$ and higher order are negligible.

(ii) Terms in $(\frac{dc}{dx})$ are small in comparison to other terms in the equations.

equation (1) becomes:

$$\frac{1}{\gamma}\phi_o\left[1+\frac{\delta^*}{r_o}\frac{u_2}{u_3}\right] = \frac{\delta^*}{u_3}\frac{\partial}{\partial\eta^*}(\bar{u}\frac{\partial\bar{u}}{\partial x}) + \frac{\delta^*}{r_o}\cdot\frac{\delta^*}{u_3}\frac{\partial}{\partial\eta^*}(\bar{u}\eta^*\frac{\partial\bar{u}}{\partial x})$$

$$+ FR_{\delta^*}\left[2\frac{u_1}{u_3}\bar{u} + \frac{\delta^*}{u_3 r_o}\frac{\partial}{\partial\eta^*}(\bar{u}^2\eta^*) - 2\frac{\bar{u}^2}{u_3}\frac{\delta^*}{r_o}\right] + \frac{1}{u_3}\frac{\partial}{\partial\eta^*}(\bar{v}\frac{\partial\bar{u}}{\partial\eta^*}$$

$$- \frac{\delta^*}{r_o}\cdot\frac{1}{u_3}\frac{\partial}{\partial\eta^*}(\bar{u}\,\bar{v}) - \frac{\delta^*}{r_o}\cdot\frac{\delta^*}{u_3}\cdot\frac{\partial\bar{u}^2}{\partial x} - \frac{\bar{u}^2\delta^{*2}}{u_3}\cdot\frac{dc}{dx} \qquad (3)$$

As $y\to\infty$, $\frac{\partial u}{\partial x}\to\frac{dU_1}{dx}$ so that we can write $\frac{\partial u}{\partial x} = f(y)\frac{dU_1}{dx}$ such that $f(y)\to 1$ as $y\to\infty$. The continuity equation is

$$\frac{1}{(1-cy)}\frac{\partial u}{\partial x} - \frac{cy}{(1-cy)} = -\frac{\partial v}{\partial y} \qquad (4)$$

whence we can obtain that:

$$\frac{\partial\bar{v}}{\partial\eta^*} \doteq \frac{\delta^*}{r_o}\bar{u} - (1+\eta^*\frac{\delta^*}{r_o})\ f(\eta^*)\ FR_{\delta^*}$$

and

$$\bar{v} = \bar{v}_o + \frac{\delta^*}{r_o}\int_o^{\eta^*}\bar{u}d\eta^* - FR_{\delta^*}\int_o^{\eta^*}(1+\frac{\delta^*}{r_o}\eta^*)\ f(\eta^*)d\eta^* \qquad (5)$$

We shall, further, make the following simplifications:

(i) $\bar{u} \doteq (A\eta^*)^{1/n}$

(ii) In the absence of severe wall and free stream influences, the effective viscosity ϕ_o is satisfactorily approximated by

$$\phi_o = 0.016\gamma = \frac{\delta^*}{u_3}\frac{\partial}{\partial\eta^*}(\bar{u}\frac{\partial\bar{u}}{\partial x}) \qquad \text{(from equation (3))}.$$

APPENDIX

The combined momentum equation (2.13), which is essentially the so-called vorticity transport equation is

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}\cdot\frac{\partial}{\partial y}\left(\frac{1}{1-cy}\right)+\frac{\partial^2}{\partial y^2}\left[\nu\frac{\partial u}{\partial y}-\overline{u'v'}\right] \tag{1}$$

$$+\frac{\partial}{\partial y}\left(\frac{c\overline{u'v'}}{1-cy}\right)$$

$$=\frac{\partial}{\partial y}\left[\frac{u}{(1-cy)}\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{cuv}{1-cy}\right)-\frac{\partial}{\partial x}\left(\frac{cu^2}{1-cy}\right)$$

Let $u = \overline{u}\,\overline{U}$,

$v = \overline{v}\,\overline{U}$,

$y = \eta^*\delta^*$

If we assume that after correction for intermittency, the dimensionless effective viscosity, $\nu_e/U_,\delta^*$, is fairly constant across a large portion of the outer region boundary layer, the left hand side of equation (1) becomes:

$$\frac{1}{\gamma}\frac{U_1^{\,2}}{\delta^{*2}}\cdot\frac{\nu_e}{U_1\delta^*}\left[\frac{\partial^3\overline{u}}{\partial\eta^{*3}}+\frac{\delta^*}{r_o}\cdot\frac{\partial^2\overline{u}}{\partial\eta^{*2}}\right] \tag{2}$$

where $\frac{1}{\rho}\frac{\partial p}{\partial x}\cdot\frac{\partial}{\partial y}\left(\frac{1}{1-cy}\right)$ has been neglected in comparison with other terms in equation (1).

If $\frac{\nu_e}{U_1\delta^*} = \phi_o$, $\frac{\partial^i\overline{u}}{\partial\eta^{*i}} = u_i$, and if:

$$F = \frac{\nu}{U_1^{\,2}}\frac{dU_1}{dx} \quad\text{and}\quad R_{\delta^*} = \frac{U_1\delta^*}{\nu}$$

Then, with the following assumptions:

With these simplifications, equation (3) reduces to the following:

$$\frac{1}{\gamma}\,\phi_o\left[1 + \frac{\delta^*}{r_o}\,\eta^*\,\frac{1}{(1/n-2)}\right] = 0.016 + 0.016\eta^*\,\frac{\delta^*}{r_o}$$

$$+ \bar{v}_o\,\frac{\eta^*}{(1/n-2)}\left[1 - \frac{\delta^*}{r_o}\,\frac{\eta^*}{(1/n-1)}\right] + FR_{\delta^*}\left[1 - \eta^*\,\frac{\delta^*}{r_o}\right]G'(\eta^*)$$

$$+ \frac{\delta^*}{r_o}\,H'(\eta^*) \qquad (6)$$

i.e.,

$$\phi_o = 0.016\gamma\left[1 - \frac{(3-1/n)}{(1/n-2)}\,\eta^*\,\frac{\delta^*}{r_o}\right]\left[1 + \bar{v}_o Q'(\eta^*) + FR_{\delta^*}G(\eta^*) + \frac{\delta^*}{r_o}\,H(\eta^*)\right] \qquad (7)$$

Let $\bar{v}_o Q'(\eta^*) + FR_{\delta^*}G(\eta^*) + \frac{\delta^*}{r_o}\,H(\eta^*) \equiv (b\bar{V}_o - dFR_{\delta^*} + m\,\frac{\delta^*}{r_o})\,f(\eta^*)$

and $n \approx 7$

where b , d , m are numerical constants. Then

$$\phi_o \doteq 0.016\gamma\left[1 + 12\cdot 1\eta\,\frac{\delta^*}{r_o}\right]\left[1 + (b\bar{V}_o - dFR_{\delta^*} + m\,\frac{\delta^*}{r_o})\,f\,(\eta)\right]. \qquad (8)$$

The constants b, d, m and the function $f(\eta)$ can be determined by comparing equation (8) with experimental results. The effect of (FR_{δ^*}) in equation (8) is so small that only for very severe pressure gradients (corresponding to very high β in Reference (3)) is that term of any importance. In that case, d is taken as unity.

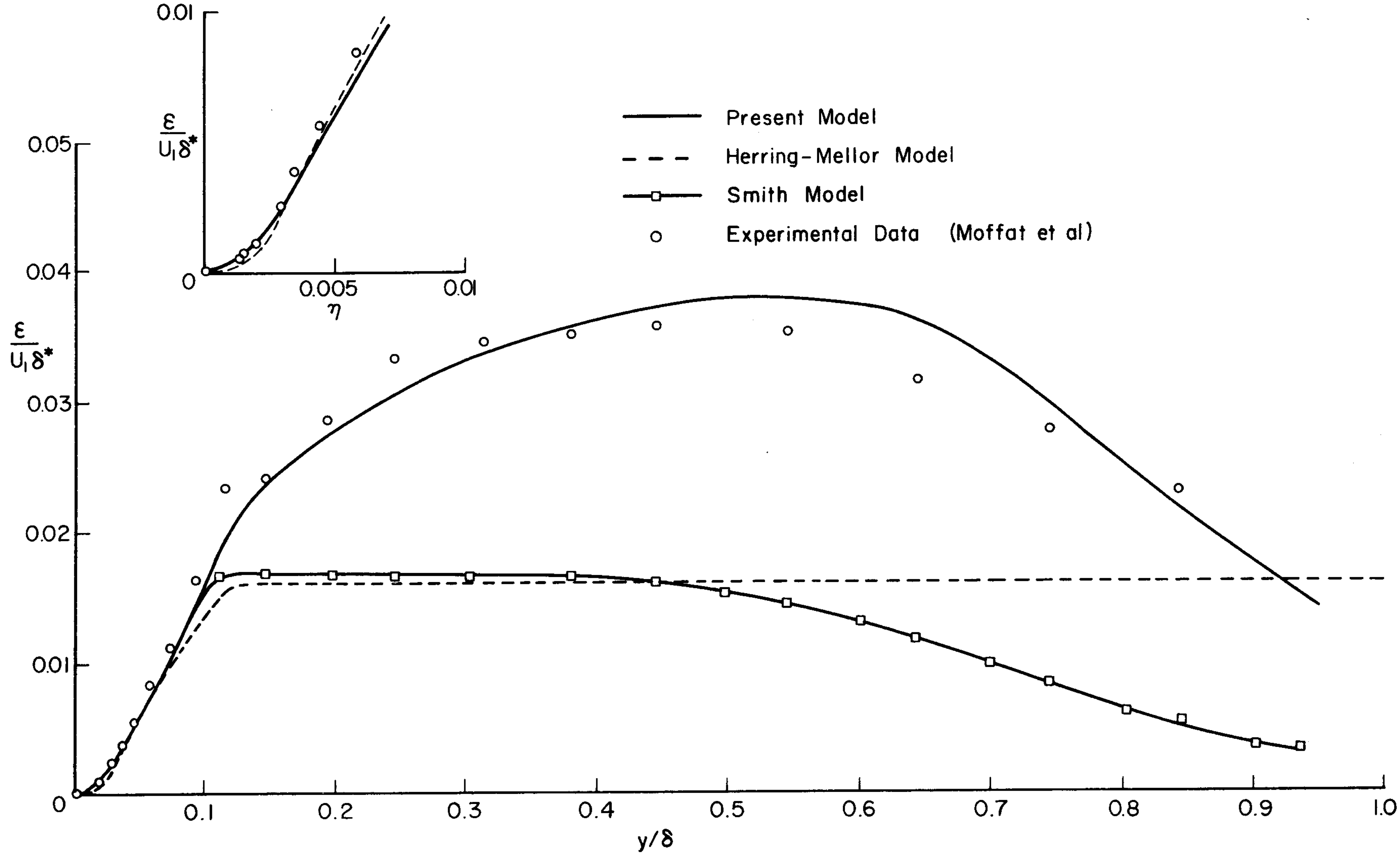

Fig. I - Some Eddy Viscosity Models

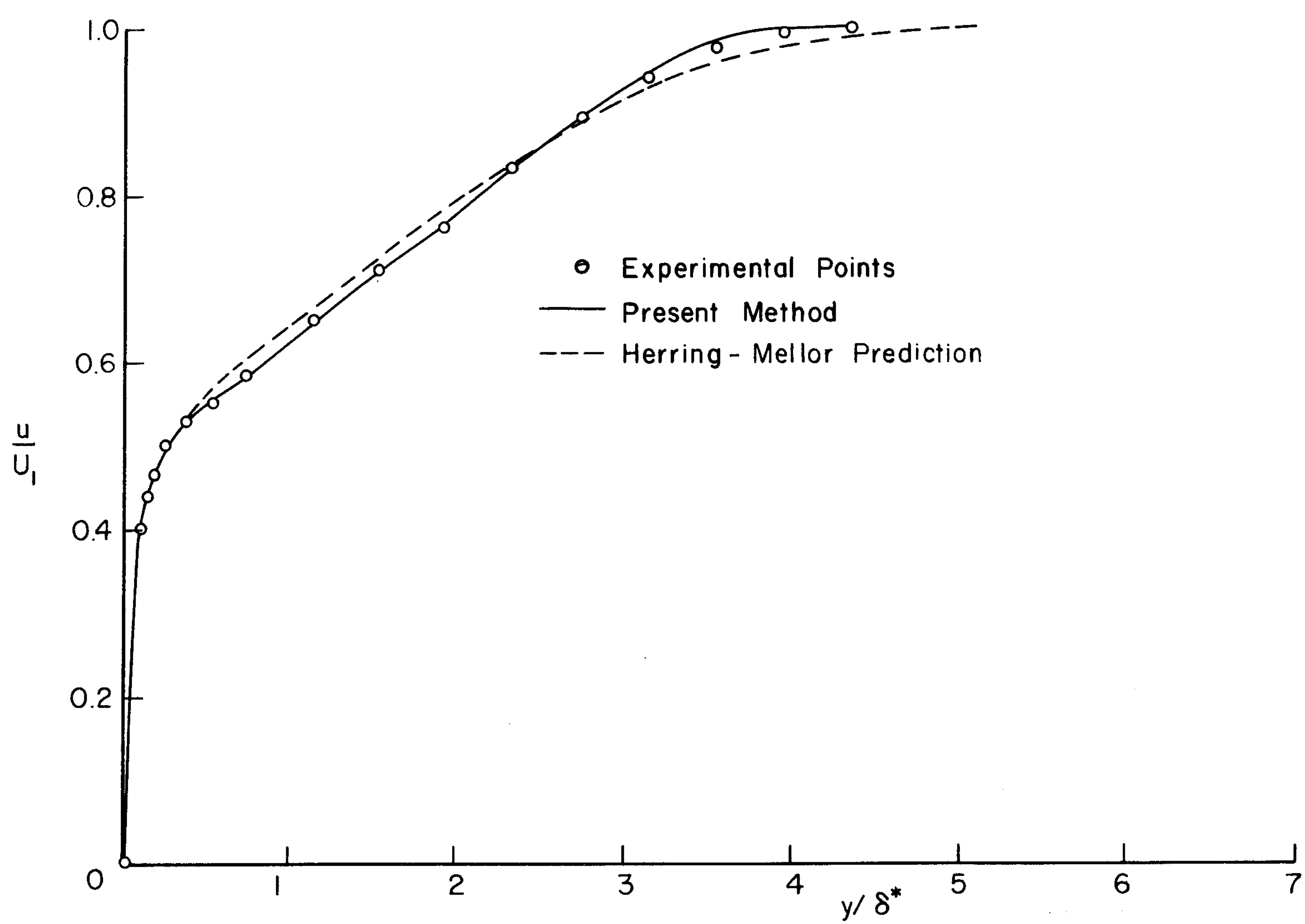

Fig. 3 Moses Flow Case 6 (Velocity Profile)

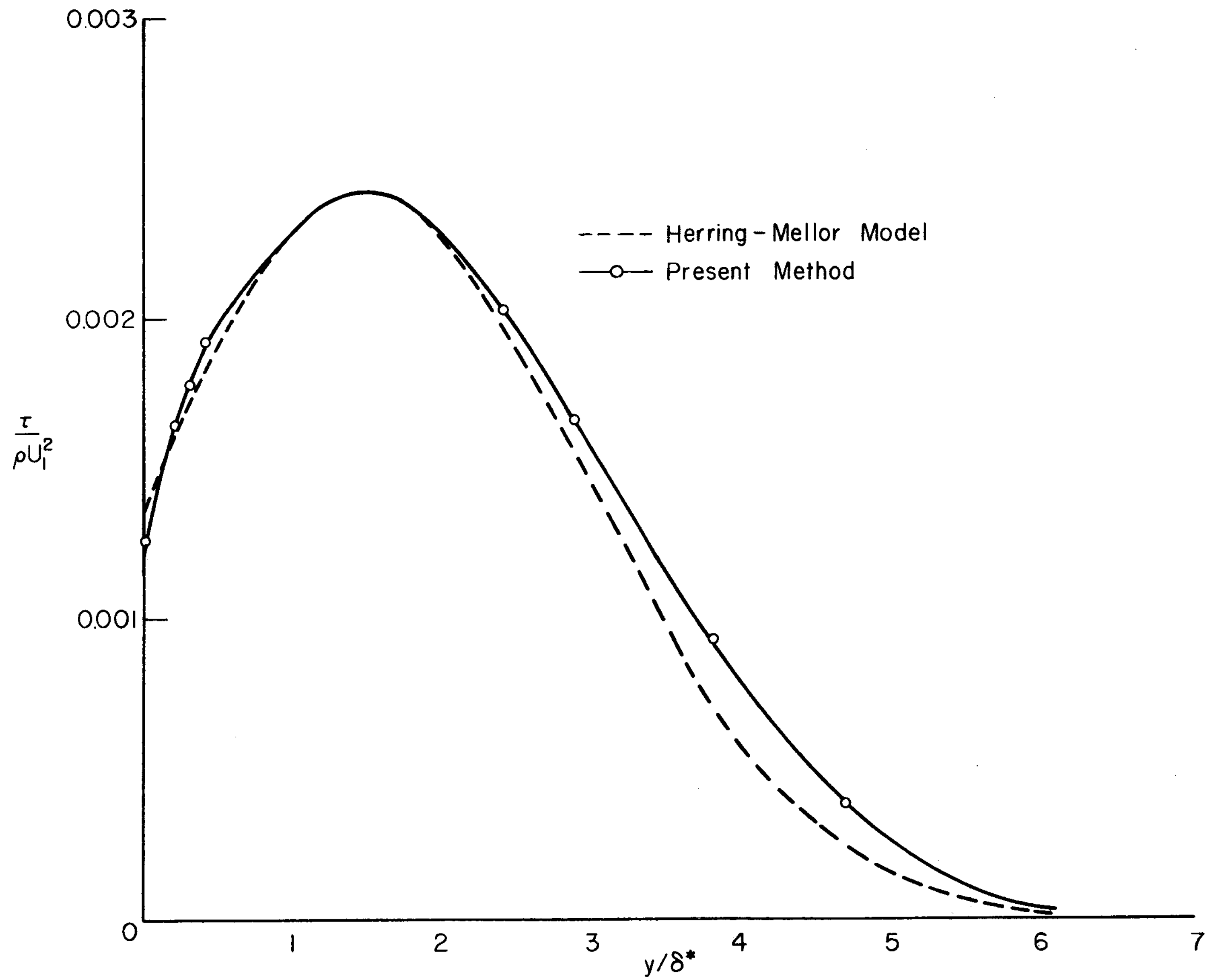

Fig. 4 Moses Flow Case 6 (Shear Stress Curves)

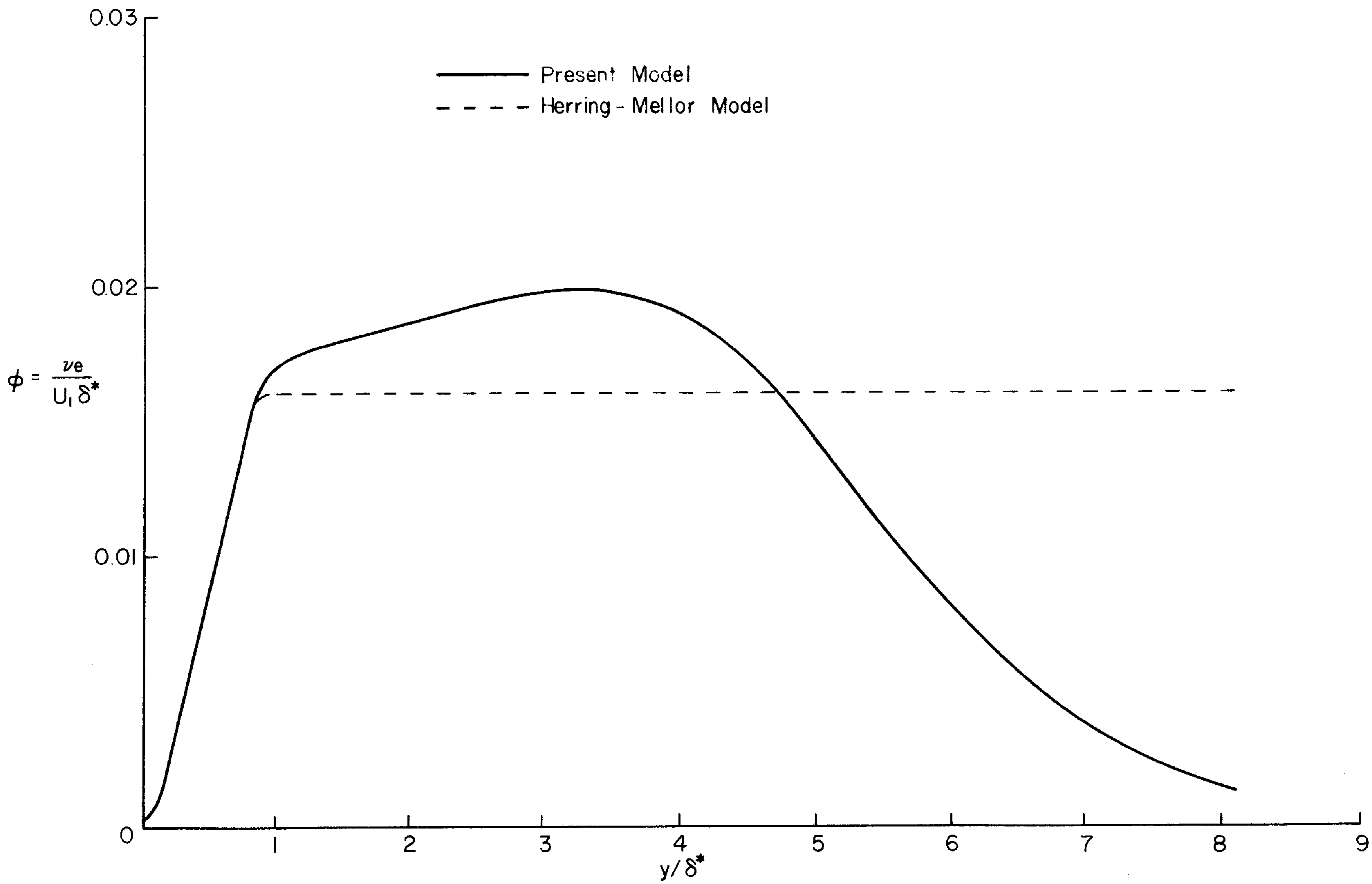

Fig. 5 Effective Viscosity for Moses Flow, Case 6

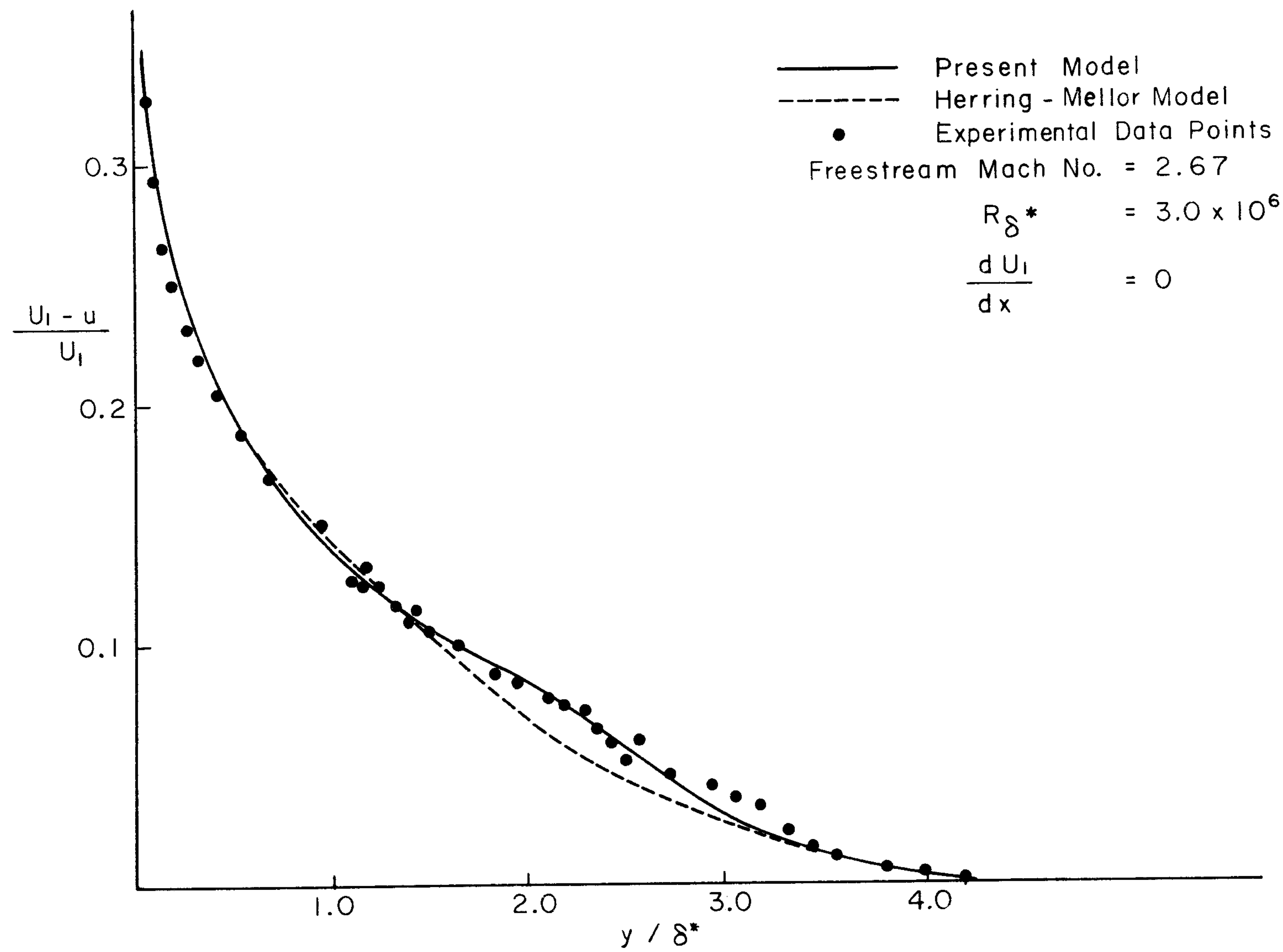

Fig. 7 Velocity Profile : Moore and Harkness

NOMENCLATURE

A, A_+	:	Turbulence functions in Van Driest and Smith eddy viscosity models.
F	:	Pressure gradient parameter $(= \frac{\nu}{U_1^2} \frac{dU_1}{dx})$
g, ϕ	:	Functions.
k, k_s	:	Roughness height; Nikuradse' roughness scale.
ℓ, ℓ_o	:	Mixing length, value at wall.
p	:	Pressure.
r_o	:	Wall radius of curvature.
R	:	Roughness parameter $(= \frac{U_\tau \kappa_s}{\nu})$
$R_{\delta*}$	:	Displacement thickness Reynolds number.
U_1, U_1^+	:	Free stream velocity, U_1/U_τ .
U_τ	:	Friction velocity $(= (\frac{\tau_o}{\rho}))$
u, u^+	:	Streamwise velocity , u/U_τ
v_o, v_o^+	:	Vertical velocity, v_o/U_τ at the wall.
x	:	Streamwise space coordinate.
y, y^+	:	Transverse space coordinate, $\frac{U_\tau y}{\nu}$
δ, δ^+	:	Boundary layer thickness, $\frac{U_\tau \delta}{\nu}$
$\delta*$	:	Displacement thickness
δ_ℓ, δ_ℓ^+	:	Viscous sublayer thickness, $\frac{U_\tau \delta_\ell}{\nu}$
μ	:	Molecular viscosity
ν	:	Kinematic viscosity
ν_e	:	Effective viscosity $(= \nu + \varepsilon)$
ν_a	:	Apparent viscosity in rough wall flows.
ρ	:	Density
γ	:	Intermittency factor
τ	:	Shear stress

η, η^*	:	Transverse dimensionless space coordinate (y/δ), y/δ^*
ε	:	Eddy-viscosity in turbulent flows.
κ	:	Von-Karman constant.

LIST OF FIGURES