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Re-examination of Eulerian-Lagrangian Turbulence Relationships

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Abstract

Eulerian-Lagrangian relationships calculated from Monte-Carlo simulation of particle movements are compared with estimates from analytical approximations based on Lagrangian kinematic relationships. Results of a one-dimensional numerical simulation and a three-dimensional analytical approximation for isotropic homogeneous turbulence are discussed. The Monte-Carlo simulation is mathematically and physically similar to the analytical approximation despite the fact that the numerical simulation is fundamentally incorrect for three-dimensional turbulence.

In a recent paper prepared by Lee and Stone,[1] Monte Carlo techniques were used to predict one-dimensional diffusion in a stationary, homogeneous field of turbulence. An analytical expression to be used to predict Lagrangian statistics from Eulerian statics was also presented. The analytical solution for cloud growth compared favorably with the results from the Monte Carlo simulation, and both results agreed with Lagrangian statistics computed by Baldwin and Johnson's[2] three-dimensional algorithms. It is indeed impressive that a one-dimensional turbulence model approximated cloud dispersion as well as a full three-dimensional model. In this paper we re-examine Lee and Stone's expressions and try to relate the traits of these two models.

The common assumption adopted in both models is that the general Eulerian space-time correlation may be represented as the product of spatial and convective time correlations. Lee and Stone approximated the Lagrangian autocorrelation function at small time increments by the Eulerian space-time correlation, and they assumed that the velocity fluctuations are normally distributed with zero mean and standard deviation σ. They used the expression

$$R_L(\delta t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \exp(-\delta t/T_c) \int_0^\infty n^2 e^{-an} e^{-\frac{n^2}{2}} \, dn, \qquad (1)$$

where $a = \frac{\sigma \, \delta t}{L}$,

$n = N(0,1)$, and

T_c, L are the convective Eulerian time and length scales.

The Lagrangian autocorrelation function at small time δt is thus

$$R_L(\delta t) = \exp(-\delta t/T_c)\ \{\exp(a^2/2)\cdot(1+a^2)(1-\mathrm{erf}(a/\sqrt{2})) - \frac{\sqrt{2}}{\sqrt{\pi}}\ a\} \quad , \qquad (2)$$

where erf signifies an error function.

Since a first order autoregressive process successfully describes the motion of a diffusing air particle, the autocorrelation function must satisfy the following relationship,[3]

$$R_L(t) = (R_L(\delta t))^k, \quad t = k\delta t\ . \qquad (3)$$

Therefore, $R_L(t)$ can be re-expressed as

$$R_L(t) = \exp(-t/T_c)\{\exp(a^2/2)\cdot(1+a^2)(1-\mathrm{erf}(a/\sqrt{2})) - \frac{\sqrt{2}}{\sqrt{\pi}}\ a\}^{t/\delta t} \qquad (4)$$

The Lagrangian integral time scale, T_L, is defined as

$$T_L = \int_0^\infty R_L(t)dt \qquad (5)$$

If we introduce the Eulerian parameter $\alpha = a/\Delta t_*$, which was used by Philip[4] and Baldwin and Johnson[2], then the ratio of Lagrangian to Eulerian integral time scales will be

$$T_L/T_c = \{1- \frac{1}{\Delta t_*}\ \ln[\exp(\alpha^2\Delta t_*^2/2)\cdot(1+\alpha^2\Delta t_*^2)(1-\mathrm{erf}(\ \frac{\alpha\Delta t_*}{\sqrt{2}})) - \frac{\sqrt{2}}{\sqrt{\pi}}\ \alpha\Delta t_*]\}^{-1}\ . \qquad (6)$$

Lee and Stone obtained $R_L(\delta t)$ by expanding the exponentials, $\exp(-\delta t/T_c)$ and $\exp(-an)$, in Equation 1 for small values of $\delta t \to 0$. T_L/T_c was obtained in conjuction with the random force model.[5] Their result is

$$T_L/T_c = (1+\ (\ \frac{8}{\pi}\ )^{\frac{1}{2}}\ \alpha)^{-1} \qquad (7)$$

Equation 6 is a more exact solution to the Monte Carolo simulation, yet the differences found are negligible. Table 1 displays values calculated from Equations 6 and 7 by using the tabulated error function in Abramowitz and Stegun.[6] It is not surprising that Equation 6 successfully predicts results from a Monte Carlo simulation, since it is the natural consequence of a Markov process.

Baldwin and Johnson employed Corrsin's independence hypothesis[7]. They proposed a numerical iterative procedure to estimate the Lagrangian autocorrelation function based on Taylor's[8] integral relationship. The method has been extended to more general turbulence fields including anisotropy and uniform shear effect by Li and Meroney.[9] The general Eulerian space-time correlation is separated into two parts such that

$$R_E(r,\theta,t) = F_1(t/T_c)[(1-\tfrac{1}{2}\left|\tfrac{r}{L}\right|\sin^2\theta)\ \exp(-\left|\tfrac{r}{L}\right|)] \ . \tag{8}$$

The bracket term stands for the general Eulerian spatial correlation function which is obtained from the classical Karman-Howarth[10] relationship by assuming that the f-function is $\exp(-\left|\frac{r}{L}\right|)$. $F_1(t/T_c)$ is the convective autocorrelation function. For comparison with Lee and Stone's analysis, an exponential form of $F_1(t/T_c)$ is adopted such that $F_1(t/T_c) = \exp(-t/T_c)$; hence, $R_L(t)$ becomes

$$R_L(t) = \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} e^{-t/T_c} (2\Pi\sigma^2)^{-\frac{3}{2}} e^{-\left|\frac{r}{L}\right|} (1-\tfrac{1}{2}\left|\tfrac{r}{L}\right| \sin^2\theta)$$

$$\cdot\ e^{-\frac{r^2}{2\sigma^2}} r^2 \sin\theta\ d\phi\ d\theta\ dr \ , \tag{9}$$

and σ^2 is obtained from Taylor's integral relationship,

$$\sigma^2(t) = 2\overline{u^2} \int_0^t \int_0^{t_1} R_L(\tau)\, d\tau\, dt_1 \quad .$$

Adopting the following nomenclature

$$\zeta = \frac{r}{L} \quad ; \quad t_* = t/T_c; \qquad \alpha = \frac{\sqrt{\overline{u2}}\, T_c}{L}$$

one obtains

$$R_L(t_*) = \exp(-t_*)\ \{\ e^{\alpha^2 I}\ (1-\mathrm{erf}\sqrt{\alpha^2 I})(1+4\alpha^2 I + \frac{4}{3}\alpha^4 I^2)$$

$$- \frac{2}{3}\ \frac{\sqrt{\alpha^2 I}}{\sqrt{\pi}}\ (5 + 2\alpha^2 I)\ \}, \tag{10}$$

and

$$I(t_*) = \int_0^{t_*} \int_0^{t_{1*}} R_L(\tau_*)\, d\tau_*\, dt_{1*} \ .$$

T_L/T_c may be computed from Equations 10 and 5 for various α and are also tabulated in Table 1. For a small value of α the estimated T_L/T_c agrees with Monte Carlo simulation predictions. As α increases the deviation becomes appreciable. Lee and Stone warned of the possibility that the one-dimensional model inadequately represents the spatial variation of the correlation function in three-dimensional homogeneous turbulence. Indeed their results agree with these more exact calculations at small α.

Suppose that instead of using Equation 8, we replace it with the simplified expression $R_E(r,\theta,t) = e^{-|t/T_c|}\ e^{-|\frac{r}{L}|}$ used by Lee and Stone.

The Lagrangian autocorrelation function is then

$$R_L(t_*) = e^{-t_*} \{e^{\alpha^2 I} (1+2\alpha^2 I)(1-\mathrm{erf}\sqrt{\alpha^2 I}) - \frac{2}{\sqrt{\pi}} \sqrt{\alpha^2 I} \} , \qquad (11)$$

where $I(t_*)$ retains its earlier definition.

If Δt_* is a small value such that $\Delta t_* \to 0$, $I(t_*)$ will approach its asymptotic value,

$$I(\Delta t_*) = \frac{\Delta t_*^{\,2}}{2} \quad .$$

Substituting this approximation for $I(\Delta t_*)$ into Equation 11 recovers Equation 2 with $a = \alpha \Delta t_*$. Values for T_L/T_c calculated from Equation 11 are also listed in Table 1. The results are significantly larger than estimates from Equation 4 and Equation 10. Deviations between estimates from Equation 10 and Equation 11 solely result from the simplification introduced for the general Eulerian spatial correlation function. T_L/T_c is overestimated, when one assumes that the f-function is the same as the g function in an isotropic homogeneous turbulence field.

Differences between estimates from Equation 4 and Equation 11 are due to the different approaches employed. By virtue of Lagrangian kinematics of a fluid particle, the independence hypothesis is applicable only when t is so large that there is no relation between the fluctuating velocity and the particle position. Equation 10 represents results based on such theory.

Equation 2 may be interpreted in terms of the independence hypothesis and Equation 11 by ignoring the influence of fluctuating velocities on particle position at small times. For small δt, (see

Figure 1), particles released at $t=0$ are most likely to arrive at $x = U\delta t$, but they will actually scatter about $x = U\delta t$. When one assumes $R_L(2\delta t) = R_L^2(\delta t)$, the independence hypothesis is essentially utilized twice. Hence, the assumption neglects the contribution to the autocorrelation from those fluid particles which scatter about $x = U\delta t$. At large t, Equation 4 underestimates the Lagrangian autocorrelation as compared to Equation 11. Table 1 shows that T_L/T_c values derived from Equation 11 are larger than time ratios calculated from Equation 6. The resultant overestimations and underestimations tend to compensate; hence, the expressions result in close agreement between Lee and Stone's one-dimensional analysis and Baldwin and Johnson's three-dimensional approach.

The discussions presented in this paper attempt to physically visualize the intrinsic differences between Monte Carlo simulations and methods based on Corrsins'indepdence hypothesis. The Monte Carlo simulations reproduce the results of the independence approximation at small time, but are based on principles which contradict the basic assumptions of the independence approximation, and the one-dimensional Markov process does not reproduce the expected variation of spatial correlation required for three-dimensional isotropic homogeneous turbulence. The estimated Lagrangian-Eulerian time scale ratio calculated from either approach varys less than the magnitudes observed during atmospheric dispersion experiments.

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Table 1. The estimated time scale ratio.

	α	0.1	0.3	0.5	1.0	1.5	2.0	4.0
$\frac{T_L}{T_c}$	Eqn. 6	0.8624	0.6764	0.5564	0.3856	0.2977	0.2391	0.1378
	Eqn. 7	0.8624	0.6763	0.5562	0.3852	0.2947	0.2386	0.1354
	Eqn. 10	0.8635	0.6837	0.5702	0.4087	0.3216	0.2664	0.1612
	Eqn. 11	0.8946	0.7439	0.6417	0.4853	0.3949	0.3350	0.2140

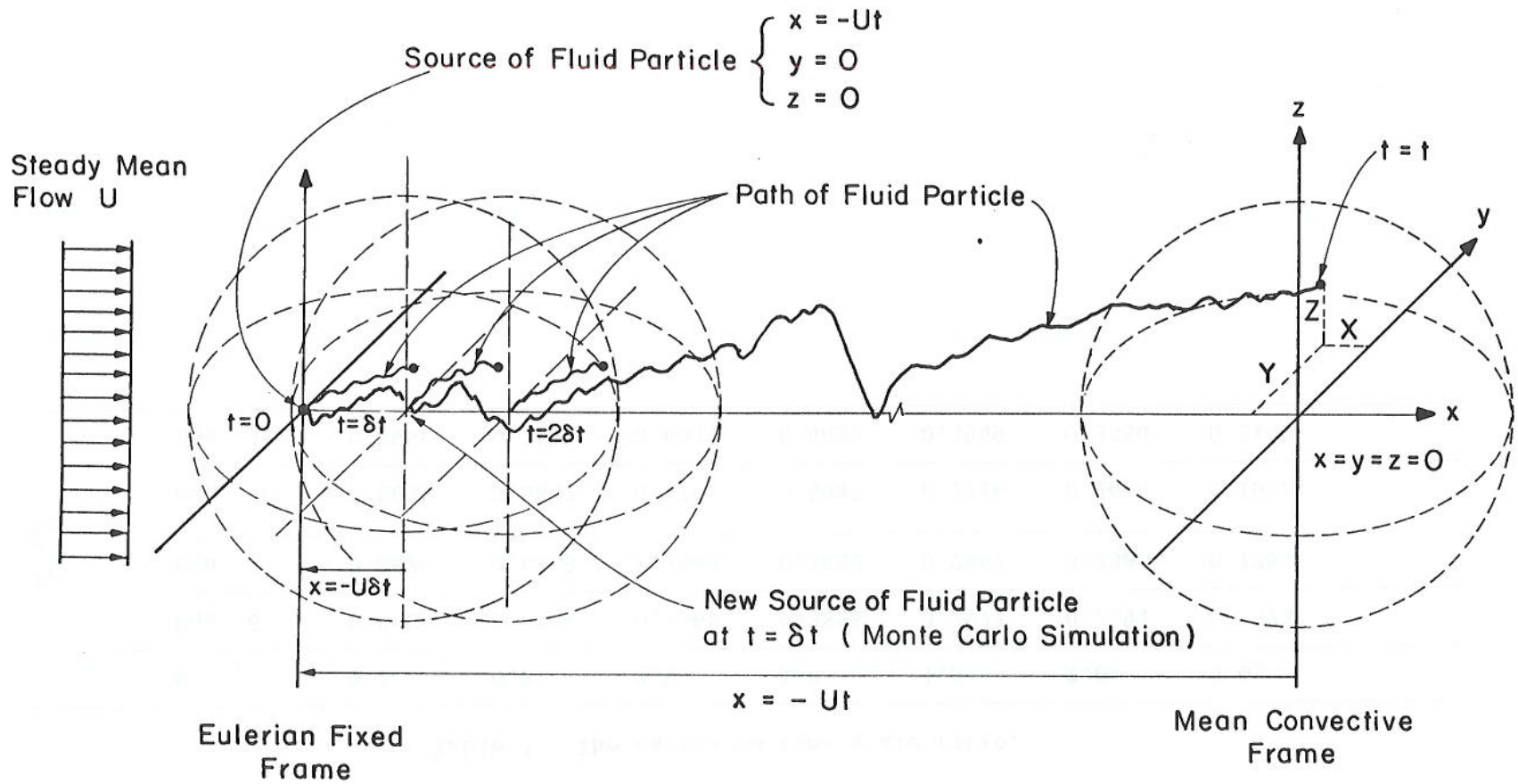

Figure 1. Reference Frame