Longitudinal vortex instabilities in laminar boundary layers over curved heated surfaces

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An analysis has been performed of penetrative convective instabilities arising from the combined action of thermal and centrifugal buoyancy forces. The objective has been to examine the effect of various mean temperature and velocity profiles on the critical limit and convective penetration of the disturbances. The linearized perturbation equations have been solved employing an approximate technique. The close analogy between streamline curvature and thermal stratification effects has been demonstrated. It is found that for parallel layers of fluid along curved heated walls, a unique stability curve for neutral disturbances may be obtained if the quantity plotted along the abscissa is $\mathrm{Ra} + \kappa N_G^2$ where Ra is the Rayleigh number, N_G is the Görtler number, and κ is a constant which expresses the relative importance of the mean temperature and velocity profiles.

I. INTRODUCTION

This discourse considers penetrative convective instabilities resulting from the combined action of thermal and centrifugal buoyancy forces. These instabilities are assumed to take the form of steady three-dimensional vortices oriented in the streamwise direction and are similar to the disturbances observed in the flow between rotating concentric cylinders. The latter instability manifests itself in the form of regularly spaced toroidal vortices stacked around the inner cylinder. This phenomenon was first examined by Taylor[1] who formulated the motion in mathematical terms, analyzed its stability, and verified the analysis in quite conclusive fashion. Görtler[2] and later Smith[3] investigated the vortex mode of motion along a plate with concave curvature and indicated the presence of a system of parallel counter rotating vortices aligned in the mean flow direction. Furthermore, their analyses clearly indicated that only flows with concave curvature were susceptible to this type of instability. Experimental verification was subsequently obtained by Görtler,[2] Liepmann,[4] and Tani.[5] The parameter governing the stability of the flow is the Görtler number $R_\delta(k\delta)^{1/2}$, where R_δ is the Reynolds number based on the boundary layer thickness δ, and k is the curvature of the wall.

The analogy between flows with concave curvature and buoyancy due to unstable stratification was pointed out by Görtler[6] and more recently by Yih[7] and by Bradshaw.[8] Terada[9] and Sparrow *et al.*,[10] have observed the vortex mode of motion in the flows of liquids down inclined heated plates.

The occurrence of a closely analogous phenomenon in the atmosphere is fairly well documented. The large-scale cloud streets frequently observed in satellite photographs[11] are now accepted as direct evidence of the presence of longitudinal vortex instabilities in the earth's atmosphere. The clouds are formed as a result of the convective action of the rolls in lifting moist air to its condensation level. Further direct evidence is supplied by the experience of glider pilots,[12] who have made use of these "invisible highways" in the air to soar over large distances. Kuo[13] analyzed the stability of plane Couette flow with a suitable gradient of potential temperature so as to model the atmospheric boundary layer. However, his boundary conditions required the physically unrealistic situation of a rigid upper bounding surface.

The present study allows for the fact that convective instabilities arising in an unstable layer may penetrate into a neighboring stable region. The analysis presented herein assumes that the disturbances are small, thereby permitting the linearization of the equations of motion. Strictly speaking therefore, this study deals not with fully developed penetrative convection in the true sense of the word but with the onset of convection or "marginal" convection. The penetrative action of the instabilities into the stable region may be due to two causes, viz., (a) the nonvanishing of the vertical velocity components of the disturbances at the interface causing inertial penetration or (b) momentum being transferred into the upper layer by viscous interaction of the perturbations with the adjoining stable fluid. In Taylor's experiments with counter rotating cylinders, the fluid adjoining the outer cylinder was in stable equilibrium, since the square of the circulation increased outwards. This is the condition for equilibrium according to the well-known Rayleigh criterion. Nevertheless, Taylor[1] observed a weak secondary system of vortices in the stable region driven by the primary system of vortices through viscous traction.

Inertial penetration has been studied extensively (see, for example, Stix,[14] and Whitehead and Chen[15]), while penetration by viscous entrainment has been studied recently by Rintel.[16] The linear analysis described herein, assumes that the penetration is of the second type and follows closely the work of Rintel. In this approximation, the instabilities generated in a fluid layer of thickness δ, penetrate to a total height d into neighboring stable fluid. A quantity c called the penetration coefficient and defined as $c = d/\delta$ provides an estimate of the degree of penetration.

Solutions have been obtained for a variety of flows along heated curved walls with stable fluid overhead. In this context "stable" refers to stability with respect to velocity gradient, (i.e., where Rayleigh's inviscid stability criterion is satisfied) as well as to the conventional interpretation of stable temperature stratification. The aim of this work is to demonstrate more clearly the analogy existing between flows with concave curvature and unstable stratification. Consequently, the Tollmien-Schlichting wave-type disturbances pertinent to transition are not accounted for in this analysis. However, for heated or curved flow-fields the Squire theorem does not necessarily hold[17]; hence, three-dimensional disturbances may be the more unstable mode. Furthermore, these stationary convective motions have been observed to persist in turbulent fluid by Tani[5] where the problem of transition does not arise. The specific cases of the parallel flows whose stability are examined herein are:

(a) Parallel flow with free surface along curved heated walls; (b) boundary layer type flow with wall curvature and heating from below bounded above by a fluid with differing stable gradients of temperature and velocity.

Details of the cases examined will be discussed later.

II. THEORETICAL DEVELOPMENT

Consider an unstably stratified parallel flow over a curved surface. The unstable layer is considered to be bounded above by fluid of neutral or arbitrarily specified stability. It is assumed that the disturbances generated in the lower layer of thickness δ penetrate to a height d. The penetration coefficient is then defined as $c = d/\delta$. The parameter c thus provides a measure of the extent of penetration as indicated in Fig. 1. In this respect it is closely allied to the "effective depth" defined by Kuo.[13]

We start with the Navier–Stokes equations of motion and the energy equation, expressed in a curvilinear coordinate system (Fig. 1). Using the Boussinesq approximation, one can derive the following equations for the perturbations $\tilde{p}$, $\tilde{T}$, $\tilde{u}_i$ of pressure, temperature, and the three components of velocity, respectively,

$$\frac{\partial \tilde{u}}{\partial t} + \tilde{v}\frac{\partial U}{\partial y} + k\tilde{v}U + \tilde{w}\frac{\partial U}{\partial z} = \nu\left(\frac{\partial^2 \tilde{u}}{\partial y^2} + \frac{\partial^2 \tilde{u}}{\partial z^2} + k\frac{\partial \tilde{u}}{\partial y}\right),$$

$$\frac{\partial \tilde{v}}{\partial t} - 2k\tilde{u}U = g\beta\tilde{T} - \frac{1}{\rho}\frac{\partial \tilde{p}}{\partial y} + \nu\left(\frac{\partial^2 \tilde{v}}{\partial y^2} + \frac{\partial^2 \tilde{v}}{\partial z^2} + k\frac{\partial \tilde{v}}{\partial y}\right), \quad (1)$$

$$\frac{\partial \tilde{w}}{\partial t} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial z} + \nu\left(\frac{\partial^2 \tilde{w}}{\partial y^2} + \frac{\partial^2 \tilde{w}}{\partial z^2} + k\frac{\partial \tilde{w}}{\partial y}\right),$$

$$\frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} + k\tilde{v} = 0, \quad (2)$$

$$\frac{\partial \tilde{T}}{\partial t} + \tilde{v}\frac{\partial T}{\partial y} + \tilde{w}\frac{\partial T}{\partial z} = \frac{\nu}{Pr}\left(\frac{\partial^2 \tilde{T}}{\partial y^2} + \frac{\partial^2 \tilde{T}}{\partial z^2} + k\frac{\partial \tilde{T}}{\partial y}\right), \quad (3)$$

where k denotes the curvature, ν is the kinematic viscosity, β is the volume expansion coefficient, and g is the gravitational acceleration. U and T refer, respectively, to the tangential velocity component and temperature of the primary flow. One can analyze an arbitrary disturbance into a set of normal modes,

$$\tilde{u} = u_p(y)\cos\alpha z\exp(\beta_1 t), \qquad \tilde{v} = v_p(y)\cos\alpha z\exp(\beta_1 t),$$

$$\tilde{w} = w_p(y)\sin\alpha z\exp(\beta_1 t), \qquad \tilde{T} = T_p(y)\cos\alpha z\exp(\beta_1 t)$$

$$\tilde{p} = p_p(y)\cos\alpha z\exp(\beta_1 t),$$

where α is a lateral wavenumber.

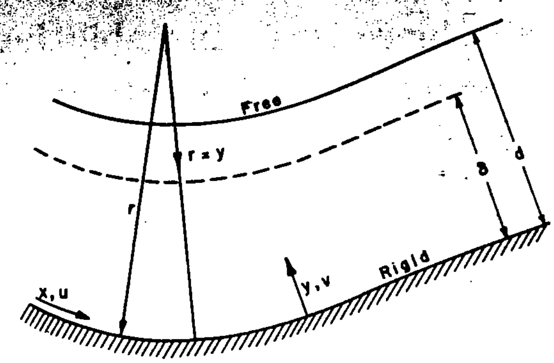

FIG. 1. Orthogonal curvilinear coordinate system.

Since we only consider neutral disturbances, we substitute the above perturbations into Eqs. (1)–(3) with $\beta_1 = 0$. The justification for this step lies in the validity of the principle of the "exchange of stabilities" for such flows.[15,18] The primary or "mean" flow is taken as being two dimensional, hence $\partial U/\partial z = \partial T/\partial z = 0$. This results in the following system of differential equations:

$$v_p(\partial U/\partial y + kU) = \nu(D^2u_p + kDu_p - \alpha^2 u_p), \quad (4)$$

$$-2kUu_p = g\beta T_p - Dp_p/\rho + \nu(D^2v_p + kDv_p - \alpha^2 v_p), \quad (5)$$

$$0 = \alpha(p_p/\rho) + \nu(D^2w_p + kDw_p - \alpha^2 w_p), \quad (6)$$

$$0 = Dv_p + kv_p + \alpha w_p, \quad (7)$$

$$v_p(\partial T/\partial y) = (\nu/P_r)(D^2T_p + kDT_p - \alpha^2 T_p); \quad (8)$$

here, $D \equiv d/dy$. By eliminating p_p and w_p and discarding higher-order curvature terms, Eqs. (4) to (8) may be reduced to

$$(D^2 - \alpha^2)^2 v_p = (\alpha^2/\nu)(2kUu_p - g\beta T_p), \quad (9)$$

$$(D^2 - \alpha^2)u_p = (-v_p/\nu)(\partial U/\partial y + kU), \quad (10)$$

$$(D^2 - \alpha^2)T_p = (Pr/\nu)(\partial T/\partial y)v_p. \quad (11)$$

Defining dimensionless quantities $\phi = \alpha\delta$, $v = v_p(\delta/\nu)$, $u = u_p(\delta/\nu)$, $w = w_p(\delta/\nu)$, $\mathrm{Gr} = g\beta\delta^3\Delta T/\nu^2$, $R_\delta = U_\infty\delta/\nu$, $f = U/U_\infty$, $T_*' = \partial T/\partial y\, 1/\Delta T$, $K = k\delta$, $y_1 = y/\delta$, where ΔT is the temperature difference across the fluid layer δ, Eq. (9). (10), and (11) become

$$(D^2 - \phi^2)^2 v = \phi^2(2KfR_\delta u - \mathrm{Gr}T), \quad (12)$$

$$(D^2 - \phi^2)u = -vR_\delta(f' + Kf), \quad (13)$$

$$(D^2 - \phi^2)T = v\,\mathrm{Pr}T_*'. \quad (14)$$

D now denotes d/dy_1 and primes denote $\partial/\partial y_1$.

As stated earlier, the penetration coefficient c is defined as the ratio of the penetrated height d to the thickness δ of the unstable layer. We therefore define a new dimensionless coordinate

$$\xi = y_1/c, \quad \Phi = c\phi, \quad K_c = cK, \quad \mathrm{Gr}_c = c^3\mathrm{Gr},$$

and a modified Reynolds number $R_{c\delta} = cR_\delta$. Eliminating the second term on the right-hand side of Eq. (13), since the product Kf is small in comparison with f', one obtains, with the above dimensionless parameters, the final form of the differential equations for the perturbations, viz.,

$$(D^2-\Phi^2)^2 v=\Phi^2[2K_c f R_{c\delta} u-(\mathrm{Gr}_c/c)T], \quad (15)$$

$$(D^2-\Phi^2)u=vR_{c\delta}f'(c\xi), \quad (16)$$

$$(D^2-\Phi^2)T=vc\,\mathrm{Pr}T'(c\xi), \quad (17)$$

where

$$D\equiv d/d\xi.$$

The boundary conditions assumed for Eqs. (15) to (17) are the "fixed-free" boundaries as explained in Chandrasekhar.[18] That is, the lower boundary is assumed to be a rigid plane with infinite thermal conductivity, while the upper boundary exerts no stress although possessing infinite thermal conductivity. With these boundary conditions we must have that at

$$\xi=0, \quad u=v=T=0,$$
$$\xi=1, \quad v=T=0.$$

Additionally, from the continuity equation (7) we may obtain the auxiliary boundary conditions $Dv=0$ at $\xi=0$ and $D^2v=0$ at $\xi=1$, since $w=0$ at $\xi=0$ and $Dw=u=0$ at $\xi=1$.

Equations (15)–(17) are solved approximately using a technique devised by Chandrasekhar.[18] The parallel and normal to the wall components of the perturbation velocity are expanded in a series of functions satisfying the boundary conditions for a rigid wall at $\xi=0$ and a free surface boundary at $\xi=1$:

$$u=\sum_{n=1}^{\infty}B_n x_n, \qquad v=\sum_{n=1}^{\infty}A_n z_n. \quad (18)$$

The temperature perturbation, since it satisfies similar boundary conditions as u is written as

$$T=\sum_{n=1}^{\infty}C_n x_n, \quad (19)$$

where we choose

$$\lambda_n x_n=\sin n\pi\xi,$$
$$\lambda_n^2 z_n=\lambda_n x_n+[2n\pi/(\sinh 2\Phi-2\Phi)]\sinh\Phi\xi$$
$$-\xi\sinh\Phi\cosh(\Phi\xi-\Phi),$$
$$\lambda_n=n^2\pi^2+\Phi^2.$$

It may be readily verified that the functions chosen satisfy the boundary conditions and also that

$$(D^2-\Phi^2)x_n=-\lambda_n x_n, \qquad (D^2-\Phi^2)z_n=\lambda_n x_n.$$

When these expressions are substituted into the differential equations and the coefficients B_n and C_n are eliminated, the following eigenvalue system is obtained for A_n;

$$A_n=c^3\Phi^2\left(8N_G\sum_{m=1}^{\infty}\frac{X_{nm}'}{\lambda_m}\sum_{l=1}^{\infty}\frac{A_lY_{ln}'}{\lambda_l}+2\,\mathrm{Ra}\frac{1}{\lambda_n}\sum_{m=1}^{\infty}\frac{A_m}{\lambda_m}Y_{mn}^0\right),$$

where $N_G=R_\delta(k\delta)^{1/2}$ is the Görtler number and Ra = Gr × Pr is the Rayleigh number with

$$X_{nm}'=\lambda_n\lambda_m\int_0^1 f x_m x_n\,d\xi,$$

$$Y_{ln}'=\lambda_l\lambda_n\int_0^1 f' z_l x_n\,d\xi,$$

$$Y_{mn}^0=\lambda_m\lambda_n\int_0^1 T' z_m x_n\,d\xi.$$

The standard method of evaluating the Fourier coefficient has been used to arrive at the previous equation. It may now be rewritten as

$$A_n=c^3\Phi^2\sum_{l=1}^{\infty}A_l(8N_G^2P_{ln}+2\,\mathrm{Ra}Q_{ln})=c^3\Phi^2\sum_{l=1}^{\infty}A_lR_{ln}. \quad (20)$$

Here,

$$Q_{ln}=Y_{ln}^0/\lambda_l\lambda_n, \qquad P_{ln}=\sum_{m=1}^{\infty}(X_{nm}'Y_{lm}'/\lambda_l\lambda_m).$$

In matrix notation we have the familiar eigenvalue problem

$$[A]=\Phi^2c^3[R][A].$$

The equation of neutral stability is then

$$|\,\delta_{nm}-c^3\Phi^2R_{nm}\,|=0.$$

To simplify the numerical evaluation of Eq. (20), it is rewritten in the form

$$A_n=c^3\Phi^2N_G\sum_{l=1}^{\infty}A_l(8P_{ln}+2R_NQ_{ln}), \quad (21)$$

where R_N is defined as Ra/N_G^2.It therefore expresses the relative importance of buoyancy forces to centrifugal inertial forces with viscous damping as an over-all effect. Equation (21) was solved for various values of the R_N number.

III. OUTLINE OF SOLUTION PROCEDURE

Numerical evaluation of the eigenvalue problem was performed on a CDC 6400 computer. The method consisted of minimizing the Görtler or Rayleigh number as a function of the wavenumber Φ and penetration coefficient c. The method is well described by Rintel.[16] A complete neutral stability curve may be generated by varying Φ keeping c at its critical value. The results thus obtained are scaled with the critical c value. A more accurate representation would be obtained if c is minimized at each point on the stability curve. This, however, ceases to be economical in terms of computer time. For cases of combined heating with wall curvature, Eq. (21) was used with R_N being treated as a parameter. For purposes of numerical evaluation, the infinite series expansion in Eqs. (18) and (19) was truncated to thirty terms and the matrices were limited to fifth order. For the cases where penetration was into neutrally

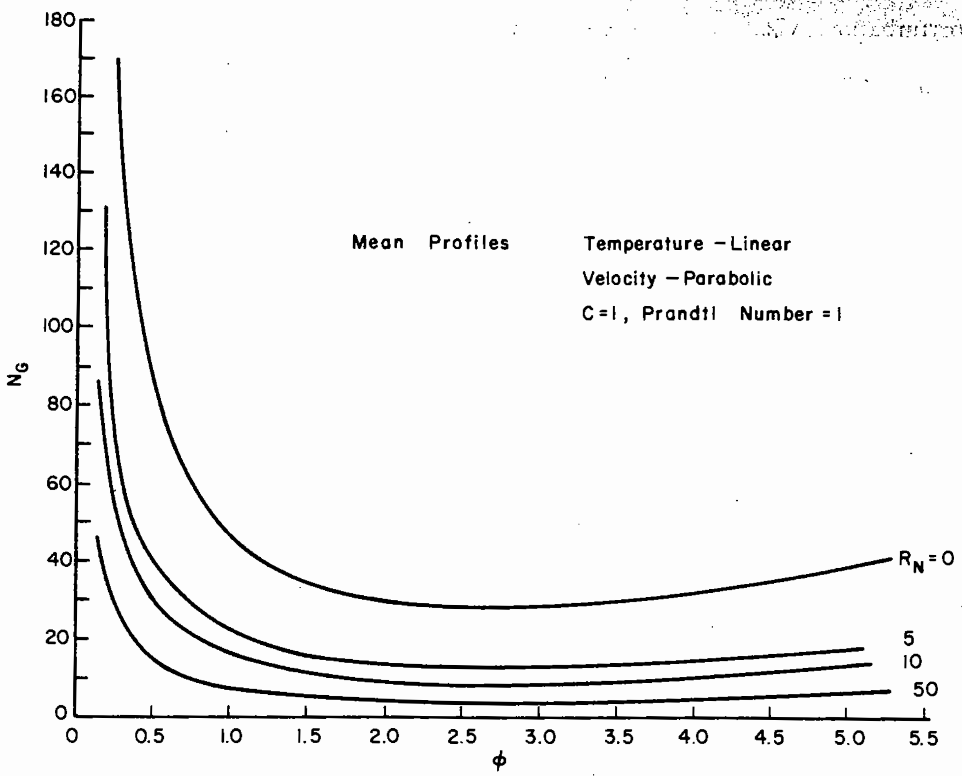

FIG. 2. Case (a), plot of Görtler number vs wavenumber for different R_N.

stable fluid, the order of the matrices used was increased to nine. A check, performed by increasing the truncation order of both the series and the matrices, revealed that this provided sufficient accuracy.

The mean flow profiles for the cases examined are listed below.

(a) Parallel flow with free surface along a curved heated wall

$$f(y) = y(2-y), \quad T_*(y) = y$$

was assumed for the mean velocity and temperature profiles, respectively, with unit Prandtl number. Since the fluid layer was assumed to have a free surface, the penetration was taken as zero implying $c = 1$.

(b) Boundary layer type flow with combined wall curvature and heating

$$f(y) = y(2-y), \qquad 0 \leq \xi \leq 1/c,$$

$$f(y) - \chi y(2-y) + 1 - \chi, \quad 1/c \leq \xi \leq 1.$$

The same mean temperature profile was assumed except that χ was taken as one for the velocity profile and three for the temperature profile. A value of 0.7 for the Prandtl number was used in evaluating this case. A semi-empirical adjustment for this was made in evaluating the integral $Y_{mn}{}^0$ by defining a new variable $\xi_1 = \xi\,\mathrm{Pr}^{1/3}$ according to Eckert and Drake.[19] Hence, when the integration of ξ is carried over the range 0 to 1, the entire thermal profile is integrated across simultaneously.

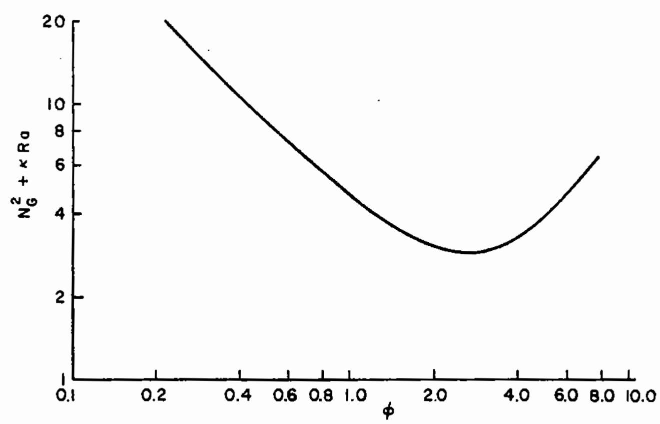

FIG. 3. Case (a), neutral stability curve.

TABLE I. Comparison of κ values obtained in case (b)

$N_{G_{cr}}$	R_N	Ra	Φ_{cr}	C_{cr}	κ
21.76	0	0.0	2.6943	1.4512	
11.57	4	536.23	2.7123	1.356	0.633
8.83	8	623.35	2.7153	1.344	0.634

IV. DISCUSSION OF RESULTS

The cases for which the critical conditions were evaluated have been listed earlier. Figure 2 illustrates the results obtained for case (a), i.e., the stability of a parallel free surface flow along a curved heated wall with zero penetration. The curve for $R_N = 0$ represents the case of pure centrifugal instability in a boundary layer along a concave wall with no heating. Increasing values of R_N, or in other words, increased heating at the wall results in a downward destabilizing shift of the neutral stability curve. The critical wavenumber at the onset of instability is relatively unaffected by the increased heating.

The analogy between curvature and buoyancy is well displayed in Fig. 3 which is a composite neutral stability

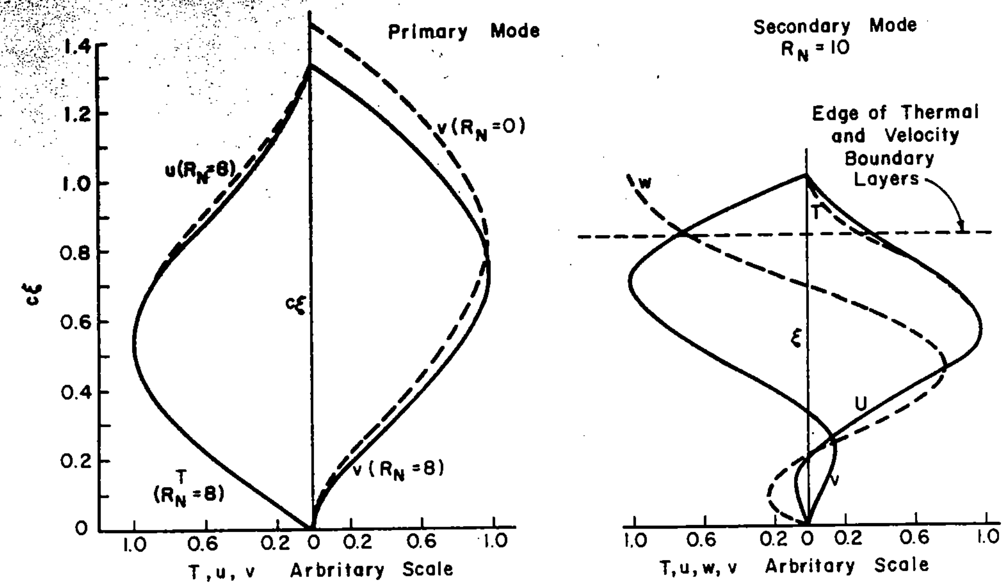

FIG. 4. Examples of velocity and temperature perturbations obtained for case (b).

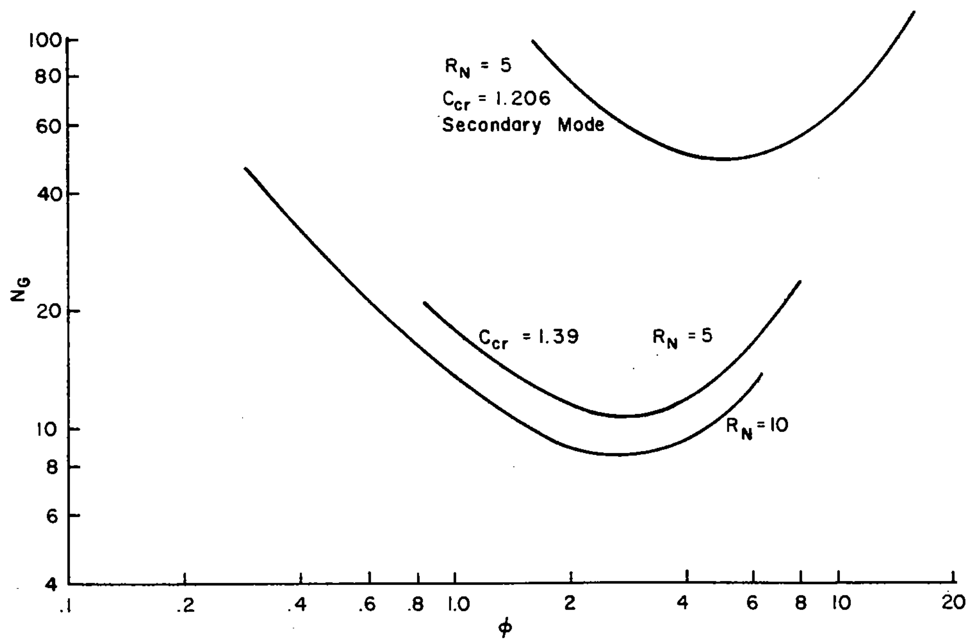

FIG. 5. Neutral stability curves for case (b).

curve obtained by plotting Φ, the disturbance wavenumber, vs $N_G^2 + \kappa\,\mathrm{Ra}$. This result was arrived at as follows: A previous simple analyses, by the authors, of curved flows with thermal stratification, had yielded the parameter $N_G^2 + \mathrm{Ra}$ as a stability criterion for linear profiles. It was, therefore, intuitively expected that nonlinear profiles would perhaps yield the slightly more general parameter $N_G^2 + \kappa\,\mathrm{Ra}$, where κ now, would account for the differences between the thermal and velocity profiles. Consequently, a number of calculations of the stability boundaries for various values of the parameter R_N were performed. It was then easy to calculate κ and establish that $N_G^2 + \kappa\,\mathrm{Ra}$ was indeed a unique parameter by checking several points on the calculated curves. A similar result was arrived at by Lindberg[20] who found that the thermal Rayleigh number and an analogously defined "concentration Rayleigh number" added linearly to form a stability parameter.

A secondary dependence of κ would be on the degree of penetration into the stable fluid, a limiting factor in establishing this dependence being the computer time available. To examine this effect, a small number of calculations was run on case (b) and the critical values obtained in a first approximation in the minimization are given in Table I, where the values of N_G and Ra are based on the thickness of the unstable layer δ. Since the deviation in κ is not large, it appears that the same type of relation holds so that one may write $N_{Gcr}{}^2 = N_{Gcr}{}^2|_{\mathrm{Ra}=0} - \kappa\,\mathrm{Ra}$. The difference in the first value of c is probably due to the absence of Prandtl number effects. The eigenfunctions obtained are plotted in Fig. 4 together with the eigenfunctions of the second modal instability with $\mathrm{Pr} = 1$. The neutral stability curves for $R_N = 5$ and 10 and $\mathrm{Pr} = 1$ are plotted in Fig. 5. They are, of course, scaled with the critical value of c. Figure 6 contains curves of the variation of critical Görtler number

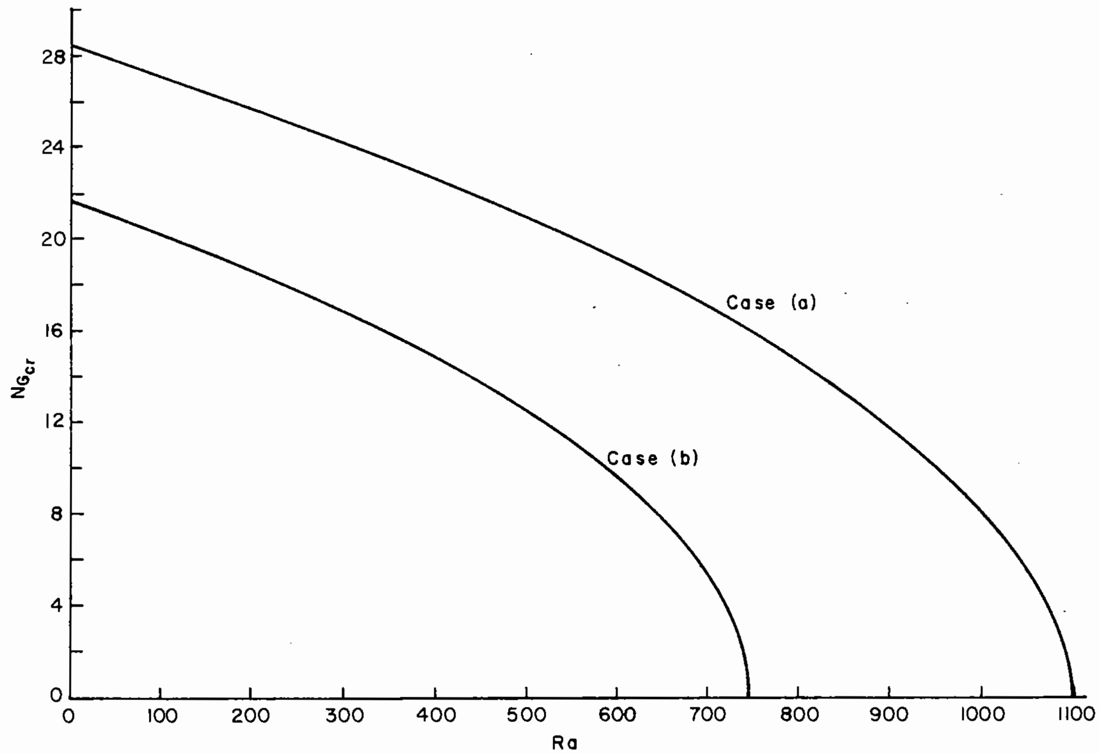

FIG. 6. Variation of critical Görtler number with Rayleigh number for cases (a) and (b).

with Rayleigh number at the point of first instability. The curves are, of course, parabolas.

V. CONCLUSIONS

The object of this analysis was to explore the stability of parallel layers of fluid under the simultaneous influence of curvature and heating. The simple linear theory indicates that the two effects are additive, demonstrating the close analogy between streamline curvature and buoyancy.

Future work should include the interaction of the Tollmien–Schlichting wave instabilities with truly three-dimensional convective disturbances. The disturbances analyzed here have been quasi-two-dimensional in that no variations in the streamwise direction have been assumed. Further calculations with infinite boundary conditions, i.e., where the fluid is unbounded vertically, should provide additionally useful results.

ACKNOWLEDGMENTS

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[1] G. I. Taylor, Phil. Trans. R. Soc. Lond. A **223,** 289 (1923).
[2] H. Görtler, NACA TM 1375 (1954).
[3] A. M. O. Smith. Q. Appl. Math. **3,** 233 (1955).
[4] H. W. Liepmann, NACA ACR No. 3H30 (1943).
[5] I. Tani, J. Geophys. Res. **67,** 3075 (1962).
[6] H. Görtler, Ing.-Arch. **28,** 71 (1959).
[7] C. S. Yih, *Dynamics of Non-homogeneous Fluids* (Macmillan, New York, 1965), Chap. 6.
[8] P. Bradshaw, J. Fluid Mech. **36,** 177 (1969).
[9] S. Terada, Tokyo Imperial University Research Report No. 31, (1928).
[10] E. M. Sparrow and R. B. Husar, J. Fluid Mech. **37,** 251 (1969).
[11] A. Krueger and S. Fritz, Tellus **13,** 1 (1961).
[12] J. Kuettner, Tellus **11,** 267 (1959).
[13] H. L. Kuo, Phys. Fluids **6,** 195 (1963).
[14] M. Stix, Tellus **22,** 517 (1970).
[15] J. A. Whitehead and M. Chen, J. Fluid Mech. **40,** 549 (1970).
[16] L. Rintel, Phys. Fluids **10,** 848 (1967).
[17] K. S. Gage and W. H. Reid, J. Fluid Mech. **33,** 21 (1968).
[18] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Oxford University Press, Oxford, 1961), Chap. 2.
[19] E. R. G. Eckert and R. M. Drake, *Heat and Mass Transfer* (McGraw-Hill, New York, 1959), 2nd ed. p. 175.
[20] W. Lindberg, Ph.D. dissertation, Colorado State University (1970).