Useful Formulas for Loans

Edwin Chong

October 27, 2010

Given:

- \( B \): amount borrowed
- \( N \): loan horizon (number of months)
- \( R \): monthly interest rate (fraction, usually annual interest rate divided by 12)

Fixed Monthly Payment

**Goal:** Calculate the fixed monthly payment such that after \( N \) payments, the loan is exactly paid off. Call this number \( U \).

Let \( B_k \) be the loan balance (or principal) after \( k \) monthly payments, where \( k = 1, \ldots, N \). Then, the sequence \( B_1, B_2, \ldots \) satisfies

\[
B_k = (1 + R)B_{k-1} - U, \quad k = 1, 2, \ldots, N,
\]

with \( B_0 = B \). The standard formula for the solution to this linear difference equation is

\[
B_k = (1 + R)^k B - U \sum_{j=0}^{k-1} (1 + R)^j = (1 + R)^k B - U \frac{(1 + R)^k - 1}{R}.
\]

If we set \( B_N = 0 \) and solve for \( U \), we get this:

\[
\text{Monthly payment: } U = BR \frac{(1 + R)^N}{(1 + R)^N - 1}
\]

Note: In Excel, \((1+R)^N\) is implemented using the formula \( \text{POWER}(1+R,N) \). An alternative formula, which contains only a single “power,” is:

\[
\text{Monthly payment: } U = \frac{BR}{1 - (1 + R)^{-N}}
\]

If \( N \) is very large, then \( U \) is approximately \( BR \), which is the monthly interest on the initial loan \( B \). In general, \( U \) is larger than \( BR \). For example, in the first month, the monthly payment must include the interest \( BR \) and some additional amount to reduce the principal.
Monthly Principal Reduction

Goal: Calculate the amount that the principal is reduced when making the $k$th payment.

We first calculate the balance after $k$ payments by substituting the formula for $U$ into the expression for $B_k$ above. We get

$$B_k = B \left( \frac{(1 + R)^N - (1 + R)^k}{(1 + R)^N - 1} \right), \quad k = 0, 1, \ldots, N.$$ 

Hence, the amount of principal reduction associated with the $k$th payment is $B_{k-1} - B_k = U - RB_{k-1}$, which simplifies to this:

$$\text{Principal reduction} = BR\frac{(1 + R)^{k-1}}{(1 + R)^N - 1}$$

Notice that if $N$ is large and $k$ is small compared to $N$, then this number is a small fraction of $BR$, the interest on the initial loan. But it grows geometrically with $k$.

Total Interest Paid

Goal: Calculate the total interest paid. Call this number $T$.

The interest associated with the $k$th payment is $B_{k-1}R$. Hence, the total interest paid is

$$T = \sum_{k=1}^{N} BR \left( \frac{(1 + R)^N - (1 + R)^{k-1}}{(1 + R)^N - 1} \right).$$

Simplifying, we get this:

$$\text{Total interest paid: } T = B \left( NR \frac{(1 + R)^N}{(1 + R)^N - 1} - 1 \right)$$

At this point, we realize that this expression looks familiar. In fact, it can be written as this:

$$T = NU - B$$

which surely makes sense: The total amount paid over $N$ months minus what was borrowed is the total interest paid.