Useful Formulas for Loans

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Given:

- B: amount borrowed
- N: loan horizon (number of months)
- R: monthly interest rate (fraction, usually annual interest rate divided by 12)

Fixed Monthly Payment

Goal: Calculate the fixed monthly payment such that after N payments, the loan is exactly paid off. Call this number U.

Let B_k be the loan balance (or *principal*) after k monthly payments, where $k = 1, \ldots, N$. Then, the sequence $B_1, B_2, \ldots$ satisfies

$$B_k = (1 + R)B_{k-1} - U, \qquad k = 1, 2, \ldots, N,$$

with $B_0 = B$. The standard formula for the solution to this linear difference equation is

$$B_k = (1 + R)^k B - U \sum_{j=0}^{k-1} (1 + R)^j = (1 + R)^k B - U \frac{(1 + R)^k - 1}{R}.$$

If we set $B_N = 0$ and solve for U, we get this:

$$\boxed{\text{Monthly payment:} \quad U = BR \frac{(1 + R)^N}{(1 + R)^N - 1}}$$

Note: In Excel, $(1+R)^N$ is implemented using the formula POWER($1+R$,N). An alternative formula, which contains only a single "power," is:

$$\boxed{\text{Monthly payment:} \quad U = \frac{BR}{1 - (1 + R)^{-N}}}$$

If N is very large, then U is approximately BR, which is the monthly interest on the initial loan B. In general, U is larger than BR. For example, in the first month, the monthly payment must include the interest BR and some additional amount to reduce the principal.

Monthly Principal Reduction

Goal: Calculate the amount that the principal is reduced when making the kth payment.

We first calculate the balance after k payments by substituting the formula for U into the expression for B_k above. We get

$$B_k = B\left(\frac{(1+R)^N - (1+R)^k}{(1+R)^N - 1}\right), \qquad k = 0, 1, \ldots, N.$$

Hence, the amount of principal reduction associated with the kth payment is $B_{k-1} - B_k = U - RB_{k-1}$, which simplifies to this:

$$\boxed{\text{Principal reduction} = BR\frac{(1+R)^{k-1}}{(1+R)^N - 1}}$$

Notice that if N is large and k is small compared to N, then this number is a small fraction of BR, the interest on the initial loan. But it grows geometrically with k.

Total Interest Paid

Goal: Calculate the total interest paid. Call this number T.

The interest associated with the kth payment is $B_{k-1}R$. Hence, the total interest paid is

$$T = \sum_{k=1}^{N} BR\left(\frac{(1+R)^N - (1+R)^{k-1}}{(1+R)^N - 1}\right).$$

Simplifying, we get this:

$$\boxed{\text{Total interest paid:} \quad T = B\left(NR\frac{(1+R)^N}{(1+R)^N - 1} - 1\right)}$$

At this point, we realize that this expression looks familiar. In fact, it can be written as this:

$$\boxed{T = NU - B}$$

which surely makes sense: The total amount paid over N months minus what was borrowed is the total interest paid.