Convergence of Sequences of Random Variables

Preliminary Concepts and Notation

If $\{x_n\} = \{x_1, x_2, \ldots\}$ is a sequence of real numbers, we say that $\{x_n\}$ converges to to a number x^* if for all $\varepsilon > 0$, there exists N such that $|x_n - x^*| < \varepsilon$ for all $n \geq N$. In this case, we write $x_n \to x^*$ or $\lim_{n\to\infty} x_n = x^*$.

In the following, let $\{X_n\} = \{X_1, X_2, \ldots\}$ be a sequence of random variables, and X^* some random variable.

Convergence Almost Surely

We say that $\{X_n\}$ converges to X^* almost surely (or a.s.) if

$$P\{X_n \to X^*\} = 1.$$

In this case, we write $X_n \to X^*$ a.s., or $\lim_{n\to\infty} X_n = X^*$ a.s., or $X_n \xrightarrow{\text{a.s.}} X^*$.

Convergence In Probability

We say that $\{X_n\}$ converges to X^* in probability if for all $\varepsilon > 0$,

$$P\{|X_n - X^*| \geq \varepsilon\} \to 0.$$

In this case, we write $X_n \to X^*$ in probability, or $\lim_{n\to\infty} X_n = X^*$ in probability, or $X_n \xrightarrow{P} X^*$.

Convergence In Mean Square

We say that $\{X_n\}$ converges to X^* in mean square (or m.s.) if

$$E[(X_n - X^*)^2] \to 0.$$

In this case, we write $X_n \to X^*$ m.s., or $\lim_{n\to\infty} X_n = X^*$ m.s., or $X_n \xrightarrow{\text{m.s.}} X^*$.

Convergence In Distribution

Let F_n be the distribution function of X_n, and F^* the distribution function of X^*. We say that $\{X_n\}$ converges to X^* in distribution if

$$F_n(x) \to F^*(x)$$

for all x such that F^* is continuous at x. In this case, we write $X_n \to X^*$ in distribution, or $\lim_{n\to\infty} X_n = X^*$ in distribution, or $X_n \xrightarrow{D} X^*$.