Convergence of Sequences of Random Variables

Preliminary Concepts and Notation

If \( \{x_n\} = \{x_1, x_2, \ldots \} \) is a sequence of real numbers, we say that \( \{x_n\} \) converges to to a number \( x^* \) if for all \( \varepsilon > 0 \), there exists \( N \) such that \( |x_n - x^*| < \varepsilon \) for all \( n \geq N \). In this case, we write \( x_n \rightarrow x^* \) or \( \lim_{n \to \infty} x_n = x^* \).

In the following, let \( \{X_n\} = \{X_1, X_2, \ldots \} \) be a sequence of random variables, and \( X^* \) some random variable.

Convergence Almost Surely

We say that \( \{X_n\} \) converges to \( X^* \) almost surely (or a.s.) if
\[
P\{X_n \rightarrow X^*\} = 1.
\]
In this case, we write \( X_n \rightarrow X^* \) a.s., or \( \lim_{n \to \infty} X_n = X^* \) a.s., or \( X_n \xrightarrow{\text{a.s.}} X^* \).

Convergence In Probability

We say that \( \{X_n\} \) converges to \( X^* \) in probability if for all \( \varepsilon > 0 \),
\[
P\{|X_n - X^*| \geq \varepsilon\} \rightarrow 0.
\]
In this case, we write \( X_n \rightarrow X^* \) in probability, or \( \lim_{n \to \infty} X_n = X^* \) in probability, or \( X_n \xrightarrow{\text{p.}} X^* \).

Convergence In Mean Square

We say that \( \{X_n\} \) converges to \( X^* \) in mean square (or m.s.) if
\[
E[(X_n - X^*)^2] \rightarrow 0.
\]
In this case, we write \( X_n \rightarrow X^* \) m.s., or \( \lim_{n \to \infty} X_n = X^* \) m.s., or \( X_n \xrightarrow{\text{m.s.}} X^* \).

Convergence In Distribution

Let \( F_n \) be the distribution function of \( X_n \), and \( F^* \) the distribution function of \( X^* \). We say that \( \{X_n\} \) converges to \( X^* \) in distribution if
\[
F_n(x) \rightarrow F^*(x)
\]
for all \( x \) such that \( F^* \) is continuous at \( x \). In this case, we write \( X_n \rightarrow X^* \) in distribution, or \( \lim_{n \to \infty} X_n = X^* \) in distribution, or \( X_n \xrightarrow{\text{D}} X^* \).