Analysis and Control of Robot Manipulators with Redundancy

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The redundancy of robot manipulators plays an important role in increasing their flexibility and versatility. However, in order to take full advantage of the redundancy, more analysis must be done and effective control algorithms should be developed. A quantitative measure of manipulatability which is applicable to both redundant and nonredundant manipulators is proposed. Based on this measure, a control algorithm for utilizing the redundancy for the singularity avoidance is established. A control algorithm for avoiding obstacles is also discussed.

1 Introduction

As more industrial robots are introduced into various manufacturing areas, demands for flexibility and versatility of their performance are also intensifying. In order to meet these demands, robot manipulators tend to have more degrees of freedom. Manipulators with more degrees of freedom than necessary for a specified class of tasks can be called redundant for that class of tasks.

The redundancy can also be seen in many situations. For example, a manipulator with six degrees of freedom on a traveling vehicle is redundant for the class of tasks which need positioning and orienting the end effector (6 degrees of freedom). A group of two or more manipulators cooperating together in performing a single task will usually be redundant, too. These systems can also be regarded as redundant manipulators.

Redundant manipulators have more ability than nonredundant ones in many aspects, such as avoiding collisions with obstacles in the working space, avoiding singular states where the manipulators lose some degrees of freedom, keeping away from the motion limits of joints, reaching into holes, etc. In order to make full use of their ability, however, effective control algorithms should be developed.

It has been pointed out by some researches that the redundancy could be used for avoiding singularities [1, 2]. Several anthropomorphic manipulators with 7 degrees of freedom have been constructed [3, 4] and their inverse kinematic problems have been studied [4, 5].

Control problems of redundant manipulators have also been discussed in [6–8]. However, most of the examples taken up in these works have been on the maximum availability, i.e., on keeping all the joint variables within the physical limits. Other pos-

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sibilities such as singularity avoidance and obstacle avoidance, seem to have not been payed much attention.

In this paper a quantitative measure of manipulatability is proposed, and using this measure a control algorithm of redundant manipulators for avoiding singularities is established. A control algorithm for avoiding obstacles is also discussed.

The paper is organized as follows. In the following section, a quantitative measure of manipulatability is defined for a given arm posture of a manipulator. In section 3, two different approaches for developing control algorithms of redundant manipulators are discussed. The first approach utilizes an additional performance criterion. The second approach specifies an additional desirable trajectory [9]. A basic control algorithm for the first approach is described in section 4. An application of this algorithm to singularity avoidance is given in section 5, and an application to obstacle avoidance is discussed in section 6.

2 Measure of Manipulability

It will be beneficial for design and control of robots and for task planning if we have a quantitative measure of manipulating ability of robot arms in positioning and orienting the end effectors. In this section one such measure is proposed and its property is investigated.

We consider a manipulator with n degrees of freedom whose joint variables are denoted θ_i, $i = 1, 2, \ldots, n$. We assume that a class of tasks we are interested in can be described by m variables r_j, $j = 1, 2, \ldots, m$ $(m \leqslant n)$, and that the relation between θ_i and r_j are given by

$$r = f(\theta), \tag{2.1}$$

where $\theta = [\theta_1, \theta_2, \ldots, \theta_n]^{\mathrm{T}}$ is the joint vector, $r = [r_1, r_2, \ldots, r_m]^{\mathrm{T}}$ is the manipulation vector, and the superscript T denotes the transpose. Differentiating (2.1) with respect to time yields

$$\dot{r} = J(\theta)\dot{\Theta}, \tag{2.2}$$

where $\dot{r} = dr/dt \in R^m$ (m-dimensional Euclidian space), $\dot{\theta} = d\theta/dt \in R^n$, and $J(\theta) = \partial f(\theta)/\partial\theta \in R^{m\times n}$ (the set of all $m \times n$ real matrices). The matrix $J(\theta)$ is called the Jacobian.

We assume that the following condition is satisfied:

$$\max_{\theta} \operatorname{rank} J(\theta) = m. \tag{2.3}$$

Failing to satisfy this condition usually means that the selection of manipulation variables is redundant and the number of these variables m can be reduced. When condition (2.3) is satisfied, we say that the degree of redundancy of this manipulator is $(n - m)$. More detailed discussion on the degree of redundancy and a related concept of redundant space can be found in [9].

If, for some θ,

$$\operatorname{rank} J(\theta) < m, \tag{2.4}$$

hen we say that the manipulator is in a singular state. This state is not desirable since the manipulation vector r cannot move in a certain direction. However, it is not only the actual singular state which is undesirable. A neighboring region of any singular state is also undesirable because the manipulation vector r can only be moved very slowly in a certain direction. In other words, the manipulating ability of the robot in this region is very limited.

In order to analyze this problem in more detail, we propose the following quantitative measure of manipulatability:

Definition A scalar value w given by

$$w = \sqrt{\det(J(\theta)J^{\mathrm{T}}(\theta))} \tag{2.5}$$

is called the ***measure of manipulatability*** at state θ with respect to manipulation vector r.

A physical meaning of this measure will be given in the following. It is known [10] that for any matrix $J \in R^{m \times n}$ there exist orthogonal matrices $U \in R^{m \times m}$ and $V \in R^{n \times n}$ such that

$$J = U \Sigma V^{\mathrm{T}}, \tag{2.6}$$

where

$$\Sigma = \left[\begin{array}{cccc|c} \sigma_1 & & 0 & & \\ & \sigma_2 & & & 0 \\ 0 & & \ddots & & \\ & & & \sigma_m & \end{array}\right] \in R^{m \times n} \tag{2.7}$$

with

$$\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_m \geqslant 0. \tag{2.8}$$

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The decomposition (2.2) is called the singular value decomposition and $\sigma_1, \sigma_2, \ldots, \sigma_m$ are called singular values of J. From (2.5) and (2.6) we have

$$w = \sigma_1 \cdot \sigma_2 \cdots \sigma_m. \tag{2.9}$$

We can also show that the subset S_v of the realizable velocity $\dot{r}$ in the space R^m using joint velocity $\dot{\theta}$ such that $\|\dot{\theta}\| = \dot{\theta}_1^2 + \dot{\theta}_2^2 + \cdots + \dot{\theta}_n^2 \leqslant 1$ is an ellipsoid with principal axes $\sigma_1 u_1, \sigma_2 u_2, \ldots, \sigma_m u_m$, where $u_i \in R^m$ is the ith column vector of U, i.e., $[u_1 u_2 \cdots u_m] = U$. This ellipsoid can be called the *manipulatability ellipsoid* and could be a good means for the analysis, design, and control of robot manipulators. The volume of this ellipsoid is given by

$$d\sigma_1 \cdot \sigma_2 \cdots \sigma_m, \tag{2.10}$$

where the constant d is given by

$$d = \begin{cases} (2\pi)^{m/2}/(2 \cdot 4 \cdot 6 \cdots (m-2) \cdot m) & \text{when } m \text{ is even} \\ 2(2\pi)^{(m-1)/2}/(1 \cdot 3 \cdot 5 \cdots (m-2) \cdot m) & \text{when } m \text{ is odd.} \end{cases} \tag{2.11}$$

Therefore, w is equal to the volume of the manipulatability ellipsoid except for the constant coefficient d.

Note that when $m = n$, i.e., when we consider nonredundant manipulators, the measure w reduces to

$$w = |\det J(\theta)|. \tag{2.12}$$

Paul and Stevenson [11] used this measure (2.12) with $m = n = 3$ for analysis of robot wrists with respect to their ability to orient the end effectors.

Consider a 3 degrees of freedom arm which moves in the (x, y) plane as shown in figure 1. we assume that only the position of the end effector is of concern. The Jacobian matrix which relates $\theta = [\theta_1 \ \theta_2 \ \theta_3]^T$ to $r = [x, y]^T$ is given by

$$J = \begin{bmatrix} l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \end{bmatrix}, \tag{2.13}$$

where $c_1 = \cos\theta_1$, $c_{12} = \cos(\theta_1 + \theta_2)$, $c_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$, $s_1 = \sin\theta_1$, $s_{12} = \sin(\theta_1 + \theta_2)$, $s_{123} = \sin(\theta_1 + \theta_2 + \theta_3)$.

The manipulatability ellipsoid for this arm is given in figure 2 for three different joint states. It can be seen from the figure that although the position of the end effector is the same, the ellipsoid varies for different θ. The measure of manipulatability w for θ_c is larger than those for θ_a and θ_b. Hence we judge that the arm posture θ_c is the best among the three. For this same position of the end effector, the manipulatability can vary from 0.051 to 0.110.

3 Subtasks with Order of Priority

In this section the concept of task decomposition into several subtasks [9] is described. This concept is then related to two different approaches to the utilization of redundancy.

Most of the complicated tasks given to the robots with many degrees of freedom could be decomposed into several subtasks with order of priority. For example, in the case of welding, the task could be devided into the hand position control and the hand orientation control, the former being more important than the latter. The task in which the hand position and/or hand orientation should track the desired trajectory and the arm should avoid the obstacles in the working area or singular states as much as possible is another example.

For these tasks, it is natural to try first to perform the subtask with the first priority. If there is any ability left to the manipulator after achieving the first subtask, then we try to perform the second subtask as much as possible. Then we pick up the third subtask, if there is one, and so on. The existence of remaining ability at any stage means the robot is redundant for subtasks up to that stage.

Suppose that the first subtask is described by a manipulation vector r_1 and second subtask is described as an optimization with respect to a performance criterion. Then we can apply the same approach taken in [12]. This approach will be described in the following section.

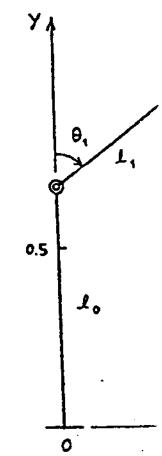

Figure 1
Three degree

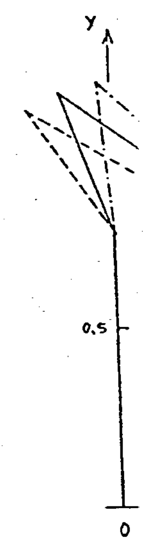

Figure 2
Arm postu
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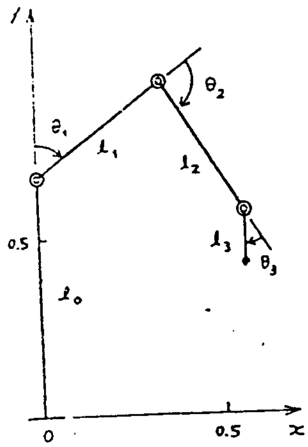


Figure 1
Three degrees of freedom arm, $l_0 = 0.67$, $l_1 = l_2 = 0.432$, $l_3 = 0.15$ (m).

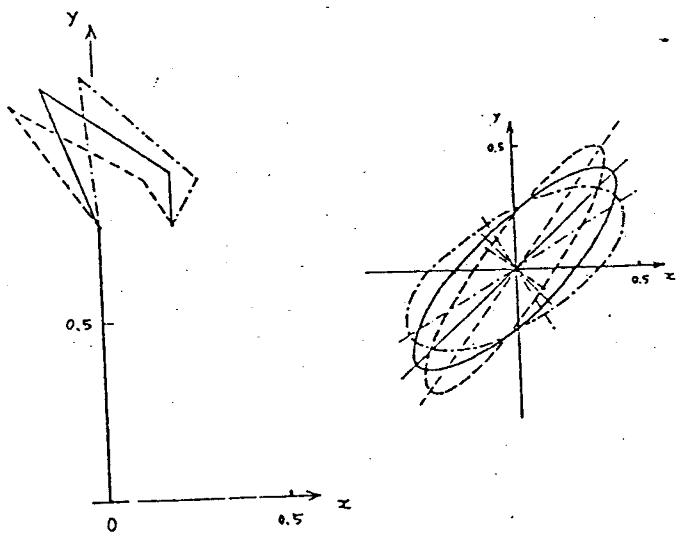


Figure 2
Arm posture and manipulatability ellipsoid: ---, $\theta = \theta_a = [-34.1, 155.9, 28.2]^T$, $w = 0.082$; ———, $\theta = \theta_b = [-20.1, 146.4, 53.7]^T$, $w = 0.101$; -·-·-, $\theta = \theta_c = [-4.7, 138.8, 75.9]^T$, $w = 0.110$.

For the case where the first and second subtasks are both described by manipulation vectors r_1 and r_2, a control algorithm has been given in [9].

4 Utilization of Redundancy for Optimizing Given Performance Criterion

We assume that the first subtask is to track a desired trajectory $r^*(t)$ of the manipulation vector r. We also assume that a scalar performance criterion for the second subtask is given by

$$p = q(\theta). \tag{4.1}$$

The second subtask is described as keeping the value of this criterion as large as possible. A control algorithm for achieving this task will be given in this section.

The main idea used here is quite the same as that used in [12] for developing a steering law for a control moment gyro system with 6 degrees of freedom in controlling a spacecraft orientation (3 degrees of freedom). Three redundant degrees of freedom of the control moment gyro system were utilized for the momentum distribution to avoid the singularities.

The general solution of (2.2) is given by

$$\dot{\theta} = J^+ \dot{r} + (I - J^+ J)k, \tag{4.2}$$

where J^+ is the pseudoinverse of the Jacobian matrix J [13] and $k \in R^n$ is an arbitrary constant vector. The second term of the right-hand side of (4.2) represents the redundancy left after performing the first subtask.

The time derivative of p is given by

$$\dot{p} = \xi^{\mathrm{T}} \dot{\theta}, \tag{4.3}$$

where

$$\xi = [\xi_1, \xi_2, \ldots, \xi_n]^{\mathrm{T}}, \tag{4.4}$$

$$\xi_l = \frac{\partial q(\theta)}{\partial \theta_l}, \quad l = 1, 2, \ldots, n. \tag{4.5}$$

Under the assumption that the first subtask is perfectly performed, we obtain from (4.3) and (4.2)

$$\dot{p} = \xi^{\mathrm{T}} J^+ \dot{r}^* + \xi^{\mathrm{T}} (I - J^+ J)k. \tag{4.6}$$

In order to achieve the second subtask, we propose to select k as

$$k = \xi k_1, \tag{4.7}$$

where k_1 is a positive constant. Hence the basic equation for the control algorithm is given by

$$\dot{\theta} = J^+ \dot{r}^* + (I - J^+ J)\xi k_1. \tag{4.8}$$

The reason for the selection of (4.7) is as follows. From (4.6) and (4.7) we obtain

$$\dot{p} = \xi^{\mathrm{T}} J^{+} \dot{r}^{*} + \xi^{\mathrm{T}}(I - J^{+}J)\xi k_1 \tag{4.9}$$

and the second term in the right-hand side of (4.9) becomes nonnegative due to the fact that $(I - J^{+}J)$ is idempotent, contributing to an increase of the value p.

The selection (4.7) can also be characterized as follows. Let

$$\dot{\theta}_0 = \dot{\theta}|_{\dot{r}=0} = (I - J^{+}J)k,$$

$$\dot{p}_0 = \dot{p}|_{\dot{r}=0} = \xi^{\mathrm{T}}(I - J^{+}J)k;$$

then the value of k which maximizes $\dot{p}_0$ under the condition

$$\|\dot{\theta}_0\| \leqslant k_1[\xi^{\mathrm{T}}(I - J^{+}J)\xi]^{1/2}$$

is given by (4.7). In practical applications, in order to prevent $\dot{\theta}$ from becoming excessive, a limitation for k_1 should be given. One canditate for this will be $\|\dot{\theta}_0\| \leqslant k_3\dot{\theta}_H$, i.e.,

$$k_1 \leqslant [\xi^{\mathrm{T}}(I - J^{+}J)\xi]^{-1/2} k_3 \dot{\theta}_H,$$

where $\dot{\theta}_H$ is the hardware limit for the joint angle rate and k_3 is a constant ($0 \leqslant k_3 \leqslant 1$).

Note that the algorithm of the form (4.8) was originally given in [12] for the attitude control spacecrafts. Essentially the same algorithm appeared also in [6] in the context of redundant mechanical systems including manipulators.

The application of the proposed algorithm to two kinds of second subtasks, i.e., singularity avoidance and obstacle avoidance, will be discussed in the following two sections.

5 Singularity Avoidance

In case of singularity avoidance, we propose to adopt the measure of manipulatability (2.5) as the performance criterion for the second subtask. In other words, we try not only to avoid the real singularities but also to keep the ability of manipulation as much as possible. In this case the gradient vector ξ can be obtained as follows. Let

$$G = JJ^{\mathrm{T}} = [g_{ij}], \quad i, j = 1, 2, \ldots, m. \tag{5.1}$$

Then the performance criterion is $p = \sqrt{\det G}$ and we have

$$\begin{aligned} \xi_l &= \frac{1}{2\sqrt{\det G}} \sum_{i,j=1}^{m} \Delta_{ij} \cdot {}_l g'_{ij} \\ &= \frac{1}{2}\sqrt{\det G} \sum_{i,j=1}^{m} [G^{-1}]_{ij} ({}_l J'_i J_j^{\mathrm{T}} + {}_l J'_j J_i^{\mathrm{T}}), \end{aligned} \tag{5.2}$$

where

Δ_{ij} = cofactor of a_{ij} for G,
${}_l g'_{ij} = \partial(g_{ij}(\theta))/\partial\theta_l$,
$[G^{-1}]_{ij}$ = the (i,j) element of the inverse of G,
J_i = the ith row of J,
${}_l J'_i$ = the derivative of J_i with respect to θ_l.

Consider a 3 degrees of freedom manipulator in the (x, y) plane shown in figure 1 with $l_0 = 0.7$, $l_1 = 0.6$, $l_2 = 0.85$, $l_3 = 0.2$ (m). We assume that there is no hardware limit as for the rotation of all joints. The given task is to move the end point along a desired trajectory while trying to avoid the singularity by utilizing the redundancy.

Simulation results are given in figure 3. The initial joint vector was $\theta_I = [-90°, 175°, 0°]^T$, which corresponds to the end point position $r_I = [0.45, 0.84]^T$. The desired trajectory was given by

$$r^* = r_I - \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} t, \quad 0 \leqslant t \leqslant 10.$$

Figure 3a is the case where the simple pseudo-inverse control law

$$\dot{\theta} = J^+ \dot{r}^* \tag{5.3}$$

was applied. Figure 3b is the case where the proposed control law (4.8) with $k_1 = 5.0$ was used. Figure 4 shows the change of the performance criterion.

From the figures it can be seen that, although the conventional control law (5.3) has little tendency of not getting into the singularity [1], it has no ability of getting away from the singularity. This can also be understood from the fact that when the command $\dot{r}^* = 0$, θ remains the same even if the present state is quite close to the singularity. On the other hand, when the control law (4.8) is used, the state θ goes away quickly from the singularity. Figure 4 tells us that in about 4 sec the state is around the maximum manipulatability state for the required position of the end point.

6 Obstacle Avoidance

A control algorithm was developed in [9] for the case where the first subtask is a trajectory control and the second subtask is an obstacle avoidance. The main idea there was to take θ as the second manipulation vector and to teach in advance just one constant arm posture θ_r which is desirable for avoiding collision with the obstacle.

It will be shown in this section that the same idea can be formulated in the framework of section 4. Let the performance criterion for the second subtask be

$$p = g(\theta) = \tfrac{1}{2}(\theta - \theta_r)^T H_2 (\theta - \theta_r), \tag{6.1}$$

where $H_2 = \mathrm{diag}(h_{2i}) \in R^{n \times n}$, and $h_{2i} > 0$ are constants. The condition (6.1) means that the arm should try to come close to the taught arm posture θ_r as much as possible by utilizing the redundancy left after the realization of the first subtask.

From (6.1) we have

$$\xi = \partial g(\theta)/\partial \theta = -H_2(\theta - \theta_r). \tag{6.2}$$

From (4.8) and (6.2) we obtain

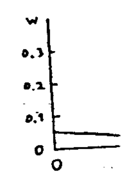
$$\dot{\theta} = J^+ \dot{r}^* - (I - J^+ J) H_2 (\theta - \theta_r) k_1. \tag{6.3}$$

Now we replace the desired velocity $\dot{r}^*$ by a modified desired velocity $\dot{r}^*_M$ given by

Skip

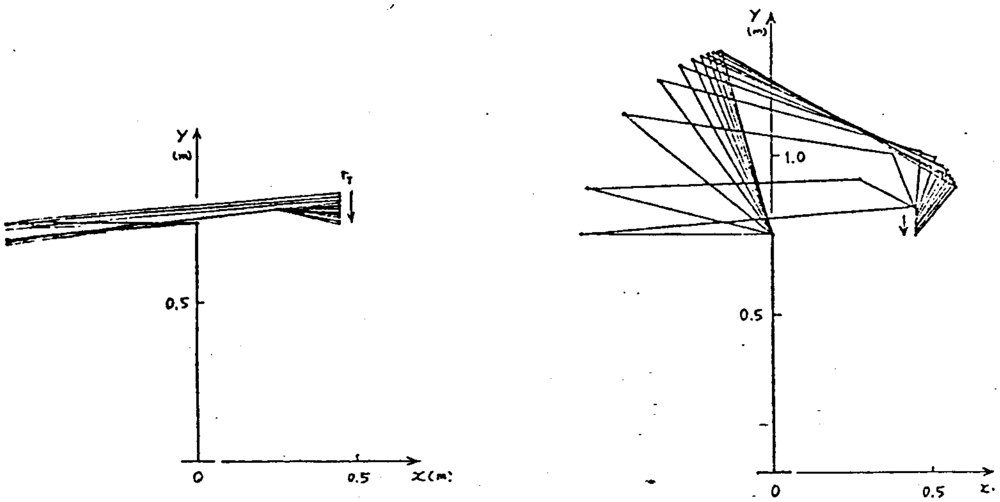
(a)

Figure 3
Simulation
(b) control



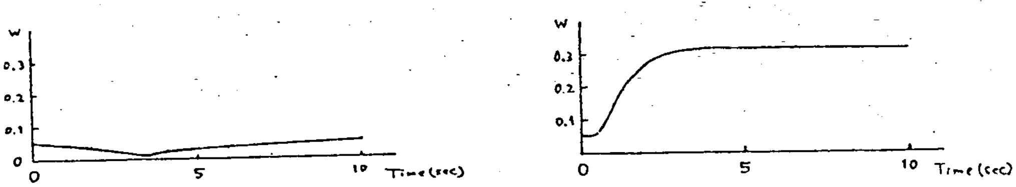
(a)

Figure 4
Measure



(a) (b)

Figure 3
Simulation result for singularity avoidance: (a) control law $\dot{\theta} = J^{+}\dot{r}^{*}$;
(b) control law $\dot{\theta} = J^{+}\dot{r}^{*} + [I - J^{+}J]\xi k_1$ with $k = 5.0$.



(a) (b)

Figure 4
Measure of manipulatability: (a) control law $\dot{\theta} = J^{+}\dot{r}^{*}$; (b) control law $\dot{\theta} = J^{+}\dot{r}^{*} + [I - J^{+}J]\xi k_1$.

$$\dot{r}_M^* = \dot{r}^* - H_1(r - r^*), \tag{6.4}$$

where $H_1 = \text{diag}(h_{1i}) \in R^{m \times m}$, and $h_{1i} > 0$. This modification is done for the purpose of coping with any possible error in the first subtask of tracking the trajectory $r^*(t)$ using the error feedback term $H_1(r - r^*)$. Therefore the control law is given by (6.4) and (6.3) with $\dot{r}^*$ replaced by $\dot{r}_M^*$.

This law is exactly the same as the one developed in the subsection "Example of Trajectory Control with Provisions for an Obstacle" of [9] (see (18) and (17) in [9]). Hence the experimental result reported in [9] can also be regarded as an example which shows the effectiveness of the approach in this section. This experimental result is introduced in the following.

The manipulator used in the experiment is an articulated arm, UJIBOT, with 7 degrees of freedom driven by dc servomortors. Figure 5 shows a comprehensive view of the UJIBOT system. The hand position is expressed by $r = [x, y, z]^T$ in the Cartesian coordinate system with the origin at the intersection of the first and second joint axes at the top.

The task is the following. From the initial state shown in figure 6a, which corresponds to $\theta_I = [2.48, -24.6, -22.1, 15.9, -29.1, 3.30, -49.4]^T$ (deg) and $r_I = [0.616, -0.348, -1.038]^T$ (m), the hand position is to be moved in the y direction at the constant rate of 0.1 (m/sec) for 6.58 sec.

Figure 6b and figure 6c show the experimental result when the combination of the simple pseudoinverse control law (5.3) and desired trajectory modification (6.4) with $H_1 = \text{diag}[3, 3, 3]$ and $k_1 = 1$ was used. Although the arm posture changed quite smoothly, the arm collided into the obstacle.

Since the arm has 4 redundant degrees of freedom, the obstacle avoidance is taken up as the second subtask. The reference arm posture θ_r shown in figure 7 was selected by an operator. The result of the utilization of control algorithm (6.3) and (6.4) with $H_1 = \text{diag}[3, 3, 3]$ and $H_2 = \text{diag}[0.3, 0.3, \ldots, 0.3]$ is shown in figure 8. The manipulator successfully avoided the collision with the obstacle.

7 Conclusion

The redundancy of robot manipulators has been studied in this paper. With the purpose of quantifying the ability of redundant manipulators in manipulating their end effectors, a measure of manipulatability has been proposed. This measure is applicable not only to redundant manipulators but also to nonredundant ones.

Based upon a task description as a set of subtasks with priority order, it has been pointed out that there are two different approaches for developing control algorithms of redundant manipulators. The first approach utilizes an additional performance criterion and the second approach specifies an additional desirable trajectory.

Attention has been focused on the first approach, and the basic control algorithm for this approach has been given along with its physical meaning. The application of this algorithm to two kinds of subtasks, i.e., singularity avoidance and obstacle avoidance, has been discussed.

Figure 5
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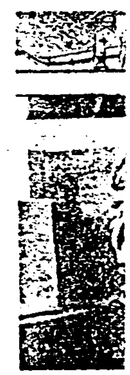


Figure 6
Trajectory

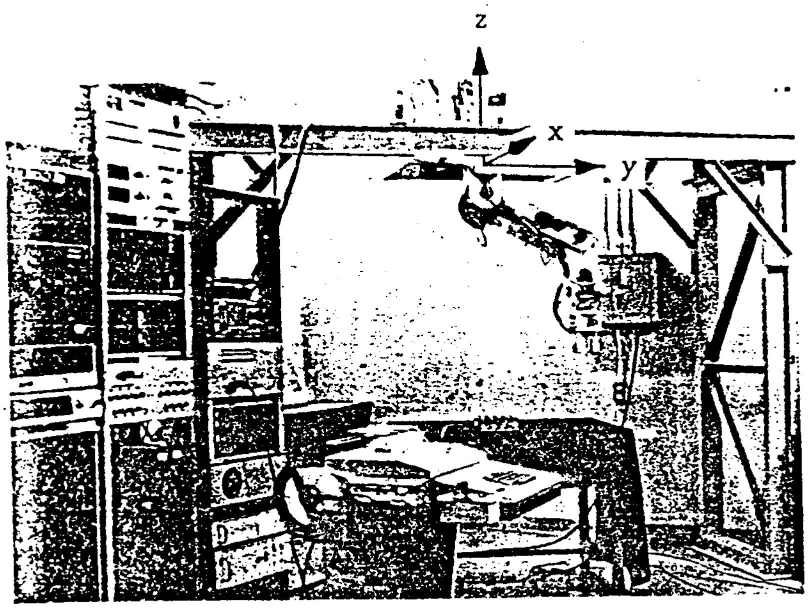


Figure 5
Comprehensive view of UJIBOT.

Figure 6
Trajectory control without provision for obstacle avoidance.

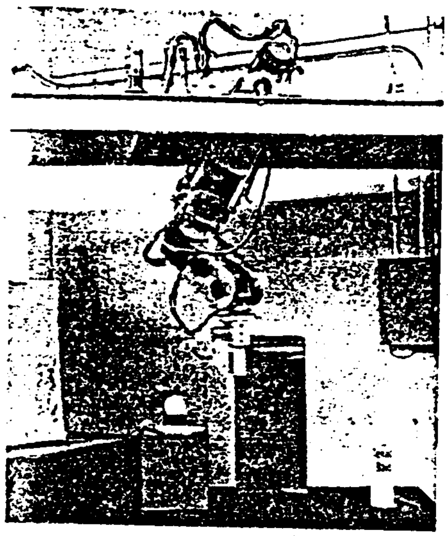
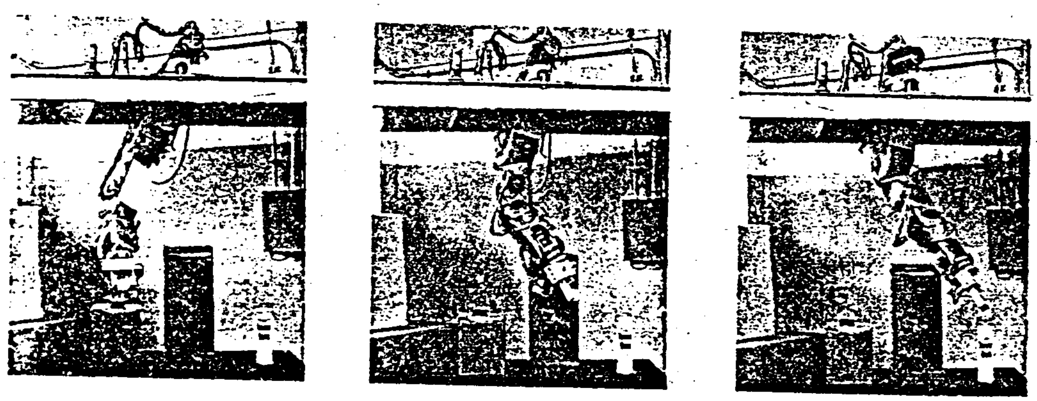


Figure 7
Reference arm posture θ_r.



(a) (b) (c)

Figure 8
Trajectory control with provision for obstacle avoidance.

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The concept of a manipulatability ellipsoid is now being studied and might be a topic of future papers.

Although results presented in this paper clearly show the effectiveness of active redundancy utilization, much more research and development will be necessary for the exploitation of the enormous potential ability of redundant manipulators.

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