

### 3.3 Reverse Position Analysis

The goal of the reverse position analysis is to obtain a relationship between the cartesian description of the manipulator end-effector and the manipulator degrees of freedom. With this relationship, given the position and orientation of a grasped pipe, the joint positions required to achieve this specification can be determined. The first three joint parameters  $\theta_1$ ,  $\theta_2$  and  $d_3$  will be determined using a geometric solution [Paul, 1981] and the wrist parameters  $\theta_6$ ,  $\theta_7$ , and  $\theta_8$  will be determined using an algebraic solution [Paul, 1981]. These type of analyses were used instead of a general recursive vector approach [Lipkin and Duffy, 1985] because they result in less complex expressions which often have physical interpretations.

$\theta_1$  is solved for by projecting the jaw pivot point "P" onto the global x-y plane as shown in Figure 3.4. From simple geometry, the first joint angle and the

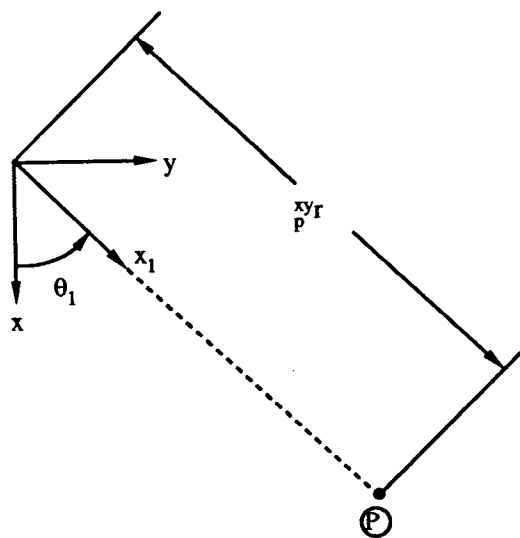


Figure 3.4 Manipulator Base Plane

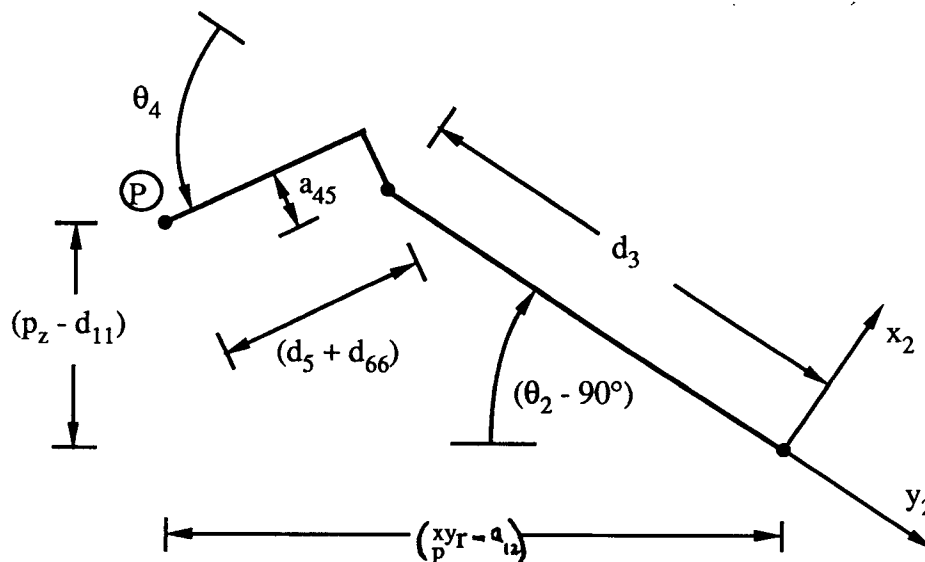
projected arm radius can be expressed as:

$$\theta_1 = \tan^{-1} \left( \frac{p_y}{p_x} \right) \quad (3.21)$$

$${}^{xy}r_p = \sqrt{p_x^2 + p_y^2} \quad (3.22)$$

$\theta_2$  and  $d_3$  are solved for by projecting the manipulator RPRP boom onto the frame 2 x-y plane as shown in Figure 3.5. The following loop closure equations are then used to determine these joint parameters:

$$\begin{aligned} ({}^{xy}r_p - a_{12}) &= d_3 c_{2-(90)} - a_{45} s_{2-4-(90)} + d_{566} c_{2-4-(90)} \\ &= d_3 s_2 + a_{45} c_{2-4} + d_{566} s_{2-4} \\ (p_z - d_{11}) &= d_3 s_{2-(90)} + a_{45} c_{2-4-(90)} + d_{566} s_{2-4-(90)} \\ &= -d_3 c_2 + a_{45} s_{2-4} - d_{566} c_{2-4} \end{aligned} \quad (3.23)$$



**Figure 3.5** Manipulator Boom Plane

Solving each loop closure equation for  $d_3$  yields:

$$\begin{aligned} d_3 &= \frac{1}{s_2} [(x_{y_{r_p}} - a_{12}) - d_{566}s_{2-4} - a_{45}c_{2-4}] \\ &= \frac{1}{c_2} [(d_{11} - p_z) - d_{566}c_{2-4} + a_{45}s_{2-4}] \end{aligned} \quad (3.24)$$

Defining:

$$\begin{aligned} \alpha &= (x_{y_{r_p}} - a_{12}) - d_{566}s_{2-4} - a_{45}c_{2-4} \\ \beta &= (d_{11} - p_z) - d_{566}c_{2-4} + a_{45}s_{2-4} \end{aligned} \quad (3.25)$$

Equation 3.24 may then be expressed as:

$$c_2\alpha - s_2\beta = 0 \quad (3.26)$$

Expanding this result yields:

$$c_2(x_{y_{r_p}} - a_{12} - d_{566}s_{2-4} - a_{45}c_{2-4}) - s_2(d_{11} - p_z - d_{566}c_{2-4} + a_{45}s_{2-4}) = 0$$

or

$$c_2(x_{y_{r_p}} - a_{12}) + s_2(p_z - d_{11}) + d_{566}s_4 - a_{45}c_4 = 0$$

or

$$a c_2 + b s_2 = d$$

where:

$$\begin{aligned} a &= x_{y_{r_p}} - a_{12} & b &= p_z - d_{11} \\ d &= a_{45}c_4 - d_{566}s_4 \end{aligned} \quad (3.27)$$

Using the standard closed-form solution of this trigonometric equation [Lipkin and Duffy, 1984] yields:

$$c_2 = \frac{ad \pm b(a^2 + b^2 - d^2)^{1/2}}{(a^2 + b^2)} \quad s_2 = \frac{bd \pm a(a^2 + b^2 - d^2)^{1/2}}{(a^2 + b^2)} \quad (3.28)$$

The values corresponding to the positive sign yields the solution which satisfies the physical constraints on  $\theta_2$  – the negative sign solution is physically unrealizable.

Therefore,

$$\theta_2 = \tan^{-1} \left( \frac{bd + ae}{ad - be} \right)$$

where:

$$e = (a^2 + b^2 - d^2)^{1/2} \quad (3.29)$$

Equation 3.24 is also used to solve for  $d_3$ :

$$s_2^2 + c_2^2 = 1$$

yields:

$$d_3 = (\alpha^2 + \beta^2)^{1/2} \quad (3.30)$$

To solve for the wrist degrees of freedom  $\theta_6$ ,  $\theta_7$ , and  $\theta_8$ , the overall manipulator transformation is separated into its positional and orientational components as:

$${}^0T_8 = {}^0T_p {}^pT_8 \quad (3.31)$$

where:

${}^0T_8$  = desired position and orientation of the jaws (specified)

${}^0T_p$  = position of the jaw pivot point (known from  $\theta_1$ ,  $\theta_2$ ,  $d_3$  solution)

${}^pT_8$  = orientation of the jaws (unknown)

The desired orientation information can be expressed in terms of known quantities as:

$${}^pT_8 = {}^0T_p^{-1} {}^0T_8 \quad (3.32)$$

where  ${}^0T_p^{-1}$  can be found from:

$$T = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} & r \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow T^{-1} = \begin{bmatrix} \hat{x}^T & -r \cdot \hat{x} \\ \hat{y}^T & -r \cdot \hat{y} \\ \hat{z}^T & -r \cdot \hat{z} \\ 0 & 1 \end{bmatrix} \quad (3.33)$$

Thus,

$${}^0T_p = \begin{bmatrix} n^p & o^p & a^p & p \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow {}^0T_p^{-1} = \begin{bmatrix} (n^p)^T & \\ (o^p)^T & p^{-1} \\ (a^p)^T & \\ 0^T & 1 \end{bmatrix} \quad (3.34)$$

where:

$$p^{-1} = \begin{pmatrix} p_x^{-1} \\ p_y^{-1} \\ p_z^{-1} \end{pmatrix} = \begin{pmatrix} -p \cdot n^p \\ -p \cdot o^p \\ -p \cdot a^p \end{pmatrix} = \begin{pmatrix} -a_{45} - d_{3s4} + a_{12}c_{2-4} - d_{11}s_{2-4} \\ 0 \\ -d_{566} - d_{3c4} + a_{12}s_{2-4} + d_{11}c_{2-4} \end{pmatrix} \quad (3.35)$$

Denoting the specified manipulator transformation as:

$${}^0T_8 = \begin{bmatrix} \hat{x}_8 & \hat{z}_8 \times \hat{x}_8 & \hat{z}_8 & r_p \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} n & o & a & p \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.36)$$

and using the expression for  ${}^0T_p^{-1}$  in Equation 3.34, the desired orientation transformation can now be expressed as:

$${}^pT_8 = {}^0T_p^{-1} {}^0T_8 = \begin{bmatrix} n^p \cdot n & n^p \cdot o & n^p \cdot a & n^p \cdot p + p_x^{-1} \\ o^p \cdot n & o^p \cdot o & o^p \cdot a & o^p \cdot p + p_y^{-1} \\ a^p \cdot n & a^p \cdot o & a^p \cdot a & a^p \cdot p + p_z^{-1} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.37)$$

Since this transformation represents orientation only, the expressions in the last column must be identically zero. This can be verified from the expression for  $\mathbf{p}^{-1}$  in Equation 3.35 – the components of  $\mathbf{p}^{-1}$  are in fact the negatives of the respective dot products.

The desired wrist joint angles can be calculated by comparing the matrix in Equation 3.37 with the previous expression for PT<sub>8</sub> given in Equation 3.14. Comparing the ratio of the "2,3" and "1,3" elements yields:

$$\begin{aligned}\theta_6 &= \tan^{-1} \left( \frac{s_6 s_7}{c_6 s_7} \right) = \tan^{-1} \left( \frac{\mathbf{OP} \cdot \mathbf{a}}{\mathbf{nP} \cdot \mathbf{a}} \right) \\ &= \tan^{-1} \left( \frac{(a_x s_1 - a_y c_1)}{c_{2-4} (a_x c_1 + a_y s_1) + a_z s_{2-4}} \right) = \tan^{-1} \left( \frac{\gamma}{\eta} \right)\end{aligned}\quad (3.38)$$

Comparing the "3,3" elements yields:

$$c_7 = -\mathbf{aP} \cdot \mathbf{a} = -s_{2-4} (a_x c_1 + a_y s_1) + a_z c_{2-4} = \chi \quad (3.39)$$

and comparing the "1,3" elements and the "2,3" elements yields:

$$s_7 = \frac{\mathbf{nP} \cdot \mathbf{a}}{c_6} = \frac{1}{c_6} (\delta c_{2-4} + a_z s_{2-4}) = \frac{\eta}{c_6} \quad (3.40)$$

and

$$s_7 = \frac{\mathbf{OP} \cdot \mathbf{a}}{s_6} = \frac{1}{s_6} (a_x s_1 - a_y c_1) = \frac{\gamma}{s_6} \quad (3.41)$$

where  $\theta_6$  is given by Equation 3.38. The possible mathematical singularities when  $\theta_6 = 0^\circ$  or  $90^\circ$  in these expressions can be avoided with the following manipulation:

$$\begin{aligned}c_6^2 + s_6^2 &= 1 \\ \Rightarrow \\ s_7 &= (\eta^2 + \gamma^2)^{1/2}\end{aligned}\quad (3.42)$$

Strictly, there are two solutions for  $s_7$  corresponding to the  $\pm$  values of the square root, but the physical range of motion of  $\theta_7$  prescribes a positive result. Using Equations 3.39 and 3.42, the desired joint angle can be expressed as:

$$\theta_7 = \tan^{-1} \left[ \frac{(\eta^2 + \gamma^2)^{1/2}}{\chi} \right] \quad (3.43)$$

Comparing the the ratio of the "3,2" and "3,1" elements yields:

$$\begin{aligned} \theta_8 &= \tan^{-1} \left( -\frac{s_7 s_8}{s_7 c_8} \right) = \tan^{-1} \left( -\frac{\mathbf{a}^P \cdot \mathbf{o}}{\mathbf{a}^P \cdot \mathbf{n}} \right) \\ &= \tan^{-1} \left[ -\frac{s_{2-4} (o_x c_1 + o_y s_1) - o_z c_{2-4}}{s_{2-4} (n_x c_1 + n_y s_1) - n_z c_{2-4}} \right] = \tan^{-1} \left( -\frac{\epsilon}{\zeta} \right) \end{aligned} \quad (3.44)$$

The parameters used in Equations 3.38 – 3.44 are summarized below:

$$\begin{aligned} \gamma &= a_x s_1 - a_y c_1 & \eta &= \delta c_{2-4} + a_z s_{2-4} \\ \delta &= a_x c_1 + a_y s_1 & \chi &= \delta s_{2-4} - a_z c_{2-4} \\ \epsilon &= s_{2-4} (o_x c_1 + o_y s_1) - o_z c_{2-4} \\ \zeta &= s_{2-4} (n_x c_1 + n_y s_1) - n_z c_{2-4} \end{aligned} \quad (3.45)$$

It should be pointed out that there are no mathematical singularities or multiple solutions whatsoever (within the physical ranges of motion of the joints) in any of the reverse position analyses presented in this section.