B

Laplace Transform

$$F(s) = \mathcal{L}\,[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

TABLE B.2 LAPLACE TRANSFORM THEOREMS

Name	Theorem
Derivative	$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0^+)$
*n*th-Order derivative	$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - s^{n-1} f(0^+) - \cdots f^{n-1}(0^+)$
Integral	$\mathcal{L}\left[\int_0^t f(\tau)\,d\tau\right] = \frac{F(s)}{s}$
Shifting	$\mathcal{L}\,[f(t-t_0)u(t-t_0)] = e^{-t_0 s}F(s)$
Initial value	$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$
Final value	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$
Frequency shift	$\mathcal{L}\,[e^{-at}f(t)] = F(s+a)$
Convolution integral	$\mathcal{L}^{-1}[F_1(s)F_2(s)] = \int_0^t f_1(t-\tau)f_2(\tau)\,d\tau = \int_0^t f_1(\tau)f_2(t-\tau)\,d\tau$

Time function $e(t)$	Laplace transform $E(s)$	z-Transform $E(z)$
$u(t)$	$\frac{1}{s}$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
$\frac{t^2}{2}$	$\frac{1}{s^3}$	$\frac{T^2 z(z+1)}{2(z-1)^3}$
t^{k-1}	$\frac{(k-1)!}{s^k}$	$\lim_{a\to 0}(-1)^{k-1}\frac{\partial^{k-1}}{\partial a^{k-1}}\left[\frac{z}{z-e^{-aT}}\right]$
e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
te^{-at}	$\frac{1}{(s+a)^2}$	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
$t^{k-1}e^{-at}$	$\frac{(k-1)!}{(s+a)^k}$	$(-1)^k\frac{\partial^k}{\partial a^k}\left[\frac{z}{z-e^{-aT}}\right]$
$1-e^{-at}$	$\frac{a}{s(s+a)}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$t-\frac{1-e^{-at}}{a}$	$\frac{a}{s^2(s+a)}$	$\frac{z[(aT-1+e^{-aT})z+(1-e^{-aT}-aTe^{-aT})]}{a(z-1)^2(z-e^{-aT})}$
$1-(1+at)e^{-at}$	$\frac{a^2}{s(s+a)^2}$	$\frac{z\{[1-e^{-aT}-aTe^{-aT}]z+[e^{-2aT}+(aT-1)e^{-aT}]\}}{(z-1)(z-e^{-aT})^2}$
$e^{-at}-e^{-bt}$	$\frac{b-a}{(s+a)(s+b)}$	$\frac{(e^{-aT}-e^{-bT})z}{(z-e^{-aT})(z-e^{-bT})^2}$
$\sin bt$	$\frac{b}{s^2+b^2}$	$\frac{z\sin bT}{z^2-2z\cos bT+1}$
$\cos bt$	$\frac{s}{s^2+b^2}$	$\frac{z(z-\cos bT)}{z^2-2z\cos bT+1}$
$t\sin bt$	$\frac{2bs}{(s^2+b^2)^2}$	
$t\cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$	
$e^{-at}\sin bt$	$\frac{b}{(s+a)^2+b^2}$	$\frac{ze^{-aT}\sin bT}{z^2-2ze^{-aT}\cos bT+e^{-2aT}}$
$e^{-at}\cos bt$	$\frac{s+a}{(s+a)^2+b^2}$	$\frac{z^2-ze^{-aT}\cos bT}{z^2-2ze^{-aT}\cos bT+e^{-2aT}}$
$1-e^{-at}\left(\cos bt+\frac{a}{b}\sin bt\right)$	$\frac{a^2+b^2}{s[(s+a)^2+b^2]}$	$\frac{z(Az+B)}{(z-1)(z^2-2ze^{-aT}\cos bT+e^{-2aT})}$

$$A = 1-e^{-aT}\left[\cos bT+\left(\frac{a}{b}\right)\sin bT\right]$$

$$B = e^{-2aT}+e^{-aT}\left[\left(\frac{a}{b}\right)\sin bT-\cos bT\right]$$

Definition

The **hyperbolic sine function**, denoted by **sinh**, and the **hyperbolic cosine function**, denoted by **cosh**, are given by

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

where x is any real number.

```
>> help residue
 RESIDUE Partial-fraction expansion (residues).
    [R,P,K] = RESIDUE(B,A) finds the residues, poles and direct term of
    a partial fraction expansion of the ratio of two polynomials B(s)/A(s).
    If there are no multiple roots,
       B(s)       R(1)       R(2)             R(n)
       ----  =  -------- + -------- + ... + -------- + K(s)
       A(s)     s - P(1)   s - P(2)         s - P(n)
    Vectors B and A specify the coefficients of the numerator and
    denominator polynomials in descending powers of s.  The residues
    are returned in the column vector R, the pole locations in column
    vector P, and the direct terms in row vector K.  The number of
    poles is n = length(A)-1 = length(R) = length(P). The direct term
    coefficient vector is empty if length(B) < length(A), otherwise
    length(K) = length(B)-length(A)+1.

    If P(j) = ... = P(j+m-1) is a pole of multplicity m, then the
    expansion includes terms of the form
                 R(j)        R(j+1)                R(j+m-1)
               -------- + ------------   + ... + ------------
               s - P(j)   (s - P(j))^2           (s - P(j))^m

    [B,A] = RESIDUE(R,P,K), with 3 input arguments and 2 output arguments,
    converts the partial fraction expansion back to the polynomials with
    coefficients in B and A.
```

Based on the preceding definitions, we may now state Mason's gain formula. The formula gives the transfer function from a source (input) node to a sink (output) node *only* and may be stated as

$$T = \frac{1}{\Delta}\sum_{k=1}^{p} M_k\Delta_k = \frac{1}{\Delta}(M_1\Delta_1 + M_2\Delta_2 + \cdots + M_p\Delta_p) \tag{2-19}$$

where T is the gain (transfer function) from the input node to the output node, p is the number of forward paths, and

$\Delta = 1 -$ (sum of all *individual* loop gains)

+ (sum of the products of the loop gains of all possible combinations of nontouching loops taken two at a time)

− (sum of the products of the loop gains of all possible combinations of nontouching loops taken three at a time)

+ (sum of the products of the loop gains of all possible combinations of nontouching loops taken four at a time)

− (...)

M_k = path gain of the kth forward path

Δ_k = value of Δ for that part of the flow graph not touching the kth forward path

4.1 TIME RESPONSE OF FIRST-ORDER SYSTEMS

In this section the time response of first-order systems is investigated. The transfer function of a general first-order system can be written as

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_0}{s + a_0} \tag{4-1}$$

where $R(s)$ is the input function and $C(s)$ is the output function; this notation is common.

A more common notation for the first-order transfer function is

$$G(s) = \frac{C(s)}{R(s)} = \frac{K}{\tau s + 1} \tag{4-2}$$

since physical meaning can be given to both K and τ. We call (4-2) the standard form of the first-order system. Of course, in (4-1) and (4-2),

$$a_0 = \frac{1}{\tau} \qquad b_0 = \frac{K}{\tau} \tag{4-3}$$

4.2 TIME RESPONSE OF SECOND-ORDER SYSTEMS

In this section we investigate the response of second-order systems to certain inputs. We assume that the system transfer function is of the form

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_0}{s^2 + a_1 s + a_0} \tag{4-17}$$

However, as in the first-order case, the coefficients are generally written in a manner such that they have physical meaning. The *standard form* of the second-order transfer function is given by

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4-18}$$

6.1 ROUTH–HURWITZ STABILITY CRITERION

The Routh–Hurwitz criterion is an analytical procedure for determining if all roots of a polynomial have negative real parts and is used in the stability analysis of linear time-invariant systems. The criterion gives the number of roots with positive real parts, and applies to all LTI systems for which the characteristic equation is a polynomial set to zero. This requirement excludes a system that contains an ideal time delay (transport lag). For this special case, which is covered later, the Routh–Hurwitz criterion cannot be employed.

The Routh–Hurwitz criterion applies to a polynomial of the form

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \tag{6-4}$$

where we can assume with no loss of generality that $a_0 \neq 0$. Otherwise, the polynomial can be expressed as a power of s multiplied by a polynomial in which $a_0 \neq 0$. The power of s indicates roots at the origin, the number of which is evident; hence, only the latter polynomial need be investigated using the Routh–Hurwitz criterion. We assume in the following developments that a_0 is not zero.

The first step in the application of the Routh–Hurwitz criterion is to form the array below, called the Routh array, where the first two rows are the coefficients of the polynomial in (6-4).

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$			
s^2	k_1	k_2			
s^1	l_1				
s^0	m_1				

The column with the powers of s is included as a convenient accounting method. The b row is calculated from the two rows directly above it, the c row, from the two rows directly above it, and so on. The equations for the coefficients of the array are as follows:

$$b_1 = \frac{-1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} \qquad b_2 = \frac{-1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}, \cdots$$

$$c_1 = \frac{-1}{b_1}\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix} \qquad c_2 = \frac{-1}{b_1}\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix}, \cdots \tag{6-5}$$

and so on. Note that the determinant in the expression for the ith coefficient in a row is formed from the first column and the $(i + 1)$ column of the two preceding rows.