Temperature Distribution Finite-Difference Boundary-Value Example

Bar and heat transfer info:

$L := 10$	length	$T_A := 40$	temperature at left end
$k := 0.001$	conduction/convection constant	$T_B := 200$	temperature at right end
$n := 20$	number of intervals	$T_a := 20$	ambient temperature

ODE:

$$\frac{d^2}{dx^2}T + k\cdot\left(T_a - T\right) = 0$$

BC's: $T(0) = T_A$ $T(L) = T_B$

Theoretical solution (using methods of Differential Equations):

$$\frac{d^2}{dx^2}T - k\cdot T = -k\cdot T_a$$

homogeneous solution:

$s^2 - k = 0$ $s = \sqrt{k}, -\sqrt{k}$

$$T_h(x) = C_1\cdot e^{\sqrt{k}x} + C_2\cdot e^{-\sqrt{k}\cdot x}$$

particular solution:

$$T_p(x) := T_a$$

general solution:

$$T(x) = T_h(x) + T_p(x) = C_1\cdot e^{\sqrt{k}\cdot x} + C_2\cdot e^{-\sqrt{k}\cdot x} + T_a$$

applying the BCs:

$$T(0) = T_A = C_1 + C_2 + T_a$$

$$T(L) = T_B = C_1\cdot e^{\sqrt{k}\cdot L} + C_2\cdot e^{-\sqrt{k}\cdot L} + T_a$$

using algebra:

$$C_1 := \frac{T_B - T_a - (T_A - T_a)\cdot e^{-\sqrt{k\cdot L}}}{\left(e^{\sqrt{k\cdot L}} - e^{-\sqrt{k\cdot L}}\right)} \qquad C_1 = 257.246$$

$$C_2 := T_A - T_a - C_1 \qquad C_2 = -237.246$$

using MathCAD:

Given

$$T_A = C_1 + C_2 + T_a$$

$$T_B = C_1\cdot e^{\sqrt{k\cdot L}} + C_2\cdot e^{-\sqrt{k\cdot L}} + T_a$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} := \mathrm{Find}(C_1, C_2) \rightarrow \begin{pmatrix} 257.24642704932558079 \\ -237.24642704932558079 \end{pmatrix}$$

<u>true analytical solution</u>:

$$T_t(x) := C_1\cdot e^{\sqrt{k\cdot x}} + C_2\cdot e^{-\sqrt{k\cdot x}} + T_a$$

<u>MathCAD numerical solution</u>:

Given

$$\frac{d^2}{dx^2}T(x) + k\cdot\left(T_a - T(x)\right) = 0$$

$$T(0) = T_A \qquad T(L) = T_B$$

$$T_M := \mathrm{Odesolve}(x, L)$$

ODE (finite difference form):

$\Delta x := \frac{L}{n}$ interval size

$\alpha := 2 + k \cdot \Delta x^2$ $\beta := k \cdot \Delta x^2 \cdot T_a$

$-T_{i-1} + \alpha \cdot T_i - T_{i+1} = \beta$ $i = 1 .. (n-1)$

Matrix form of finite difference equations:

$AT = B$

$i := 0 .. (n-2)$ $j := 0 .. (n-2)$ $m := 1 .. (n-2)$

$A_{i,j} := 0$ $A_{i,i} := \alpha$ $A_{m,m-1} := -1$ $A_{m-1,m} := -1$

$B_0 := \beta + T_A$ $B_m := \beta$ $B_{n-2} := \beta + T_B$

A =

	0	1	2	3	4	5	6	7	8	9
0	2	-1	0	0	0	0	0	0	0	0
1	-1	2	-1	0	0	0	0	0	0	0
2	0	-1	2	-1	0	0	0	0	0	0
3	0	0	-1	2	-1	0	0	0	0	0
4	0	0	0	-1	2	-1	0	0	0	0
5	0	0	0	0	-1	2	-1	0	0	0
6	0	0	0	0	0	-1	2	-1	0	0
7	0	0	0	0	0	0	-1	2	-1	0
8	0	0	0	0	0	0	0	-1	2	-1
9	0	0	0	0	0	0	0	0	-1	2

B =

	0
0	40.005
1	$5 \cdot 10^{-3}$
2	$5 \cdot 10^{-3}$
3	$5 \cdot 10^{-3}$
4	$5 \cdot 10^{-3}$
5	$5 \cdot 10^{-3}$
6	$5 \cdot 10^{-3}$
7	$5 \cdot 10^{-3}$
8	$5 \cdot 10^{-3}$
9	$5 \cdot 10^{-3}$

$X := A^{-1} \cdot B$ $T_0 := T_A$ $T_n := T_B$ $T_{i+1} := X_i$

$i := 0 .. n$ $\qquad x_i := i \cdot \Delta x$