Interpolation

Tabulated Data:

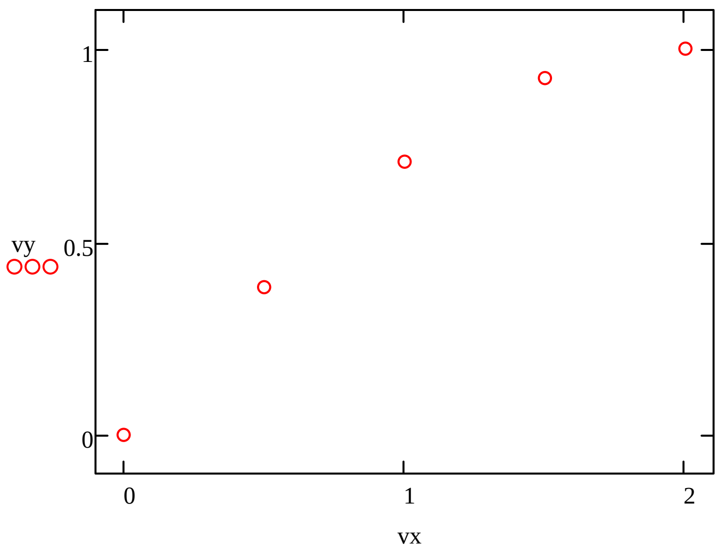
$$vx := \begin{pmatrix} 0 \\ 0.5 \\ 1 \\ 1.5 \\ 2 \end{pmatrix}$$

$$f(x) := \sin\left(\frac{\pi}{4}\cdot x\right)$$

$$i := 0..4$$

$$vy_i := f\left(vx_i\right)$$

$$vy = \begin{pmatrix} 0 \\ 0.383 \\ 0.707 \\ 0.924 \\ 1 \end{pmatrix}$$



we wish to estimate the y value for an untabulated x value of 0.7 (between 0.5 and 1)

Linear Interpolation:

$x := .7$

$x_0 := vx_1 \; y_0 := vy_1 \qquad x_1 := vx_2 \; y_1 := vy_2$

$$y\_linear := y_0 + \frac{y_1 - y_0}{x_1 - x_0}\cdot\left(x - x_0\right) \qquad y\_linear = 0.512$$

$$y_t := f(x) \qquad y_t = 0.522499$$

$$\varepsilon_t := \left|\frac{y_t - y\_linear}{y_t}\right| \qquad \varepsilon_t = 1.923\,\%$$

MathCAD method:

$linterp(vx, vy, .7) = 0.51245$ **Format-Result to 5 decimal places**

Quadratic Interpolation:

$x_2 := vx_3 \quad y_2 := vy_3$ adding another tabulated point

$$y\_quadratic := y\_linear + \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0} \cdot (x - x_0) \cdot (x - x_1) \qquad y\_quadratic = 0.525$$

$$\varepsilon_t := \left| \frac{y_t - y\_quadratic}{y_t} \right| \qquad \varepsilon_t = 0.55\,\%$$

Cubic Interpolation:

$x_3 := vx_4 \quad y_3 := vy_4$ adding another tabulated point

$$fdd1(x_i, x_j) := \frac{f(x_i) - f(x_j)}{x_i - x_j}$$

$$fdd2(x_i, x_j, x_k) := \frac{fdd1(x_i, x_j) - fdd1(x_j, x_k)}{x_i - x_k}$$

$$fdd3(x_3, x_2, x_1, x_0) := \frac{fdd2(x_3, x_2, x_1) - fdd2(x_2, x_1, x_0)}{x_3 - x_0}$$

$b_0 := f(x_0) \qquad b_0 = 0.383$

$b_1 := fdd1(x_1, x_0) \qquad b_1 = 0.649$

$b_2 := fdd2(x_2, x_1, x_0) \qquad b_2 = -0.215$

$b_3 := fdd3(x_3, x_2, x_1, x_0) \qquad b_3 = -0.044$

$y\_linear := b_0 + b_1 \cdot (x - x_0) \qquad y\_linear = 0.512$

$y\_quadratic := y\_linear + b_2 \cdot (x - x_0) \cdot (x - x_1)$

$y\_quadratic = 0.525$

$$y\_cubic := y\_quadratic + b_3 \cdot (x - x_0) \cdot (x - x_1) \cdot (x - x_2)$$

$$y\_cubic = 0.523$$

$$\varepsilon_t := \left| \frac{y_t - y\_cubic}{y_t} \right| \qquad \varepsilon_t = 0.145\,\%$$

Alternative (full) equation:

$$y\_cubic := b_0 + b_1 \cdot (x - x_0) + b_2 \cdot (x - x_0) \cdot (x - x_1) \ldots$$
$$+ b_3 \cdot (x - x_0) \cdot (x - x_1) \cdot (x - x_2)$$

use CTRL+Enter to continue a line

$$y\_cubic = 0.52326$$

Lagrange Interpolating Polynomials:

$$vx := (0.5 \quad 1 \quad 1.5 \quad 2)^T$$

$$L(vx, i, x, n) := \begin{array}{|l} L \leftarrow 1 \\ \text{for } j \in 0..n \\ \quad L \leftarrow L \cdot \dfrac{(x - vx_j)}{(vx_i - vx_j)} \quad \text{if } j \neq i \end{array}$$

checking some of the Lagrange factors [Li(x)=1 for x=xi; Li(x)=0 at x = xj where j<>i]:

$L(vx, 0, vx_0, 1) = 1$ $\quad L(vx, 0, vx_1, 2) = 0$ $\quad L(vx, 2, vx_2, 3) = 1$ $\quad L(vx, 2, vx_3, 3) = 0$

$$Lagrange\_poly(n, vx, x) := \sum_{i=0}^{n} \left(L(vx, i, x, n) \cdot f(vx_i) \right)$$

$$Lagrange\_poly(1, vx, .7) = 0.51245$$

$$Lagrange\_poly(2, vx, .7) = 0.52537$$

$$Lagrange\_poly(3, vx, .7) = 0.52326$$

Finding Coefficients of an Interpolation Polynomial:

$$f(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3$$

$$\begin{bmatrix} 1 & x_0 & (x_0)^2 & (x_0)^3 \\ 1 & x_1 & (x_1)^2 & (x_1)^3 \\ 1 & x_2 & (x_2)^2 & (x_2)^3 \\ 1 & x_3 & (x_3)^2 & (x_3)^3 \end{bmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$x := \begin{pmatrix} 0.5 \\ 1 \\ 1.5 \\ 2 \end{pmatrix} \qquad i := 0..3 \qquad y_i := f(x_i) \qquad y = \begin{pmatrix} 0.383 \\ 0.707 \\ 0.924 \\ 1 \end{pmatrix}$$

$$j := 0..3$$

$$A_{i,j} := (x_i)^j \qquad A = \begin{pmatrix} 1 & 0.5 & 0.25 & 0.125 \\ 1 & 1 & 1 & 1 \\ 1 & 1.5 & 2.25 & 3.375 \\ 1 & 2 & 4 & 8 \end{pmatrix}$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} := A^{-1} \cdot \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} \qquad \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -0.016 \\ 0.851 \\ -0.083 \\ -0.044 \end{pmatrix}$$

$$f(x) := \sum_i \left(a_i \cdot x^i\right)$$

$$f(.7) = 0.52326$$

same result as Newton and Lagrange cubic interpolations (because the polynomial is unique)

Comparing the cubic polynomial fit to the tabulated data and the theoretical function:

$xr := .5, .51 .. 2$ $\qquad f_t(x) := \sin\left(\frac{\pi}{4} \cdot x\right)$