Parameter Estimation from Compressed and Sparse Measurements

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May 11, 2015
Begin with a low-order parametric model for the measurements. The problem is to invert the measurements for the underlying parameters.

Estimates of parameters that linearly modulate the measurements (complex amplitudes of modes) and parameters that non-linearly modulate the measurements (frequency, wavenumber, delay, and/or doppler) are extracted, usually based on maximum likelihood, or some variation on linear prediction, using $\ell_2$-norm minimization.

Fisher Information, Cramér-Rao Bound (CRB), Kullback-Leibler divergence, Bayesian CRB, Threshold Effects are used to bound performance.
Question: How are fundamental limits for parameter estimation (i.e., Fisher Information, CRB, Threshold SNR, etc.) affected when original measurements are replaced by compressed ones?

Compression refers to raw subsampling or subsampled filtering.

Any compression of the measured image has consequences for resolution (or bias) and for variability (or variance).

Performance and performance bounds before and after compression of measurements.

Importantly, we treat compression of noisy data, which brings reality to the topic of compressive sampling.

Our results quantify the effects of compression on parameter estimation.
1. **Fisher Information:** What is the impact of compressive sampling on Fisher information and Cramér-Rao bound (CRB) for estimating parameters that nonlinearly modulate measurements?

**Answer** → We derive the probability distributions of the Fisher information matrix and the Cramér-Rao bound for specific random compression schemes.

**Impact** → These distributions can be used to determine whether compression is viable, and if so, at what compression ratios is the performance acceptable.
2. **Breakdown Thresholds**: What is the impact of compressive sampling on threshold SNRs at which mean-squared error in estimating parameters deviate sharply from the CRB?

**Answer** → We investigate threshold effects associated with the swapping of signal and noise subspaces in estimating signal parameters from compressed noisy data.

**Impact** → The results can be used to determine compression ratios at which the threshold SNR is suitably low.
3. **Alias Free Modal Estimation:** How can methods of linear prediction and approximate least squares be extended to modal analysis from sparsely sampled data?

**Answer** → We adapt methods of linear prediction and approximate least squares for estimating the parameters from sparse and co-prime arrays in a modal analysis problem.

**Impact** → Our results allow us to resolve aliasing ambiguities in modal analysis from sparse samples.
Publications

- **Journal Papers:**
Publications:

- **Conference Papers:**
Fisher Information and Cramér-Rao Bound

- **Observable:** $y \sim f(y; \theta)$, probabilistic model of remeasurements.
- **Fisher information matrix:**

$$
J(\theta) = E \left[ \left( \frac{\partial \log f(y; \theta)}{\partial \theta} \right) \left( \frac{\partial \log f(y; \theta)}{\partial \theta} \right)^H \right].
$$

- **Cramér-Rao Bound:** The inverse $J^{-1}(\theta)$ lower bounds the error covariance matrix for any unbiased estimator $\hat{\theta} = T(y)$ of the parameter vector $\theta$:

$$
C = E \left[ (\hat{\theta} - \theta)(\hat{\theta} - \theta)^H \right] \geq J^{-1}(\theta),
$$

provided that $E(\hat{\theta}) = \theta$. 
Complex normal measurement model:

\[ y = s(\theta) + n \in \mathbb{C}^n; \quad y \sim \mathcal{CN}_n[s(\theta), C] \]

Fisher information matrix:

\[ J(\theta) = G^H(\theta)C^{-1}G(\theta) \]
\[ = \frac{1}{\sigma^2} G^H(\theta)G(\theta), \quad \text{when} \quad C = \sigma^2 I \]

\[ G(\theta) = [g_1(\theta), \ldots, g_p(\theta)]; \quad g_i(\theta) = \frac{\partial s(\theta)}{\partial \theta_i} \]

Cramér-Rao lower bound:

\[ (J^{-1}(\theta))_{ii} = \sigma^2 (g_i^H(\theta)(I - P_{G_i(\theta)})g_i(\theta))^{-1} \]

When one sensitivity looks like a linear combination of others, performance is poor.
Compressive measurement:

\[ z = \Phi y = \Phi [s(\theta) + n] \in \mathbb{C}^m; \quad \Phi \in \mathbb{C}^{m \times n}, \quad m < n \]

Fisher information matrix

\[ \hat{J}(\theta) = \frac{1}{\sigma^2} G^H(\theta) P_{\Phi^H} G(\theta) = \hat{G}^H(\theta) \hat{G}(\theta) \]

\[ \hat{G}(\theta) = [\hat{g}_1(\theta), \ldots, \hat{g}_p(\theta)]; \quad \hat{g}_i(\theta) = P_{\Phi^H} \frac{\partial s(\theta)}{\partial \theta_i} \]

Cramér-Rao lower bound:

\[ (\hat{J}^{-1}(\theta))_{ii} = \sigma^2 (\hat{g}_i^H(\theta)(\mathbf{I} - P_{\hat{G}_i(\theta)})\hat{g}_i(\theta))^{-1} \]

Compressive measurement reduces the distance between subspaces: loss of information.
Question: What is the impact of compressive sampling on the Fisher information matrix and Cramér-Rao bound (CRB) for estimating parameters?
Consider linear compression with the class of random matrices \( \Phi \in \mathbb{C}^{m \times n} \) whose distributions are right-unitary invariant.

For this class, the pdf \( f_{\Phi}(\Phi) \) is invariant under the right-unitary transformation \( \Phi \rightarrow \Phi U \) for any \( n \times n \) unitary matrix \( U \).

Example: The rows of the compression matrix \( \Phi \) are independent, each having a spherical symmetric distribution such as:

- zero mean white Gaussian or mixture of Gaussian distributions
- zero mean white Laplace distribution
- zero mean white \( t \)-distribution
Theorem: [P., Pezeshki, Scharf, Cochran and Howard ’15]

For any right-unitary invariant random compression matrix, the normalized Fisher information matrix after compression $\mathbf{V} = \mathbf{J}^{-1/2} \hat{\mathbf{J}} \mathbf{J}^{-H/2}$ is distributed as a Type I complex multivariate beta distribution $\mathbb{C}B^I_p(m, n - m)$, with density

$$\frac{\tilde{\Gamma}_p(n)}{\tilde{\Gamma}_p(m)\tilde{\Gamma}_p(n - m)} |\mathbf{V}|^{m-p} |\mathbf{I}_p - \mathbf{V}|^{n-m-p} \quad \text{for} \quad 0 \leq \mathbf{V} \leq \mathbf{I}_p.$$

- The distribution is independent of the FIM before compression, and the parameter vector $\theta$. 
Sketch of the proof:

- For the class of right-unitary invariant distributions, the density function is of the form $g(\Phi \Phi^H)$ and $\Phi^H(\Phi \Phi^H)^{-1/2}$ is uniformly distributed on the Stiefel manifold $\mathcal{V}_m(\mathbb{C}^n)$.

- The distribution of the Fisher information matrix for this class is the same as the one in which the elements of $\Phi$ are i.i.d. $\mathcal{CN}(0, 1)$ random variables.

- Use matrix factorization in the MVN model to prove (See proof in the thesis, pages 9-12)
Remarks:

For any right-unitary invariant random compression matrix:

- The Fisher information matrix after compression \( \hat{J} \) has pdf

\[
\frac{\tilde{\Gamma}_p(n)}{\tilde{\Gamma}_p(m)\tilde{\Gamma}_p(n-m)} |J|^{p-n} |\hat{J}|^{m-p} |J - \hat{J}|^{n-m-p} \quad \text{for} \quad 0 \leq \hat{J} \leq J
\]

- The inverse of the Fisher information matrix after compression \( \hat{K} = \hat{J}^{-1} \) has pdf

\[
\frac{\tilde{\Gamma}_p(n)}{\tilde{\Gamma}_p(m)\tilde{\Gamma}_p(n-m)} |J|^{p-n} |\hat{K}|^{-n} |J\hat{K} - I_p|^{n-m-p} \quad \text{for} \quad \hat{K} \geq J^{-1}.
\]
Remarks:

For any right-unitary invariant random compression matrix:

- The normalized CRB after compression $\left( \hat{J}^{-1} \right)_{ii} / \left( J^{-1} \right)_{ii}$ is distributed as the inverse of a univariate beta random variable $B'(m - p + 1, n - m)$.

- The relative increase in the CRB due to compression $\frac{\left( \hat{J}^{-1} \right)_{ii} - \left( J^{-1} \right)_{ii}}{\left( J^{-1} \right)_{ii}}$ is distributed as $B'(n - m, m - p + 1)$.

- The average losses in the FIM and the inverse FIM (CRB) are:

$$E[\hat{J}] = \frac{m}{n} J,$$
$$E[\hat{J}^{-1}] = \frac{n - p}{m - p} J^{-1}.$$
Concentration Ellipses for the Fisher information matrices before and after compression:

Locus of all errors $\mathbf{e} = [e_1 \ e_2]^T$ for which $\mathbf{e}^T \mathbf{J} \mathbf{e} = r^2$ and $\mathbf{e}^T \hat{\mathbf{J}} \mathbf{e} = r^2$, with $r^2 = \mathbf{J}_{11}$. 
KL divergence between $\mathcal{CN}(x(\theta), C)$ and $\mathcal{CN}(x(\theta'), C)$:

$$D(\theta, \theta') = [(x(\theta) - x(\theta'))^H C^{-1}(x(\theta) - x(\theta'))].$$

- After compression with $\Phi$:

$$\hat{D}(\theta, \theta') = [(x(\theta) - x(\theta'))^H \Phi^H (\Phi C \Phi^H)^{-1} \Phi (x(\theta) - x(\theta'))].$$

- With white noise $C = \sigma^2 I$:

$$\hat{D}(\theta, \theta') = [(x(\theta) - x(\theta'))^H P_{\Phi^H} (x(\theta) - x(\theta'))].$$

**Theorem:** [P., Pezeshki, Scharf, Cochran and Howard '15]

The normalized KL divergence after compression $\frac{\hat{D}(\theta, \theta')}{D(\theta, \theta')}$ is distributed as $B^I(m, n - m)$. 

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**CS and Kullback-Leibler Divergence**

KL divergence between $\mathcal{CN}(x(\theta), C)$ and $\mathcal{CN}(x(\theta'), C)$:

$$D(\theta, \theta') = [(x(\theta) - x(\theta'))^H C^{-1}(x(\theta) - x(\theta'))].$$

- After compression with $\Phi$:

$$\hat{D}(\theta, \theta') = [(x(\theta) - x(\theta'))^H \Phi^H (\Phi C \Phi^H)^{-1} \Phi (x(\theta) - x(\theta'))].$$

- With white noise $C = \sigma^2 I$:

$$\hat{D}(\theta, \theta') = [(x(\theta) - x(\theta'))^H P_{\Phi^H} (x(\theta) - x(\theta'))].$$

**Theorem:** [P., Pezeshki, Scharf, Cochran and Howard '15]

The normalized KL divergence after compression $\frac{\hat{D}(\theta, \theta')}{D(\theta, \theta')}$ is distributed as $B^I(m, n - m)$. 

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**CS and Kullback-Leibler Divergence**
Recap

Contributions:
- We have derived distributions of the Fisher information matrix and the Cramér-Rao bound for specific random compression schemes. The results can be used as guidelines for choosing compression ratios, or to quantify the amount of loss due to compression.
- We have derived lower bounds on the probability of a subspace swap for parameter estimation from compressed noisy data. These bounds can be used to quantify the amount of increase in the threshold SNR due to compression.

Conclusions:
- Compression, whether by linear maps (eg, Gaussian or Laplace) or by subsampling (eg, co-prime) has performance consequences.
- The CR bound increases and the onset of performance breakdown (threshold SNR) increases. These increases may be quantified to determine where compressive sampling is viable.
Modal Analysis Using Sparse and Co-prime Arrays

- **Problem**: Estimating damped complex exponential modes and mode amplitudes from sparse/co-prime samples of their weighted sum.

- **Applications**:
  - DOA estimation in sensor array processing
  - Frequency and amplitude estimation in spectrum analysis
  - Range, doppler, and azimuth estimation in radar/sonar
Measurement Model

- Measurement model:

\[
y(t) = \sum_{\ell=1}^{p} x_{\ell} z_{\ell}^t + e(t) \quad \xrightarrow{\text{Sampling}} \quad y_k = \sum_{\ell=1}^{p} x_{\ell} z_{\ell}^{i_k} + e_k, \quad k = 0, 1, \ldots, m-1
\]

where \( z_{\ell} = \rho_{\ell} e^{i\theta_{\ell}} \) is a complex exponential mode with damping \( \rho_{\ell} \) and normalized frequency \( \theta_{\ell} \in (-\pi, \pi] \).

- Measurement vector:

\[
y = V(z, \mathbb{I})x + e
\]

where \( \mathbb{I} = \{i_0, i_1, \ldots, i_{m-1}\} \) is the set of sensor locations and

\[
V(z, \mathbb{I}) = \begin{bmatrix}
z_{i_0}^{i_0} & z_{i_0}^{i_1} & \cdots & z_{i_0}^{i_{m-1}} \\
z_{i_1}^{i_0} & z_{i_1}^{i_1} & \cdots & z_{i_1}^{i_{m-1}} \\
\vdots & \vdots & \ddots & \vdots \\
z_{i_{m-1}}^{i_0} & z_{i_{m-1}}^{i_1} & \cdots & z_{i_{m-1}}^{i_{m-1}}
\end{bmatrix}
\]
Maximum Likelihood and the Orthogonal Subspace

- **ML or LS estimation:**

  \[
  \min_{z,x} \| y - V(z, I)x \|^2_2
  \]

  \[\hat{x}_{ML} = V(z, I)^\dagger y\]

  \[\hat{z}_{ML} = \arg\min_z y^H P_{A(z)} y; \quad A^H V = 0\]

- For uniformly sampled data, Prony’s method (1795), linear prediction and Iterative Quadratic Maximum Likelihood (IQML) are used to solve exact ML or its modifications. Rank-reduction is used to reduce effects of noise.
Dense, Sparse, and Co-prime Sampling

- **Dense ULA:** $\mathbb{I}_u = \{0, 1 \ldots, m - 1\}$.

- **Sparse array:** $\mathbb{I}_s = \{0, d, \ldots, (m - 2)d, M\}$, and $(M, d) = 1$.

- **Co-prime array:** $\mathbb{I}_c = \mathbb{I}_1 \cup \mathbb{I}_2$, 
  $\mathbb{I}_1 = \{0, m_2, 2m_2, \ldots, (m_1 - 1)m_2\}$, $\mathbb{I}_2 = \{m_1, 2m_1, \ldots, (2m_2 - 1)m_1\}$, and $(m_1, m_2) = 1$. 
Orthogonal Subspace $\langle A \rangle$ for ULA

$\triangleright$ $m$ sensor ULA:

$$V(z, I_u) = \begin{bmatrix} z_0^1 & z_0^2 & \cdots & z_0^p \\ z_1^1 & z_1^2 & \cdots & z_1^p \\ \vdots & \vdots & \ddots & \vdots \\ z_{m-1}^1 & z_{m-1}^2 & \cdots & z_{m-1}^p \end{bmatrix}.$$  

$\triangleright$ Polynomial with roots $\{z_i\}_{i=1}^p$:

$$A(z) = \prod_{i=1}^p (1 - z_i z^{-1}) = \sum_{i=0}^p a_i z^{-i}, \quad a_0 = 1.$$  

$\triangleright$ Orthogonal subspace: $A^H V = 0$ where

$$A^H = \begin{bmatrix} a_p & a_{p-1} & \cdots & a_1 & 1 & \cdots & 0 \\ 0 & a_p & \cdots & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_p & \cdots & a_1 & 1 \end{bmatrix} \in \mathbb{C}^{(m-p) \times m}.$$
Orthogonal Subspace $\langle A \rangle$ for Sparse Array

- Sparse array with $m$ sensors, $\mathbb{I}_s = \{0, d, \ldots, (m - 2)d, M\}$:
  - Generalized Vandermonde matrix of the modes

$$V(z, \mathbb{I}_s) = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
z_1^d & z_2^d & \cdots & z_p^d \\
z_1^{2d} & z_2^{2d} & \cdots & z_p^{2d} \\
\vdots & \vdots & \ddots & \vdots \\
z_1^{(m-2)d} & z_2^{(m-2)d} & \cdots & z_p^{(m-2)d} \\
z_1^M & z_2^M & \cdots & z_p^M
\end{bmatrix} \in \mathbb{C}^{m \times p}$$

- **Question**: What is the characterization for the $(m - p)$-dimensional subspace orthogonal to the columns of $V(z, \mathbb{I}_s)$?
Consider the polynomial $A(z)$ with the roots $\{z_i^d\}_{i=1}^p$:

$$A(z) = \prod_{i=1}^p (1 - z_i^d z^{-1}) = \sum_{i=0}^p a_i z^{-i}, \quad a_0 = 1.$$

Suppose $\{z_i\}_{i=1}^p$ are roots of

$$B(z) = z^M + \sum_{i=1}^p b_i z^{(i-1)d}.$$

Note that

$$B(z_i e^{i2\pi k/d}) = z_i^M (e^{i2\pi Mk/d} - 1) \neq 0 \text{ for } 1 \leq k \leq d - 1.$$

Orthogonal subspace: $A^H V = 0$ where

$$A^H = [A_0 | \beta]^H = 
\begin{bmatrix}
a_p & a_{p-1} & \cdots & a_1 & 1 & 0 & \cdots & 0 \\
0 & a_p & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & a_p & \cdots & a_1 & 1 & 0 \\
b_p & b_{p-1} & \cdots & b_1 & 0 & \cdots & 0 & 1
\end{bmatrix}.$$
1. Estimate $\mathbf{a} = (a_1, a_2, \ldots, a_p)$ from

$$\hat{\mathbf{a}} = \arg\min_{\mathbf{a}} \mathbf{y}^H \mathbf{P}_{A_0(\mathbf{a})} \mathbf{y}$$

using IQML, modified least squares or similar methods.

2. Root $A(z) = \sum_{i=0}^{p} \hat{a}_i z^{-i}$ for $(w_1, \ldots, w_p)$, where $w_k = z_k^d$ for $k = 1, 2, \ldots, d$.

3. Form the set of candidate modes $\mathcal{R}$ by taking the $d$th roots of $(w_1, \ldots, w_p)$:

$$\mathcal{R} = \{(z_1 e^{j2\pi k_1/d}, z_2 e^{j2\pi k_2/d}, \ldots, z_p e^{j2\pi k_p/d}) \mid 0 \leq k_1, k_2, \ldots, k_p \leq d - 1\}.$$
Approximations to ML for the Sparse Array

\[ B(z) = z^M + \sum_{i=1}^{p} b_i z^{(i-1)d}, \]

\[ y_{m-1} = \sum_{\ell=1}^{p} x_{\ell} z_{\ell}^M + e_{m-1}, \]

\[ u = [y_0, y_1, \ldots, y_{p-1}]^T \text{ where } y_i = \sum_{\ell=1}^{p} x_{\ell} z_{\ell}^i + e_i. \]

4. Estimate \( \hat{b} \) from the following constrained linear prediction problem:

\[ \hat{b} = \arg \min_\zeta |y_{m-1} + \zeta^T u|^2 \quad s.t. \quad B_\zeta(z) = 0, \quad z \in \mathcal{R} \]

5. Form the polynomial \( B(z) \) from the solution \( \hat{b} \) in step 3.

6. Intersect \( \mathcal{R} \) and and the roots of \( B(z) \).
Numerical results: Performance of ML approximation

Estimating two closely spaced modes $z_1 = e^{j0.52}$ and $z_2 = 0.95e^{j0.69}$ using a ULA with 50 elements. Left: per sensor SNR = 0 dB. Right: per sensor SNR = −5 dB. Breakdown at −5 dB.
Numerical results: Performance of ML approximation

Estimating two closely spaced modes $z_1 = e^{j0.52}$ and $z_2 = 0.95e^{j0.69}$ using a sparse array with 14 elements, $d = 4$ and $M = 3$. Left: per sensor SNR = 5 dB. Right: per sensor SNR = 0 dB. Breakdown at 0 dB.
Summary

- Extension of methods of linear prediction for modal analysis from nonuniformly sampled (sparse and co-prime) time series and sensor array measurements.
- $2p$ parameterization of orthogonal subspaces for modal analysis.
- Alias-free reconstruction of complex exponential modes using IQML and constrained linear prediction.
Future Work

- Study of the distribution of the Fisher information matrix and the CRB after random compression, for the case that the parameters modulate the covariance of a complex multivariate normal model.
- Study of the effect of compressed sensing on Bayesian, Bhattacharyya and Weiss-Weinstein bounds.
- Study of the sensitivity of our parameterization of the orthogonal subspace for the sparse arrays to sensor location errors.
- Characterization of the orthogonal subspace for other sparse arrays such as nested arrays.
Breakdown Threshold and Subspace Swaps
Breakdown Threshold and Subspace Swaps

- **Threshold effect:** Sharp deviation of Mean Squared Error (MSE) performance from Cramér-Rao Bound (CRB).

- **Threshold SNR** SNR at which a threshold effect occurs with non-negligible probability.

- The main reason for this performance breakdown is known to be the occurrence of a subspace swap.

**Subspace Swap**: Event in which measured data is more accurately resolved by one or more modes of an *orthogonal subspace* to the signal subspace.

- Cares only about what the data itself is saying.
- Bound probability of a subspace swap to predict breakdown SNRs.

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Signal Model: Mean Case

Before compression:

\[ y : \mathcal{CN}_n[\mathbf{H}\alpha, \sigma^2 \mathbf{I}] \]

After compression with left-orthogonal \( \Phi \in \mathbb{C}^{m \times n}, m < n \):

\[ y : \mathcal{CN}_m[\Phi \mathbf{H}\alpha, \sigma^2 \mathbf{I}] \]

Typically \( \mathbf{H} \) is a Vandermonde matrix that carries modes in its columns.
Signal Model: Covariance Case

- Before compression:
  \[ y : \mathcal{CN}_n[0, HR_\alpha H^H + \sigma^2 I] \]

- After compression with left-orthogonal \( \Phi \in \mathbb{C}^{m \times n}, m < n \):
  \[ y : \mathcal{CN}_m[0, \Phi H R_\alpha H^H \Phi^H + \sigma^2 I] \]

Assume data consists of \( L \) iid realizations of \( y \) arranged as \( Y = [y_1, y_2, \cdots, y_L] \).
Subspace Swap Events

- **Subspace Swap Event** $E$: One or more components of the orthogonal subspace $\langle A \rangle$ resolves more energy than one or more modes of the noise-free signal subspace $\langle H \rangle$:

\[
E(q) : \min_{Q \in \mathcal{I}_{p,q}} \text{tr}(Y^H P H Q Y) < \max_{S \in \mathbb{C}^{(n-p) \times q}} \text{tr}(Y^H P A S Y),
\]

\[
E = \bigcup_{q=1}^{p} E(q).
\]
Subspace Swap Events

- **Subevent $F$:** Average energy resolved in the orthogonal subspace $\langle A \rangle$ is greater than the average energy resolved in the noise-free signal subspace $\langle H \rangle$:

\[
\min_i \text{tr}(Y^H P_{h_i} Y) \leq \frac{1}{p} \text{tr}(Y^H P_H Y) < \frac{1}{n - p} \text{tr}(Y^H P_A Y) \leq \max_i \text{tr}(Y^H P_{a_i} Y).
\]

Therefore, $F \subseteq E$ and $P(F) \leq P(E)$. 
Subspace Swap Events

- **Subevent $G$:** Energy resolved in the apriori minimum mode $h_{min}$ of the noise-free signal subspace $\langle H \rangle$ is smaller than the average energy resolved in the orthogonal subspace $\langle A \rangle$:

\[
\min_i \text{tr}(Y^H P_{h_i} Y) \leq \text{tr}(Y^H P_{h_{min}} Y) \\
< \frac{1}{n - p} \text{tr}(Y^H P_A Y) \\
\leq \max_i \text{tr}(Y^H P_{a_i} Y).
\]

Therefore, $G \subseteq E$ and $P(G) \leq P(E)$. 
Theorem: [P., Pezeshki, Scharf (GlobalSIP’13)]

- After compression:

\[
P_{ss} \geq 1 - P\left[ \frac{\text{tr}(Y^H \Phi H Y)}{p} > 1 \right] = 1 - P[F_{2pL,2(m-p)L}(\|\Phi H \alpha\|^2_2/\sigma^2) > 1]
\]

\(\|\Phi H \alpha\|^2_2/\sigma^2\) is the SNR after compression.
Theorem: [P., Pezeshki, Scharf (GlobalSIP'13)]

After compression:

\[ P_{ss} \geq 1 - P\left[ \frac{\text{tr}(\mathbf{Y}^H \mathbf{P}_{\mathbf{h}'} \mathbf{Y}/L)}{\text{tr}(\mathbf{Y}^H \mathbf{P}_A \mathbf{Y}/(m-p)L)} > 1 \right] \]

\[ = 1 - P[F_{2L,2(m-p)L} > \frac{1}{1 + \tau'^2/\sigma^2}]. \]

\[ \tau' = \| \mathbf{R}_{\alpha\alpha}^{1/2} \mathbf{H}^H \mathbf{P}^H \mathbf{h}'_{\min} \| \]

\( \tau'^2/\sigma^2 \): Effective SNR after compression

\( \mathbf{h}'_{\min} \) is the minimum mode of the signal subspace after compression

\( \mathbf{A}' \) is the \( m-p \) dimensional orthogonal subspace after compression
Dense array

\[ \frac{\lambda}{2} \]

\[ \begin{array}{cccccc}
0 & \bullet & \bullet & \bullet & \bullet & \bullet \\
& \longrightarrow & | & \leftarrow & & \\
\end{array} \]

\[ \begin{array}{ccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & (2m_2 - 1)m_1\frac{\lambda}{2} \\
& & \cdots & \cdots & \cdots & \cdots & \\
\end{array} \]

Co-prime compression [Pal and Vaidyanathan (2011)]

\[ \begin{array}{cccccc}
0 & \bullet & \bullet & \bullet & \bullet & \bullet \\
& \longrightarrow & | & \leftarrow & & \\
\end{array} \]

\[ \begin{array}{ccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & (m_1 - 1)m_2\frac{\lambda}{2} \\
& & \cdots & \cdots & \cdots & \cdots & \\
\end{array} \]

\[ \begin{array}{cccccc}
m_1\frac{\lambda}{2} & \bullet & \bullet & \bullet & \bullet & \bullet \\
& \longrightarrow & | & \leftarrow & & \\
\end{array} \]

\[ \begin{array}{ccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & (2m_2 - 1)m_1\frac{\lambda}{2} \\
& & \cdots & \cdots & \cdots & \cdots & \\
\end{array} \]

At \( m_1 = 11 \) and \( m_2 = 9 \), \( (2m_2 - 1)m_1\frac{\lambda}{2} = 187\frac{\lambda}{2} \).
Sensor Array Processing–Mean Case

Analytical lower bounds for the probability of subspace swap.

MSE and MSE bounds; Interfering source at $\theta_2 = \pi/188$; Average over 500 trials.
Analytical lower bounds for the probability of subspace swap.

MSE and MSE bounds; Interfering source at $\theta_2 = \pi/188$; Average over 500 trials.

Method of intervals: $\sigma_T^2 = P_{ss}\sigma_0^2 + (1 - P_{ss})\sigma_{CR}^2$
Analytical lower bounds for the probability of subspace swap.

MSE and MSE bounds; Interfering source at $\theta_2 = \pi/36$; 200 snapshots; Averaged over 500 trials.

Co-prime array with 12 elements ($m_1 = 5$ and $m_2 = 4$), and dense array of 36 elements.