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Noncausal Image Modeling Using Descriptor Approach

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Abstract—The problems of noncausal image modeling and subsequent image estimation are considered in this brief. The noncausal vector autoregressive (AR) model for the image process is arranged into a descriptor system. This system is then decomposed into backward and forward stable subsystems. The resulting subsystems are utilized to derive a Kalman filter by solving some types of discrete time algebraic Liapunov equations. A numerical example for noncausal image modeling is also presented.

I. INTRODUCTION

The problem of image modeling involves fitting an appropriate autoregressive (AR) or autoregressive moving average (ARMA) model to the image data. Unlike most 1-D signals that are typically time dependent, images are defined in spatial coordinates and thus causality is not a rigid requirement for the physical realizability of the image model. In fact, the region of support (ROS) of an image model can be causal, semicausal, or noncausal leading to initial value, initial-boundary value, or boundary-value problems [1], respectively.

In almost all the previous treatments [2]–[4], causal ROS was mainly used. However, by using causal models, typically close to half of the adjoining pixels in the ROS were ignored in the process of modeling. On the other hand, in noncausal modeling, correlational information is extracted from as many pixels surrounding a given pixel as possible and thus the correlations of a particular pixel with all spatially close pixels are reflected in the model. Therefore, one would expect that noncausal models provide better representation than the causal ones. This naturally leads to a more accurate filtering operation.

Until very recently image modeling and restoration using noncausal image and blur modeling was not fully investigated in the literature. However, recently there has been some interest in noncausal AR and

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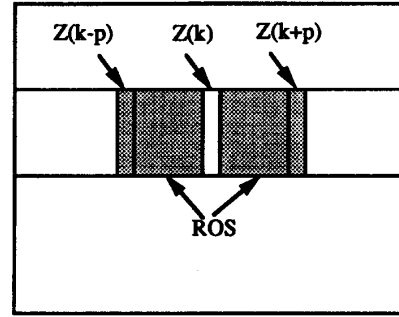


Fig. 1. ROS of noncausal model.

ARMA models mainly because noncausal models are more natural choices for many applications. In [5], [6] it is indicated that noncausal processes can be described by descriptor systems. Recursive processing of 1-D and 2-D noncausal systems is reported in [7]. In [8], 2-D spectrum estimation using noncausal models is presented. In [9], a technique for modeling and recursive state estimation for 2-D noncausal filters was proposed. Identification and restoration of images degraded by symmetric noncausal blur is treated in [10].

Perhaps the most natural way of dealing with noncausal discrete time models is through the descriptor form [5], [6]. The main goal of this brief is to develop a scheme using descriptor approach to represent an image modeled by a vector AR process with a noncausal ROS. The Kronecker canonical form decomposition is then applied to this descriptor system yielding two subsystems which perform equivalent to the original system. These subsystems may then be used to develop a Kalman filter for image restoration applications.

II. MODELING OF IMAGE AND DEGRADATION PROCESSES IN DESCRIPTOR FORM

Consider an $N \times N$ image which is scanned horizontally in strips of width W . It is assumed that the image is column wide-sense stationary [4] within each strip. Let $Z(k)$ denote the k -th, $k \in [0, N-1]$, image vector of size W in the i -th strip, i.e.,

$$Z(k) = [z_{(i-1)W,k}, z_{(i-1)W+1,k}, \dots, z_{iW-1,k}]^T$$

where $z_{m,n}$ represents the intensity value of the original image at location (m, n) and the superscript T denotes matrix transposition. Assume that the image process is modeled by a minimum variance representation (MVR) AR noncausal process of order p ,

$$Z(k) = \Phi_p^T Z(k-p) + \dots + \Phi_1^T Z(k-1) + \Phi_{-1}^T Z(k+1) + \dots + \Phi_{-p}^T Z(k+p) + U(k) \quad (1)$$

where $\Phi_{-p}, \Phi_{-p+1}, \dots, \Phi_p$ are constant $W \times W$ autoregressive parameter matrices. The ROS of this model is shown in Fig. 1.

The vector $U(k)$ is a $W \times 1$ zero mean noise vector driving the AR process. The necessary and sufficient conditions for the covariance matrix $Q_U := E[U(k)U^T(k)]$ to be minimum is that the orthogonal property, $E[U(k-l)Z^T(k)] = Q_U \delta(l)$, holds for $l \in [-p, p]$. This orthogonality principle leads to the following normal equations

$$\rho_r = \rho_{r-p} \Phi_p + \dots + \rho_{r-1} \Phi_1 + \rho_{r+1} \Phi_{-1} + \dots + \rho_{r+p} \Phi_{-p} + Q_U \delta(r) \quad (2)$$

for $r = -p, \dots, -1, 0, 1, \dots, p$, where $\rho_r := E[Z(k-r)Z^T(k)]$ which can be solved to yield the AR parameter matrices. Note that $\rho_{-r} = \rho_r^T$.

It should be noted that in contrary to the causal AR models which are driven by white noise process, noncausal MVR models are driven by colored noise process [1]. This can be shown by multiplying both sides of (1) by $U(k-l)$ and taking expectation, which yields

$$E[U(k-l)U^T(k)] = Q_U \{ \delta(l) - \sum_{\substack{i=-p \\ i \neq 0}}^p \Phi_i \delta(l-i) \}.$$

This implies that $U(k)$ is a colored process since $E[U(k-l)U^T(k)] \neq 0$ for $l \in [-p, p], l \neq 0$.

1) Descriptor State-Space Model

The noncausal vector AR model in (1) can be transformed into a descriptor system using the following local state vector

$$\bar{X}(k) = [x_p(k)^T, \dots, x_1^T(k), x_0^T(k), x_{-1}^T(k), \dots, x_{-p}^T(k)]^T \quad (3)$$

where $x_i(k) = Z(k-i)$ for $i = -p, \dots, p$. This state assignment yields the following "descriptor" or "singular" system [5], [6]

$$\begin{aligned} \bar{E}\bar{X}(k+1) &= \bar{A}\bar{X}(k) + \bar{B}U(k) \\ x_0(k) &= Z(k) = \bar{C}\bar{X}(k) \end{aligned} \quad (4a)$$

where (see (4b) below). $\bar{C} = [0 \ \dots \ 0 \ I_W \ 0 \ \dots \ 0]$ and $\bar{B} = [0 \ 0 \ \dots \ 0 \ I_W]^T$. Note that \bar{E} and \bar{A} are $(2p+1)W \times (2p+1)W$ matrices with \bar{E} being singular of rank $2p \times W$ and \bar{B}^T and \bar{C} are $W \times (2p+1)W$ matrices. It can be shown [10] that the vector AR model (1) and the descriptor system (4a) have the same transfer function matrix. A by-product of this result is that the pencil $z\bar{E} - \bar{A}$ is regular [11], since from (1) the determinant $|z\bar{E} - \bar{A}| \neq 0$. It can be shown that pW zeros of this polynomial $p(z) = |z\bar{E} - \bar{A}| = 0$ are inside the unit circle while the other pW zeros are outside the unit circle.

Although $U(k)$ in (4a) is not white, it can be modeled by a finite dimensional linear system [12] driven by a white process as

$$\begin{aligned} V(k+1) &= A_a V(k) + B_a w(k) \\ U(k) &= C_a V(k) \end{aligned} \quad (5)$$

where $w(k)$ is a zero mean white vector process and A_a, B_a and C_a are matrices of appropriate dimensions. Augmenting this system with system (4a) yields

$$\begin{aligned} EX(k+1) &= AX(k) + Bw(k) \\ Z(k) &= CX(k) \end{aligned} \quad (6)$$

where $X(k) := \begin{bmatrix} \bar{X}(k) \\ V(k) \end{bmatrix}$, $A := \begin{bmatrix} \bar{A} & \bar{B}C_a \\ 0 & A_a \end{bmatrix}$, $E := \begin{bmatrix} \bar{E} & 0 \\ 0 & I \end{bmatrix}$, $B := \begin{bmatrix} 0 \\ B_a \end{bmatrix}$ and $C := [\bar{C} \ 0]$.

The degradation due to a full-plane blur [9] and additive white Gaussian noise can be modeled as $y(k) = \bar{H}\bar{X}(k) + v(k)$, where $\{v(k)\}$ is zero mean white process and \bar{H} consists of the elements of the point spread function (PSF) of the full-plane blur. Combining the image and degradation processes gives the following descriptor system

$$\begin{aligned} EX(k+1) &= AX(k) + Bw(k) \\ y(k) &= HX(k) + v(k) \end{aligned} \quad (7)$$

where $H = [\bar{H} \ 0]$. In the state estimation for (7) the boundary conditions $X(0)$ and $X(N-1)$ are needed and can be taken to be $[y^T(0) \ \dots \ y(2p)^T, 0]^T$ and $[y^T(N-1-2p) \ \dots \ y(N-1)^T, 0]^T$ where $y(0), \dots, y(2p)$ and $y(N-1-2p), \dots, y(N-1)$ are the first and last $(2p+1)$ columns of the strip under consideration.

To develop state estimator for system (7) it is useful to transform the system into two stable subsystems using the Kronecker canonical form as described next.

2) Stable Forward-Backward Decomposition

Descriptor systems in general are not stable. In digital applications of descriptor systems it is necessary to decompose them into stable subsystems. This can be established as in the following theorem.

Theorem 1 [10]: Let $E, A \in \mathbb{R}^{n \times n}$, then there are two nonsingular real matrices P and Q such that $PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & A_b \end{bmatrix}$ and $PAQ = \begin{bmatrix} A_f & 0 \\ 0 & I_{n_2} \end{bmatrix}$ where $n_1, n_2 \geq 0, n_1 + n_2 = n$, and A_f and A_b are stable matrices.

Note that if $|zE - A|$ has no zeros on the unit circle, then the matrices A_f and A_b in Theorem 1 are strictly stable. Applying this decomposition to the system (7) yields

$$\begin{aligned} X_f(k+1) &= A_f X_f(k) + B_f w(k) \\ X_b(k) &= A_b X_b(k+1) - B_b w(k) \\ y(k) &= H_f X_f(k) + H_b X_b(k) + v(k) \end{aligned} \quad (8)$$

where $\begin{bmatrix} X_f(k) \\ X_b(k) \end{bmatrix} := Q^{-1}X(k)$, $\begin{bmatrix} B_f \\ B_b \end{bmatrix} := PB$ and $[H_f \ H_b] := HQ$.

This system is equivalent to the following forward and backward subsystems, respectively, propagating in opposite directions:

$$\begin{aligned} X_f(k+1) &= A_f X_f(k) + B_f w(k) \\ y(k) &= H_f X_f(k) + v_f(k) \end{aligned} \quad (9a)$$

and

$$\begin{aligned} X_b(k) &= A_b X_b(k+1) - B_b w(k) \\ y(k) &= H_b X_b(k) + v_b(k). \end{aligned} \quad (9b)$$

Note that the measurement noise in the forward and backward subsystems (9a) and (9b) are $v_f(k) = H_b X_b(k) + v(k)$ and $v_b(k) = H_f X_f(k) + v(k)$, respectively, which are colored.

The following example illustrates how a noncausal vector model can be decomposed into stable forward-backward subsystems using the method developed in this paper.

Example: Let us consider a zero-mean stationary image with autocorrelation function $r_z(j, l) = E[z_{i-j, k-l} z_{i, k}] = \rho^{|j|+|l|}$, where $|\rho| < 1$. This image is to be represented by a first-order noncausal vector AR model with $W = 2$. The parameter matrices can be determined by solving the following normal equations for $r = -1, 0, 1$

$$\rho_r = \rho_{r-1}\Phi_1 + \rho_{r+1}\Phi_1 + Q_U \delta(r)$$

$$\bar{E} = \begin{pmatrix} I_W & 0 & \dots & 0 & 0 \\ 0 & I_W & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I_W & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} 0 & I_W & 0 & \dots & 0 & 0 \\ 0 & 0 & I_W & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots & I_W \\ \Phi_p^T & \Phi_{p-1}^T & \dots & -I_W & \dots & \Phi_{-p}^T \end{pmatrix} \quad (4b)$$

where $\rho_r := E[Z(k-r)Z^T(k)]$ and $Z(k) = [z_{2i-2,k}, z_{2i-1,k}]^T$, i.e.,

$$\begin{aligned}\rho_1 &= \rho_0 \Phi_1 + \rho_2 \Phi_{-1} \\ \rho_0 &= \rho_{-1} \Phi_1 + \rho_1 \Phi_{-1} + Q_U \\ \rho_{-1} &= \rho_{-2} \Phi_1 + \rho_0 \Phi_{-1}.\end{aligned}\quad (10)$$

Note that the correlation matrices ρ_r are given by $\rho_r = \begin{bmatrix} \rho^{|r|} & \rho^{|r|+1} \\ \rho^{|r|+1} & \rho^{|r|} \end{bmatrix} = \rho^{|r|} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$ and $\rho_{-r} = \rho_r^T$.

The matrices ϕ_1 and ϕ_{-1} are solutions of the following system

$$\begin{bmatrix} \rho_0 & \rho_2 \\ \rho_2^T & \rho_0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_{-1} \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_1^T \end{bmatrix}$$

from which it follows that

$$\begin{bmatrix} \phi_1 \\ \phi_{-1} \end{bmatrix} = \begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 \\ \rho & 1 & \rho^3 & \rho^2 \\ \rho^2 & \rho^3 & 1 & \rho \\ \rho^3 & \rho^2 & \rho & 1 \end{bmatrix}^{-1} \begin{bmatrix} \rho & \rho^2 \\ \rho^2 & \rho \end{bmatrix} = \frac{\rho}{1+\rho^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

hence $\phi_1 = \phi_{-1} = \frac{\rho}{1+\rho^2} I$. Solving the third equation of (10) for Q_U yields $Q_U = \frac{1-\rho^2}{1+\rho^2} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$. As a result, the first order noncausal vector AR model with $W = 2$ has the following form

$$Z(k) = \begin{bmatrix} \frac{\rho}{1+\rho^2} & 0 \\ 0 & \frac{\rho}{1+\rho^2} \end{bmatrix} Z(k-1) + \begin{bmatrix} \frac{\rho}{1+\rho^2} & 0 \\ 0 & \frac{\rho}{1+\rho^2} \end{bmatrix} Z(k+1) + U(k)$$

where $U(k) = [u_{2i-2,k}, u_{2i-1,k}]^T$. Note that this vector model is equivalent to two 1-D models driven by colored noise. Arranging this model into a descriptor system gives

$$\begin{aligned}\bar{E}\bar{X}(k+1) &= \bar{A}\bar{X}(k) + \bar{B}U(k) \\ Z(k) &= \bar{C}\bar{X}(k)\end{aligned}$$

where $\bar{X}(k) = [Z^T(k-1), Z^T(k), Z^T(k+1)]^T$, (see first matrix at the bottom of the page), $\bar{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$ and

$$\bar{C} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The generalized eigenvalues which are solutions of $|\lambda\bar{E} - \bar{A}| = 0$ are found to be $\lambda = \rho, \rho, \frac{1}{\rho}, \frac{1}{\rho}, 0, 0$, i.e., four eigenvalues are inside the unit circle while the other two are outside. Using the procedure in [11] and the results of Theorem 1 we can define transformation matrices P and Q such that the descriptor system

for the noncausal image model is decomposed into stable forward-backward subsystems. These matrices are found to be (See second matrix at the bottom of the page.)

Now, it can easily be checked that $P\bar{E}Q = \begin{bmatrix} I_2 & 0 \\ 0 & \bar{A}_b \end{bmatrix}$ and $P\bar{A}Q = \begin{bmatrix} \bar{A}_f & 0 \\ 0 & I_4 \end{bmatrix}$, where $\bar{A}_f = \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix}$, $\bar{A}_b = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

I_2 and I_4 are 2×2 and 4×4 identity matrices, respectively. Therefore, we obtain the following decoupled stable subsystems

$$\bar{X}_f(k+1) = \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix} \bar{X}_f(k) + \begin{bmatrix} \frac{1+\rho^2}{\rho} & 0 \\ 0 & \frac{1+\rho^2}{\rho} \end{bmatrix} U(k)$$

$$\bar{X}_b(k) = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \bar{X}_b(k+1)$$

$$+ \begin{bmatrix} \frac{1+\rho^2}{1-\rho^2} & 0 \\ 0 & \frac{1+\rho^2}{1-\rho^2} \\ -\frac{\rho(1+\rho^2)}{1-\rho^2} & 0 \\ 0 & -\frac{\rho(1+\rho^2)}{1-\rho^2} \end{bmatrix} U(k)$$

$$Z(k) = \begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{1}{\rho} \end{bmatrix} \bar{X}_f(k) + \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \end{bmatrix} \bar{X}_b(k)$$

where $\begin{bmatrix} \bar{X}_f \\ \bar{X}_b \end{bmatrix} = Q^{-1}\bar{X}(k)$ and

$$Q^{-1} = \begin{bmatrix} \frac{1}{1-\rho^2} & 0 & \frac{-\rho}{1-\rho^2} & 0 & 0 & 0 \\ 0 & \frac{1}{1-\rho^2} & 0 & \frac{-\rho}{\rho(1-\rho^2)} & 0 & 0 \\ \frac{-\rho^2}{1-\rho^2} & 0 & \frac{\rho}{1-\rho^2} & 0 & 0 & 0 \\ 0 & \frac{-\rho^2}{1-\rho^2} & 0 & \frac{\rho}{1-\rho^2} & 0 & 0 \\ 1 & 0 & -\frac{1+\rho^2}{\rho} & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1+\rho^2}{\rho} & 0 & 1 \end{bmatrix}.$$

As can be observed, all eigenvalues of the forward and backward subsystems are at ρ , ($|\rho| < 1$), with the exception of two eigenvalues of the backward subsystem which are at the origin of the unit circle, i.e., deadbeat. In addition, since no approximation is used in computing P and Q matrices, the subsystems preserve all the correlational information of the original model.

$$\bar{E} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\rho}{1+\rho^2} & 0 & -1 & 0 & \frac{\rho}{1+\rho^2} & 0 \\ 0 & \frac{\rho}{1+\rho^2} & 0 & -1 & 0 & \frac{\rho}{1+\rho^2} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{1-\rho^2} & 0 & \frac{-\rho}{1-\rho^2} & 0 & \frac{1+\rho^2}{1-\rho^2} & 0 \\ 0 & \frac{1}{1-\rho^2} & 0 & \frac{-\rho}{1-\rho^2} & 0 & \frac{1+\rho^2}{1-\rho^2} \\ \frac{-\rho^3}{1-\rho^2} & 0 & \frac{\rho^2}{1-\rho^2} & 0 & \frac{-\rho(1+\rho^2)}{1-\rho^2} & 0 \\ 0 & \frac{-\rho^3}{1-\rho^2} & 0 & \frac{\rho^2}{1-\rho^2} & 0 & \frac{-\rho(1+\rho^2)}{1-\rho^2} \\ 0 & 0 & 0 & 0 & \frac{1+\rho^2}{\rho} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1+\rho^2}{\rho} \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \rho & 0 & \frac{1}{\rho} & 0 & 0 & 0 \\ 0 & \rho & 0 & \frac{1}{\rho} & 0 & 0 \\ \rho^2 & 0 & \frac{1}{\rho^2} & 0 & 1 & 0 \\ 0 & \rho^2 & 0 & \frac{1}{\rho^2} & 0 & 1 \end{bmatrix}$$

Even though the above procedure is only applied to the image model and the effects of colored noise and the degradation processes are not considered, the general approach of this brief can be utilized to decompose the complete system (7) into forward and backward subsystems and then derive the corresponding reduced order Kalman filters for image restoration applications.

III. STATE ESTIMATION FOR FORWARD-BACKWARD SUBSYSTEMS

In this section, we formulate the problem of state estimation for subsystems (9) into standard reduced order Kalman filters. To achieve this, state space models for the colored processes $v_f(k)$ and $v_b(k)$ are needed.

We start by obtaining the statistics of the processes $v_f(k)$ and $v_b(k)$ which can be computed using only the statistics of the zero mean white processes, $\{w(k)\}$ and $\{v(k)\}$, which are assumed to be known. Let $P_f := E[X_f(k)X_f^T(k)]$ and $P_b := E[X_b(k)X_b^T(k)]$ be the forward and backward state covariances, respectively. Then, using the state equation in (9a) and assuming stationarity, it can easily be shown that P_f is the positive definite solution of the discrete Liapunov matrix equation $P_f = A_f P_f A_f^T + B_f Q_w B_f^T$ which is given by

$$P_f = \sum_{k=0}^{\infty} A_f^k B_f Q_w B_f^T A_f^{T k} \quad (11)$$

where $Q_w := E[w(k)w^T(k)]$. The limit of this summation exists as A_f is stable [8]. As a result, we have $\rho_{if} := E[v_f(k)v_f^T(k+i)] = Q_v \delta(i) + H_b A_b^i P_b H_b^T$ for $i = 0, 1, 2, \dots$.

Similarly using the backward state equation in (9b) yields $P_b = A_b P_b A_b^T + B_b Q_w B_b^T$. Again the stability of A_b ensures the existence of the following solution

$$P_b = \sum_{k=0}^{\infty} A_b^k B_b Q_w B_b^T A_b^{T k} \quad (12)$$

It can readily be verified that $\rho_{ib} := E[v_b(k)v_b^T(k+i)] = Q_v \delta(i) + H_f P_f A_f^T H_f^T$ for $i = 0, 1, \dots$.

Having determined the statistics of $v_f(k)$ and $v_b(k)$, we now model these processes by a q -th and s -th order AR models, respectively. Thus, for $v_f(k)$ we have

$$v_f(k) = \phi_1^T v_f(k-1) + \dots + \phi_q^T v_f(k-q) + \eta_f(k) \quad (13)$$

where $\eta_f(k)$ is a zero mean white noise process with covariance matrix Q_{η_f} . The parameters ϕ_1, \dots, ϕ_q are obtained by solving the following Yule-Walker system of equations

$$\begin{bmatrix} \rho_{0f} & \rho_{1f}^T & \dots & \rho_{qf}^T \\ \rho_{1f} & \rho_{0f} & \dots & \rho_{q-1f}^T \\ \dots & \dots & \dots & \dots \\ \rho_{qf} & \rho_{q-1f} & \dots & \rho_{0f} \end{bmatrix} \begin{bmatrix} -I \\ \phi_1 \\ \vdots \\ \phi_{q-1} \\ \phi_q \end{bmatrix} = \begin{bmatrix} -Q_{\eta_f} \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad (14)$$

Once the model parameters are identified, the following state vector can be defined

$$Z_f(k)^T = [v_f(k)^T, v_f(k-1)^T, \dots, v_f(k-q+1)^T].$$

Then (13) can be written in the following state space form

$$\begin{aligned} Z_f(k+1) &= F_f Z_f(k) + G_f \eta_f(k) \\ v_f(k) &= C_f Z_f(k) \end{aligned} \quad (15)$$

where

$$F_f = \begin{bmatrix} \phi_1^T & \phi_2^T & \dots & \phi_{q-1}^T & \phi_q^T \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I & 0 \end{bmatrix} \quad (16)$$

$$G_f = [I \ 0 \ \dots \ 0]^T$$

$$C_f = [I \ 0 \ \dots \ 0].$$

Augmenting systems (9a) and (15) yields

$$\begin{aligned} \begin{bmatrix} X_f(k+1) \\ Z_f(k+1) \end{bmatrix} &= \begin{bmatrix} A_f & 0 \\ 0 & F_f \end{bmatrix} \begin{bmatrix} X_f(k) \\ Z_f(k) \end{bmatrix} + \begin{bmatrix} B_f & 0 \\ 0 & G_f \end{bmatrix} \begin{bmatrix} w(k) \\ \eta_f(k) \end{bmatrix} \\ y(k) &= [H_f \ C_f] \begin{bmatrix} X_f(k) \\ Z_f(k) \end{bmatrix}. \end{aligned} \quad (17)$$

Similarly, we can model the process $v_b(k)$ by a finite dimensional linear system of order s ,

$$v_b(k) = \psi_1^T v_b(k+1) + \dots + \psi_s^T v_b(k+s) + \eta_b(k) \quad (18)$$

where $\eta_b(k)$ is a zero mean white noise process with covariance matrix Q_{η_b} . An analogous Yule-Walker system of equations can be arranged to yield the parameter matrices ψ_1, \dots, ψ_s . Once the model parameters are identified, the following state vector can be defined

$$Z_b(k)^T = [v_b(k)^T, v_b(k+1)^T, \dots, v_b(k+s-1)^T].$$

Then (18) can be arranged into the following state space system

$$\begin{aligned} Z_b(k) &= F_b Z_b(k+1) + G_b \eta_b(k) \\ v_b(k) &= C_b Z_b(k) \end{aligned} \quad (19)$$

where

$$F_b = \begin{bmatrix} \psi_1^T & \psi_2^T & \dots & \psi_{s-1}^T & \psi_s^T \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I & 0 \end{bmatrix}$$

$$G_b = [I \ 0 \ \dots \ 0]^T$$

$$C_b = [I \ 0 \ \dots \ 0].$$

Augmenting systems (9b) and (19) yields

$$\begin{aligned} \begin{bmatrix} X_b(k) \\ Z_b(k) \end{bmatrix} &= \begin{bmatrix} A_b & 0 \\ 0 & F_b \end{bmatrix} \begin{bmatrix} X_b(k+1) \\ Z_b(k+1) \end{bmatrix} + \begin{bmatrix} -B_b & 0 \\ 0 & G_b \end{bmatrix} \begin{bmatrix} w(k) \\ \eta_b(k) \end{bmatrix} \\ y(k) &= [H_b \ C_b] \begin{bmatrix} X_b(k) \\ Z_b(k) \end{bmatrix}. \end{aligned} \quad (20)$$

As can be seen, both systems in (17) and (20) are driven by zero mean white processes and further there is no measurement noise in the output equations. Consequently, the state vectors $X_f(k)$ and $X_b(k)$ must be estimated using standard reduced order estimators. A treatment of reduced order estimators can be found in [12].

IV. CONCLUSION

The problem of noncausal vector modeling of images is addressed in this brief. The noncausal vector image model was arranged into descriptor system which was in turn decomposed into equivalent stable forward and backward subsystems. These subsystems were used to design Kalman filters in which the forward state was estimated with the backward state considered as a colored output noise and similarly the backward state was estimated with the forward state regarded as colored output noise. The estimation of the forward and backward states can then be accomplished using standard reduced order estimators. A numerical example was given to demonstrate the decomposition process into stable forward and backward subsystems.

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Discrete Orthogonal Polynomial Deconvolution for Time-Varying Systems

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Abstract—Discrete orthogonal polynomial deconvolution (DOPD) has been demonstrated to be a robust method for obtaining the inverse solution for time-invariant systems. In this communication, extension of the method to time-varying linear systems is explored. The operator-based nature of DOPD lends itself to application to linear time-varying systems expressible as an operator matrix. The stability and noise tolerance characteristics of time-invariant DOPD are demonstrated to apply to time-varying systems. *A priori* estimation of the quality of the inverse solution is possible if the characteristics of noise in the forward solution can be estimated. For time-varying linear systems having a region of basis function support approximately congruent to the support region of the transfer function, and for which there is sufficient *a priori* knowledge of the system, DOPD provides an efficient and noise tolerant method of inverse solution.

I. INTRODUCTION

Inverse solution of linear time varying systems is an ill-conditioned problem which requires constrained or approximate solutions [1], [2]. Methods of solution may involve reduction of the time-varying

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system into time-invariant segments [3]–[5], and tend to be computationally intensive due to matrix inversion or the numerical demands of filter design [3]–[9]. Presented in this communication is the extension of the discrete orthogonal polynomial deconvolution method [10], [11] to linear time-varying systems. The stability and robustness associated with this method in the time-invariant case is demonstrated to extend to time-varying systems.

II. DISCRETE ORTHOGONAL POLYNOMIAL DECONVOLUTION FOR TIME-VARYING LINEAR SYSTEMS

Implementation of DOPD in matrix operator format for time-invariant systems has been described [10], and will be shown to extend to linear time-varying systems. A discrete linear time-varying system may be expressed in matrix operator format as

$$\sum_{k,n} B_{kn} O_n = I_k \quad (1)$$

Where I is a row vector of length N resulting from the multiplication of the $N \times N$ matrix operator \mathbf{B} with the column vector O . The matrix operator \mathbf{B} is constructed using the time-varying system transfer function h_{kn} so that the equation

$$I_k = \sum_{n=1}^N h_{kn} O_n \quad k = 1, 2, 3, \dots, N \quad (2)$$

is satisfied.

The cornerstone of DOPD is the adjoint property of square matrix multiplication operators. In general, the adjoint operator is the complex conjugate transpose of the matrix operator \mathbf{B} . For strictly real systems, the adjoint operator, \mathbf{B}^+ is the transpose of \mathbf{B} , and has the general property that the inner product of any vector x with the output of the operator \mathbf{B} acting on the vector y is equivalent to the inner product of y with the output of the adjoint operator \mathbf{B}^+ acting on x (3) [12], [13].

$$\sum_k x_k \cdot \sum_{k,n} B_{kn} y_n = \sum_k y_k \cdot \sum_{k,n} B_{kn}^+ x_n \quad (3)$$

$$k = 1, 2, 3, \dots, N$$

$$n = 1, 2, 3, \dots, N$$

When the system function h is time-invariant and symmetrical, \mathbf{B} is block-circulant and self-adjoint ($\mathbf{B}^+ = \mathbf{B}$). However the adjoint property as stated in (3) is generally applicable to linear systems expressible in square matrix form and has no implicit restrictions regarding causality or temporal characteristics of the transfer function. The time-varying characteristics of the system are incorporated into DOPD when the matrix transfer function \mathbf{B} and the adjoint operator \mathbf{B}^+ are cast. Subsequent DOPD operations proceed in identical fashion to the time-invariant case [10], but are repeated here to provide a complete description for the time-varying case.

DOPD is initiated by application of the adjoint operator B^+ to a set of IZ orthonormal basis functions p_{mk} of length N . Hartley functions, defined by

$$p_{mk} = \sin \frac{2\pi(m-1)k}{N} + \cos \frac{2\pi(m-1)k}{N}$$

$$p_{(m+1)k} = \sin \left(-\frac{2\pi(m-1)k}{N} \right) + \cos \left(-\frac{2\pi(m-1)k}{N} \right)$$

$$k = 1, 2, 3, \dots, N$$

$$m = 1, 3, 5, \dots, IZ - 1 \quad (4)$$