OPTIMAL OPERATION OF PHYSICALLY COUPLED SURFACE AND UNDERGROUND STORAGE CAPACITIES

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Dragoslav Isailović

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LIST OF SYMBOLS

Symbol	Definition
a	Constant
$\underline{A}$	Column vector of constants
A_d	Direct drainage area
A_r	Reservoir surface area
b	Constant
$\underline{B}$	Matrix of constants
c	Constant
C_r	Runoff coefficient
$\underline{C}_D$	Matrix of constants
$\underline{C}_I$	Matrix of constants
$\underline{D}$	Column vector of differences of water table elevations
$\underline{D}(t)$	Decision vector
$e(t)$	Evaporation from a unit area of water table
$f_j(p)$	Precipitation function
$f(V,W)$	Exchange flow determination function
$F_t(S_t,V_t)$	Return function at stage t
$g_i(x_i)$	Return from allocation x_i
h	Surface subsystem water table elevation
$\underline{H}$	Underground subsystem water table elevation
I	Indicator function
$\underline{I}_t$	Input vector
H_t	Sample size, number of time units
H_T	Sample size, number of intervals each of length of τ units
$p(t)$	Precipitation
$q_b(t)$	Surface outflow from the natural surface subsystem
$q_c(t)$	Channel inflow into the surface subsystem
$q_d(t)$	Direct inflow into the surface subsystem
$q_e(t)$	Evaporation from the surface subsystem

Symbol	Definition
$q_g(t)$	Loss from the system
$Q_i(t)$	Total input into the system
$Q_o(t)$	Total output from the system
$q_p(t)$	Inflow resulting from precipitation over the water storage surface area
$q_r(t)$	Recharge to the underground subsystem
$q_u(t)$	Exchange flow between the surface and underground subsystems
r	Constant
r_1	Serial correlation coefficient
$S(t)$	Surface storage content
t	Time
T	Time
$V(t)$	Underground storage content
$W(t)$	Underground storage component
x_i	Allocation of resource to i^{th} activity
X	Constraint, conditional allocation
α_i	Parameters of linear model
β_i	Parameters of the recharge model
γ	Effective porosity
δ_k	Difference, H_k-h
ε	Prediction error
$\Gamma(t)$	Function of the states of the system
$\Theta(t)$	Function of the states of the system
$\Lambda(t)$	Function of the states of the system
$\Pi(t)$	Function of the states of the system
σ^2	Variance or mean square error
$\Phi(t)$	Function of the states of the system
$\psi(t)$	Function of the states of the system
ω	Constant parameter
$\Omega(t)$	Total water content of the system

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ABSTRACT

Surface water storage reservoirs in karstified limestone areas, as well as in some other geological formations, are often connected to significant natural underground storage capacities. The underground storage represents a cost-free augmentation of the surface storage. A mathematical model of optimal reservoir management under these conditions was developed. The model is based on the assumption that water value to various users during different time periods of the year were known. Performing several levels of optimization by dynamic programming, the optimal policy of water release was obtained for hydrologic sequences, either observed or generated. The optimal policy obtained by this model can be used to establish operational rules and to find the optimal reservoir size.

The oversimplification of the contribution of the underground storage to the surface reservoir is to sum the corresponding storage capacities. However, a mathematical formulation of the physical reality is needed. An optimization model and a mathematical description of the exchange flow between the two storage subsystems were developed also.

Theories associated with the recession curve of the river hydrograph were often founded on the linear relationship between the river flow and the content of the underground storage. An implicit assumption of the model was that the water content of the surface subsystem remains unchanged over an extended period of time. When a surface reservoir is constructed, the above assumption is no longer valid. Under these conditions, a mathematical formulation of the coupled storage was developed when both storages change with time. As in the derivation of the hydrograph recession curve, it is assumed that the flow from one storage to the other is determined by their states.

The state of the underground subsystem of porous karstified limestones is usually greatly affected by natural recharge to the aquifer. The laws governing the recharge are distinct from those of classical river basins. For that reason, a mathematical model for recharge to the karstic aquifer was developed. A model is based on the fact that the autoregressive-moving average (ARMA) model was a valid description of the river flow when the system is assumed linear. A method of estimating the parameters of the recharge model from the ARMA model is given.

FOREWORD

In providing storage capacity for water regulation either surface or underground space is used. For the surface storage new reservoirs are constructed or the existing storage space is transformed and used for flow regulation. For the underground storage, aquifers have been used for flow regulation by water recharge recently. However, the use of rock voids for large storage capacity still awaits various practical methods of solution. Recently, the combined use of surface and underground storage capacities has been studied for physically non-interconnected storage capacities. By their management they are treated as operationally combined storage capacities.

In developing controlled underground storage, several important problems must be solved. Dimensions of the total groundwater environment participating in the storage has to be estimated, overall porosity and its distribution evaluated. Because the time factor is important in filling and emptying the underground storage voids, the effective porosity needs to be determined. The effective porosity is defined as that part of the total volume of voids, which can be used during usual time intervals of recharge or emptying of groundwater voids. Because difficulties exist in estimating the above characteristics, the study of complex underground storage is more difficult than the study of surface storage.

One of the least studied problems is the use of physically coupled surface and underground storage capacities. In this combined system, any fluctuation in surface storage is automatically reflected in level fluctuations of underground storage, because the two storage capacities physically interact. This paper by Dr. D. Isailović refers to the joint use of these physically interconnected, surface and underground storage capacities.

The karstified limestone and dolomite formations are among the most characteristic geological formations in which surface and underground storage capacities can be simultaneously developed and operated. Their porosity is composed of: different sizes of rock fissures, dissolved channels and caves, voids in gravel and sand depositions, and voids in depositions of other materials inside the old fissures, channels, and caves. Important factors in studying the underground storage of karst formations is the relationship between the future range of surface and underground storage level fluctuations and the history of tectonic movement of large karst blocks. In cases of deep karstification, the submerged karstified blocks may contain a large percentage of voids, and may have a large spacial extention. Up to five percent of the total rock space may be composed of voids, to be available for water storage. For a shallow karstification or for lifted karstified blocks, the percentage of voids and the total space involved are limited to a porosity of up to one percent.

In developing an underground storage for flow regulation in karst formations, two types of alternatives should be considered. Interconnected surface and underground storage capacities are developed as the primary alternatives whenever feasible, because they may be easy to accomplish economically. The control of water leakage from both capacities is usually a precondition for implementing these alternatives. When sites of economical joint surface and underground storage development in karst regions are exhausted, or when the potential water leakage is assessed to be both of a high risk and of large quantities, alternatives of using only the underground storage should be studied.

In many cases the large karst springs occur at the contact between highly karstified limestone and an imperious formation. The spring water might have eroded the imperious formation in a triangular shape, with water drainage at its lowest point. It may then be feasible to grout a rock mass above the spring (as an underground impervious dam), which will raise the water level in the karst formation as high as it may be economically feasible. The water outflow is then controlled by an outlet conduit. In some cases this control conduit may be located much below the spring level. Sufficiently large karst water storage could thus be created. The experience with coupled surface and underground karst storage developments may produce a sufficient scientific information to be applied also to developments of pure underground karst storage capacities.

The basic problem in utilizing the underground karst storage space is to involve as large a rock mass in storage as possible. If the rock mass between the upper and lower fluctuation levels is 20-50 times as large as the surface storage capacity, then 1-2 percent of the effective rock porosity will produce an underground storage capacity of 20-100 percent of the surface storage capacity.

While a surface storage capacity reacts instantaneously to opening of outflow outlets or to water inputs, the underground storage capacity has a time-delaying effect. To recharge water into the voids or to take water out of voids of an underground pervious formation, time is necessary. Therefore, any coupled underground storage to a surface storage must take time delays into account in one way or another. This problem has been treated by Dr. D. Isailović in a particular way, namely by a relationship between the interchange of water flow between surface and underground storages to total storage volume. The response hydrographs to changes in the relative storage levels of two capacities indirectly incorporate these time delays. The basic problem of physically coupled surface and underground storage capacities is how to estimate the response properties of underground storage to surface storage changes and how to develop mathematical models for the operation of two capacities as a unit. This has been presented in the paper. Several problems need to be resolved, namely how large will an underground storage capacity be to a given surface storage, what the effect of underground storage will be on floods after the surface reservoir is constructed, namely whether flood peaks coming out of karst formation into surface storage through large karst springs will be increased or decreased by the underground storage. This question of floods will reflect on design of reservoir spillway capacity and downstream flood control. Furthermore, the question arises for the influence of increased underground storage and longer water time residence in karst formations on water quality.

In presenting this paper by Dr. D. Isailović, it should be stressed that the problem of finding proper solutions to a joint operation of surface and underground storage capacities is not only related to karst formations. Other formations like sandstones, some volcanic rocks, large deposits of sand and gravel, and many fissured rocks can also produce a significant underground storage capacity by surface reservoirs. In practical terms, the problem is how large the percentage of an underground storage capacity should be in terms of surface storage capacity for the underground storage to be considered as a coupled storage to the surface storage, rather than being ignored or simply added to the surface storage.

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Chapter I
INTRODUCTION

1-1 Preliminary Remarks

Efficient management of water resources calls for a set of hierarchical decisions concerning every specific project. Surface reservoirs appear to be a dominant component of almost every important water resource system. The reservoir operation is regarded as one of the most important aspects of the storage analysis since the system performance depends, to a large extent, on the way the reservoir is operated. This analysis is a part of the process by which the reservoir size is selected.

Three groups of approaches are commonly used to analyze a storage capacity: empirical, experimental, and analytical. The empirical method refers to the application of the mass-curve analysis [Rippl, 1883] to the observed streamflow sequence. The experimental method is based on the Monte Carlo technique in generating a large number of new hydrologic samples. The mass-curve analysis is then applied to these samples to assess the storage characteristics. The analytical method utilizes mathematical theories to derive the statistical properties of the most important variables that characterize the storage.

An instructive example of water resources development under adverse conditions is construction of reservoirs in limestone regions. Karstified limestone, as well as some other geological formations, are known to have high water transmissibility. As a result, significant natural underground storage may be directly coupled with the surface reservoirs. Examples of relatively large underground storage capabilities coupled with man-made lakes are: the Lake Powell on the Colorado River (USA), [U. S. Bureau of Reclamation, 1974], the Libby Reservoir on the Kootenai River (Montana, USA) [Coffin, 1970], the Lake Nasser on the Colorado River (USA), [U. S. Bureau of Reclamation, 1974], the Libby Reservoir on the Kootenai River (Egypt, Sudan), and the Lake Bileća on the Trebisnjica River (Yugoslavia) [Mikulec and Trumić, 1970]. The account of effects of natural underground storage capacity is virtually cost free. In addition, the water of the underground storage is subject to minimum evaporation.

A sound basis for evaluating the degree of usefulness of a storage project is a monetary performance index which includes economic, social, and political criteria. With the project output quantified, an optimization scheme is usually applied to choose the best alternative from a set of feasible actions. These techniques frequently combine one or more methods of storage analysis with an extensive use of computers. Application of the optimization methods to analyze storage operation under specific conditions is the subject of this study.

1-2 Study Objective

The objective of this study was to develop a mathematical model for determining the optimal policy of water use from a single, multi-purpose surface reservoir which is physically coupled with a natural, fast responding underground storage space of significant capacity. This objective is accomplished by a dynamic programming scheme.

In order to account for the contribution of the underground storage, an appropriate mathematical description of the exchange of flows between the two interconnected storages was necessary. The usefulness of presently available models for groundwater flow in an optimization procedure may be severely limited, primarily because they require extensive computations. For this reason, simple mathematical models are developed herein. This by no means implies that more complex, other groundwater flow models are precluded in achieving the objective.

Large quantities of water may flow out of a karst aquifer. Since aquifers are fed by natural recharge, the intensity of recharge may be very high. This is a characteristic of aquifers in fissured rocks. In addition to determining the optimal policy, a recharge model for drainage basins in karstified limestones and dolomites was developed as a complementary part of the flow exchange model.

In the subsequent analysis the underground storage is assumed directly noncontrollable, that is, there is no controllable component entering into or coming out of the underground storage. The underground storage is controlled indirectly. Besides being dependent on natural stochastic inputs to the system, the underground storage depends on the behavior of the surface reservoir controlled by making decisions concerning its water release. Thus, the exchange of flows between the two interconnected storages is affected by the surface storage operation.

1-3 Scope and Organization of the Study

The material presented in this study is organized as follows: Chapter II is a brief review of literature. Chapter III defines the hydrologic system dealt with in the underlying assumptions. The next four chapters represent detailed analysis of components of the model used in the study: Chapter IV, with the mathematical model for the underground storage; Chapter V, with the mathematical formulation of the hydrologic system and a model for the recharge; Chapter VI, with the description of the system identification under frequently encountered conditions; and Chapter VII, with the dynamic programming solutions to the resource allocation problem with equality and inequality constraints. Chapter VIII describes procedures and results in the application of the models, with two sample problems under slightly different hydrologic conditions solved. Finally, Chapter IX presents concluding remarks and recommendations for further study.

Chapter II
BRIEF LITERATURE REVIEW

2-1 Reservoir Storage

Reservoirs for flow regulations have been used for several thousand years. The first attempt to determine the size of a reservoir by a scientific approach can be traced back only to the last century. The foundation of water storage analysis was laid by W. Rippl [1883]. The size of the storage required for water supply of Vienna was critically examined by the method which soon became known as the Rippl-diagram method. The technique has been used extensively throughout the world since. The method is based on the use of a time series realization, i.e., an observed sequence of streamflow data representing inflow into the reservoir. The storage capacity determined by the Rippl-diagram method is that which provides a sufficient supply over the periods of critically low flows within the observed sequence.

No objection can be made to the use of the observed streamflow sequences, for in many instances it is the only, and cetainly the best, source of information available. What can, however, be objected to is the manner in which the observed sequence is used. It soon became evident that the Rippl method when applied to an observed sequence of streamflows may prove to be inadequate since the determined reservoir size will be correct if the streamflow observations during the reservoir lifetime were identical to the sequence upon which the evaluation of the reservoir size was based. The probability that a realization of a continuous time process will repeat itself identically in any subsequent sequence is zero. Thus, the selected reservoir size may be incorrect. To account for the stochastic variation of annual flows, it was suggested as early as the beginning of this century [Hazen, 1914] that probabilistic concepts be applied to streamflow. At that time probability was not regarded as a legitimate branch of mathematics [Feller, 1968], so that half of the century passed before Hazen's idea received widespread recognition and application.

Among early developments of the theory of reservoir storage, which utilized probabilistic methods to determine water storage capacity, was that of Hurst. The long time studies of the Nile River led to the formation of an expression [Hurst; 1951, 1965] relating the ratio of the mean range, $\bar{R}$, and the standard deviation of annual flows, σ, to the time period of N years over which a sufficient water supply is to be provided by a storage capacity, namely known as the Hurst equation,

$$R/\sigma = 0.61\ N^{0.72}. \tag{2-1}$$

Even though this expression was derived from the observed data of several natural phenomena, it has become controversial.

Advances in the theory of probability and mathematical statistics and continuing attempts to apply them to practical engineering brougth about new ideas associated with the analysis of reservoir storage. There is a large number of references contributing to the development of new methods authored by Feller, Spitzer, Hurst, Morran, Lloyd, Annis, and many others. It is not the objective of this review to cite all those who have contributed significantly in bringing the theory of water storage to its present level.

A summarized description of a class of probabilistic problems that usually arise in the theory of storage was given by Morran [1959]. Another article dealing with the same subject is due to Lloyd [1967]. Probabilistic treatment of hydrologic time series that are determinative factors of the reservoir size and its efficient operation, such as surplus, deficit, range, run, etc., is described by Yevjevich [1965, 1972c], and Salas [1972]. Application of the probabilistic models within the framework of queuing theory was described by Langbein [1958]. Later, Fiering [1962, 1967] combined queuing theory and simulation in the optimal reservoir design. More details concerning various techniques of probabilistic reservoir analysis are summarized by Roefs [1968].

A particular branch of applied mathematics that has provided tools in planning and managing of water resources systems is mathematical optimization. Probabilistic and statistical methods have improved, and their application to the description of the stochastic nature of hydrologic processes has received widespread recognition. The advances in computer technology made economic the processing of large amounts of data within a short time. As a result, not only was it feasible to efficiently use the observed data, but opportunities have been created for generating new sequences of data according to the statistical properties and dependence structure inferred from historic data [Yevjevich, 1972b]. Optimization models combined with probabilistic methods and data generating techniques, all based on computer capability, were seen as potentially promising devices in analyzing various aspects of water resources systems.

Various optimization schemes, linear and nonlinear, are applied to water resources at present. A large number of problems in water resources belong to the class of problems involving sequential decision making, and these in turn "...lend themselves best to analysis and solution through the application of the dynamic programming," [Buras, 1972]. Consequently, as pointed out by Hall and Dracup [1970] and Buras [1972], dynamic programming was demonstrated to be a particularly useful technique in analyzing water storage problems.

An extensive review of the methods and techniques applied to reservoir planning and management is given by Roefs [1968] and Croley [1974]. Numerous examples of the application of different optimization schemes to water resources systems of various sizes are given by Hall and Dracup [1970] and Buras [1972]. Butcher et al. [1969] used a dynamic programming scheme to define the optimal strategy of installation of a sequence of water supply projects. Buras [1963] outlined methods employed to obtain the optimal use of pumped underground aquifer operated in conjunction with a surface reservoir. Burt [1974] devised a method for attaining the optimal management of groundwater resource in view of decisions concerning the timing and the location of surface water developments. Hall and others [1963, 1964, 1969] applied dynamic programming to problems of reservoir design and operation. Generated sequences of dependent annual streamflow were first used by Hall and Howell [1963] for analysis of a single purpose reservoir. The method was later called "generation-deterministic optimization-regression," [Roefs, 1968], and also "implicit stochastic optimization" (ISO) [Croley, 1974].

Performing the optimization over each of a number of generated samples of data, several optimal returns and the optimal policy corresponding to each of them can be determined. From those it is possible to determine the following: (1) the operational rules, and (2) the risk of failure to satisfy certain levels of the demand or the risk of failing to generate a given level of return. They also suggest that the selection of the optimal reservoir size should be based upon the results of computation performed for several arbitrarily chosen physically feasible reservoir sizes. It should be noticed that this method uses a deterministic optimization procedure to analyze the system with a stochastic input, i.e., a generated sequence of streamflow. This perhaps explains the diversity of names given to the method described.

Even though dynamic programming has been found to be well-suited for the analysis of many water resources systems, not all problems can be solved by this method because of limitations of the technique. The most frequent obstacle for the use of dynamic optimization appears to be excessive computer time and, particularly, computer memory requirements. Much work has been done to alleviate the burdens usually associated with stochastic dynamic optimization. Heidari et al. [1971] described a method of discrete dynamic optimization which considers a narrow, arbitrarily chosen, band of feasible policies. The best policy constrained by this band is selected by the method of dynamic programming. Then, a new band around the optimal policy is formed and a new optimal policy corresponding to the new band is determined. The process is repeated until the optimal policy obtained for a given band remains unchanged for two successive iterations. Although the procedure is essentially iterative, it is claimed to be very efficient in terms of both computer time and memory requirements, particularly for multidimensional systems. Another model with a similar objective was developed by Croley [1974] proposing the method of sequential stochastic optimization as an alternative to the existing methods. In essence it combines two of the most frequently used forms of dynamic stochastic optimization, eliminating some of their disadvantages. The method is particularly suitable for actual operation of water resources systems.

In spite of the fact that the literature concerning storage problems is prolific, no attempt has been made to account for the natural underground storage effects in analyses of the surface reservoir planning and operation, when the two storages are physically coupled. In some circumstances, consideration was given to the bank storage but only insofar as determining its magnitude -- the Lake Powell of the U. S. Department of the Interior [1965], or to ascertain the response of the basin storage to changes of the surface reservoir -- the Libby Reservoir, Coffin [1970].

2-2 Specific Problems Related to Karst

Karst regions are characterized by unusual features and extreme conditions which do not allow conclusions as to what is typical or average for many properties of carbonate rock [Stringfield and Le Grand, 1969]. Specific characteristics of karst watersheds such as those related to scarcity of surface streams and rugged topography were developed by natural processes as a result of the presence of soluble rock, carbonic acid, ample precipitation, rock fissures, and favorable topographic settings [Le Grand, 1973]. As in almost all catchments the time of extensive water occurrence in karst does not coincide with the highest water demand. However, the problems are here sharply accentuated by specific relationships between water and soil.

High infiltration and low surface runoff are hydrologic characteristics of most karstlands [Sweeting, 1973]. As a consequence, underground flows are large, often concentrated, sometimes reaching the proportions of underground rivers. The underground channels may converge and form extremely large springs, as frequently occurs in the Mediterranean Karst. Karst rivers can disappear as suddenly as they appear. Another peculiar feature of karst regions are karst plains (poljes) which are enclosed valleys with no surface drainage. They usually act as retention basins during rainy periods or snowmelt and experience extreme water shortages during dry seasons.

It is clear that when the methods of classical groundwater hydrology, developed for nonfissured rocks, are used for evaluation of karst water resources, the results may be misleading [Sweeting, 1973]. For that reason, as realized long ago, new methods of investigation in karst are needed. Many controversies concerning karst underground water were centered about the question, which has yet to be resolved, of the existance of a water table. Nevertheless, the distinctive nature of karst hydrology was emphasized at the Dubrovnik Symposium on Hydrology of Fractured Rocks [Sweeting, 1973].

Due to the described geologic and hydrologic conditions, economic and engineering problems related to karstlands can, in summary, be said to result from: (a) scarcity of surface water supplies; (b) poor predictability of underground water resources; (c) instability of cavernous grounds; (d) leakage of surface reservoirs; and (e) unreliable waste disposal. To resolve these problems, it was necessary to understand environmental relationships which determine the effects of engineering actions on the system. These had to be learned for every particular case through carefully planned, often elaborate, exploratory work and actual construction of water resources systems. Many aspects of the encountered field problems are described in the proceedings of the above mentioned symposium. Additional recent references concerning karst hydrology and geology can be found in publications by Stringfield and Le Grand [1969], and Sweeting, [1973]. Extensive bibliography in the subject matter is given by Herak et al. [1973].

During the last 25 years, a number of dams was constructed in regions of Yugoslav Karst [Mikulec and Trumić, 1972], and others are under construction or on the drawing board. Much experience concerning construction of reservoirs under karst conditions has been gained. One important aspect of this experience is how to successfully cope with potentially unusually large reservoir leakage, which was sometimes thought to be sufficient reason to rule out the construction of reservoirs under these conditions. In recent years it was realized that more comprehensive planning and management of water resources systems in karst is needed. The realization led to a joint U. S. - Yugoslav research project currently underway. Within the framework of this project, it is desired to investigate the aspects of underground natural storage physically coupled with the surface storage. This is intended to contribute to a better understanding of effects of the natural underground storage space on surface storage in karst areas.

Chapter III
SYSTEM DESCRIPTION AND GENERAL APPROACH TO THE PROBLEM SOLUTION

3-1 Description of the System

Consider an overall hydrologic system consisting of two subsystems: a surface subsystem and an underground subsystem which are physically interrelated in such a manner that water can flow from one subsystem to another in both directions. A simplified schematic representation of such a system is given in Fig. 3-1, where the surface subsystem is denoted by S and the underground subsystem by V. The dashed line in Fig. 3-1 can be thought of as the boundary of the total system denoted by Ω.

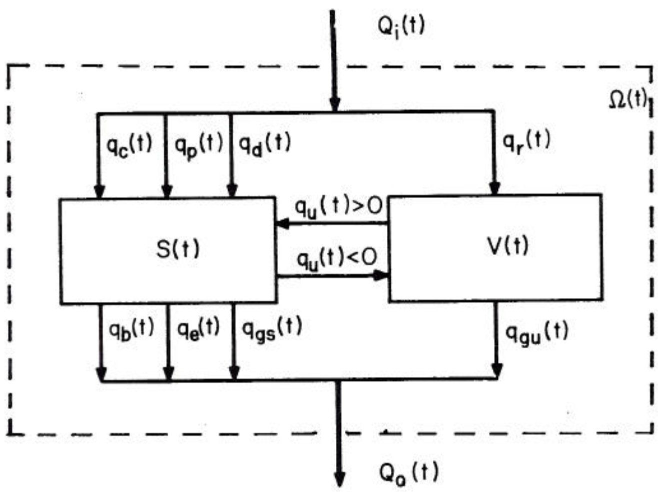

Fig. 3-1. Schematic Representation of a Hydrologic System Composed of Two Subsystems.

The water budget equation of the overall system can be expressed by the simple relation

$$Q_i(t) - Q_o(t) = \frac{d\Omega(t)}{dt}, \qquad (3\text{-}1)$$

in which $Q_i(t)$ = the rate of input into the system, $Q_o(t)$ = the rate of output from the system, and $d\Omega(t)$ = the rate of change of the system storage, all three as continuous functions of time t.

The total input into the system, $Q_i(\cdot)$, can be represented as the sum of its four major components as depicted in Fig. 3-1, namely

$$Q_i(t) = q_c(t) + q_d(t) + q_p(t) + q_r(t), \qquad (3\text{-}2)$$

in which $q_c(t)$ = the direct channel inflow into the surface subsystem, $q_d(t)$ = the direct surface inflow into the surface subsystem, $q_p(t)$ = the inflow resulting from precipitation over the water storage surface area, and $q_r(t)$ = the recharge to the underground subsystem.

Similarly, total output from the system is the sum of the components

$$Q_o(t) = q_b(t) + q_e(t) + q_g(t), \qquad (3\text{-}3)$$

in which $q_b(t)$ = the channel outflow from the surface subsystem, $q_e(t)$ = the evaporation from the water surface area of the surface subsystem, and $q_g(t) = q_{gs}(t) + q_{gu}(t)$ = the loss of water from both the surface subsystem, $q_{gs}(t)$, and the underground subsystem, $q_{gu}(t)$, respectively.

Under the given description of the system, the rate of change of the system storage, $d\Omega(t)/dt$, at the right-hand side of Eq. 3-1, consists of a summation of the corresponding changes of the surface subsystem $dS(t)/dt$ and the underground subsystem $dV(t)/dt$, namely

$$\frac{d\Omega(t)}{dt} = \frac{d}{dt}\,[\,S(t) + V(t)\,] = \frac{dS(t)}{dt} + \frac{dV(t)}{dt}\,. \qquad (3\text{-}4)$$

In Eq. 3-4 $S(t)$ = the volume of water content in the surface subsystem, and $V(t)$ = the total volume of water stored in the underground subsystem.

Substituting Eqs. 3-2 through 3-4 into Eq. 3-1, the basic water budget equation becomes

$$q_c(t) + q_d(t) + q_p(t) + q_r(t) - q_b(t) - q_e(t) - q_g(t) = \frac{dS(t)}{dt} + \frac{dV(t)}{dt}\,. \qquad (3\text{-}5)$$

In addition to the information concerning the components of surface flow and atmospheric events, a description of a complex hydrologic system also requires that the behavior of the underground subsystem be described. More specifically, the last term in Eq. 3-5, $dV(t)/dt$, must be mathematically defined in such a way that it describes the response of the underground subsystem to the other flow components.

Sufficiently accurate models of the underground flow available at present appear to be very complex and somewhat burdensome to compute [Freeze, 1971]. In addition, under the condition of nonexistent regular water table in karstified aquifers [Sweeting, 1973], these models are inapplicable. Thus, a simplified and convenient model is necessary.

The continuity condition of the underground subsystem must be satisfied, namely

$$q_r(t) - q_u(t) - q_{gu}(t) = \frac{dV(t)}{dt}\,, \qquad (3\text{-}6)$$

where $q_u(t)$ = the exchange flow between the surface and underground subsystems, $q_{gu}(t)$ = the water loss from the underground subsystem, while $q_r(t)$ and $dV(t)/dt$ are as defined previously. The mathematical formulation of Eq. 3-6 is depicted in Fig. 3-2.

Both the recharge to the underground aquifer, $q_r(t)$, and the loss, $q_{gu}(t)$, are non-negative time functions. The latter component is virtually unmeasured and hopefully insignificant. Hence, it is often dropped from computation. This need not be true

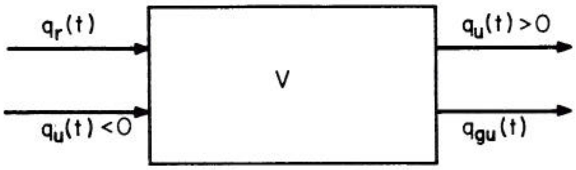

Fig. 3-2. Schematic Representation of the Flow Components of the Underground System.

under all conditions. For example, water leakage may increase significantly after a dam is constructed in areas of specific geologic conditions such as karstified limestone. However, when this is likely, measures are taken to prevent excessive leakage. In this study, $q_{gu}(t)$, is assumed controllable in the sense that it can be reduced to a negligible quantity by grouting. Thus, Eq. 3-6 can be reduced to

$$q_r(t) - q_u(t) = \frac{dV(t)}{dt} . \qquad (3\text{-}7)$$

The flow components of both the underground and surface subsystems change simultaneously with time. Depending upon their mutual relationships, various hydraulic conditions can be found. These conditions will determine whether the exchange flow, $q_u(t)$, will be either positive or negative. For example, when $q_u(t) = q_1 < 0$, the flow direction is from the surface subsystem into the underground subsystem, so that the underground water content increases since $q_r(t) - q_u(t) > 0$, hence $dV(t)/dt > 0$. However, when $q_u(t) = q_2 > 0$, the flow direction is from the underground subsystem into the surface subsystem. The rate of change of the former, $dV(t)/dt$, will depend on the magnitude of the recharge $q_r(t)$ so that: (a) the water content of the underground subsystem $V(t)$ increases whenever $q_r(t) > q_u(t)$ i.e., $dV(t)/dt > 0$; (b) $V(t)$ remains unchanged whenever $q_r(t) = q_u(t)$, namely $dV(t)/dt = 0$; and (c) $V(t)$ decreases whenever $q_r(t) < q_u(t)$ resulting in $dV(t)/dt < 0$. In addition, $q_u(t) = 0$ whenever both the following conditions are satisfied: $q_r(t) = 0$ and $dV(t)/dt = 0$.

3-2 Natural and Modified System Conditions

The system depicted in Fig. 3-1 can be thought of as a body of surface water representing the surface subsystem, while the underground subsystem consists of porous geological formations which, depending upon the hydraulic conditions of the system, can be filled either with water or with air. The quantity of water stored in the underground system has often been called the bank storage. This term will be used interchangeably with underground subsystem or underground storage throughout this paper.

With regard to the research objective, it will be necessary to distinguish between the system under natural conditions and the system under modified conditions. The hydrologic system dealt with in this paper, under natural conditions, consists of a river channel and the bank storage along the considered river reach. Under the modified conditions the system is comprised of a man-made surface reservoir and a natural underground storage physically interacting with the surface storage. From the point of view of application of the techniques developed in this paper to a hydrologic system, the natural conditions correspond to the period of reservoir planning while the period of reservoir operation is characterized by the modified conditions. Attempts are made to extract as much information as possible from the data available during the planning stage. It is, however, clear that complete knowledge of the modified system in most cases cannot be obtained until the modified conditions are created. Hence, in order to apply the results of investigations of the natural system to the modified system, some reasonable assumptions will have to be made concerning the manner and the extent to which the system modification will affect the system behavior.

To outline the change of hydrologic processes caused by the system modification, the components of the budget equation should be considered. First, assume that the underground subsystem can be neglected. Therefore, the recharge, $q_r(t)$, and the rate of change of the underground subsystem, $dV(t)/dt$, can be dropped from Eq. 3-5. Remembering that the loss of water from the natural system, $q_g(t)$, is assumed to be zero, then

$$q_c(t) + q_d(t) + q_p(t) - q_b(t) - q_e(t) = \frac{dS(t)}{dt} \qquad (3\text{-}8)$$

is obtained. When such a system is modified, that is when a surface reservoir is constructed, some components on the left-hand side of Eq. 3-8 may be affected. Yet, it is usually possible to account for the resulting changes with a satisfactory degree of accuracy since these processes are governed primarily by the hydrometeorological conditions unaffected by the system modification.

Consider Eq. 3-8 term by term. The channel inflow, $q_c(t)$, obviously will not be affected by a reservoir. The components of the direct surface inflow, $q_d(t)$, the inflow due to precipitation, $q_p(t)$, and the evaporation, $q_e(t)$, will, however, be different for the two conditions. Nevertheless, they are proportional to the areas of the system surface over which they occur. The $q_p(\cdot)$ will increase since the reservoir water area increases. Thus, for a given quantity of precipitation it can be conveniently described as

$$q_p(t) = p(t)A_r(t), \qquad (3\text{-}9)$$

where $p(t)$ is rainfall per unit area, and $A_r(t)$ is area of the reservoir surface at the given time, t. The direct surface inflow, $q_d(t)$, will, however, be smaller than for the natural system because part of the area drained directly into the system is now submerged under the reservoir. Likewise, it can be defined in terms of the rainfall and the direct drainage area $A_d(t)$ which changes with time, namely

$$q_d(t) = c_d p(t) A_d(t), \qquad (3\text{-}10)$$

where c_d = a constant parameter.

Increase of evaporation is probably the most significant change resulting from the system modification. However, it, too, can be determined when the evaporation from a unit surface is estimated. Since this remains unchanged for both conditions, it can be written

$$q_e(t) = e(t)A_r(t), \qquad (3\text{-}11)$$

where $e(t)$ = the evaporation from a unit area of water surface, with $A_r(t)$ and $q_e(t)$ as defined previously.

The remaining two components of the budget equation, namely $q_b(t)$ and $dS(t)/dt$, become, under the modified conditions, the controlled reservoir release and the rate of change of reservoir storage, respectively. With the knowledge of the previously described components, and when a control is exerted over either $q_b(t)$ or $S(t)$, the other component--$S(t)$ or $q_b(t)$--is obtained from Eq. 3-8. Thus, as demonstrated, insight into the natural system can, to a large degree, be extended to the modified conditions.

When the assumption underlying Eq. 3-8 does not hold, i.e., when the contribution of the underground subsystem is significant, it is considerably more difficult to describe the system. Even when all the previously discussed extensions are possible, it is necessary to know the rate of change of the underground subsystem, $dV(t)/dt$, and the recharge, $q_r(t)$. The latter is not affected by the system modification, and a reasonably accurate model can be developed to describe it. However, the $dV(t)/dt$ component, besides being expensive to accurately assess, may be significantly affected by the construction of the reservoir. In addition, the optimization of the reservoir operation requires that the underground volume, $V(t)$, be simulated, so that the state of the system resulting from any given (or assumed) decision can be evaluated. To accomplish the desired goal, Eqs. 3-5 and 3-7 are combined in a model capable of simulating the exchange flow, $q_u(t)$, as a function of $S(t)$ and $V(t)$, namely

$$q_c + q_d + q_b + q_r - q_b - q_e - q_g = \frac{dS}{dt} + \frac{dV}{dt}, \qquad (3\text{-}12a)$$

$$q_r - q_u = \frac{dV}{dt}, \qquad (3\text{-}12b)$$

$$q_u = q_u(V,S), \qquad (3\text{-}12c)$$

where for reasons of simplicity the argument, t, is omitted.

The above mathematical formulation is rearranged to accommodate the use of discrete variables. Introducing Eq. 3-12b into Eq. 3-12a and assuming that the exchange flow, $q_u(t)$, is properly defined by the system states, S_{t-1} and V_{t-1}, at the beginning of a given time interval, t, the following is obtained

$$S_t = S_{t-1} + q_t^c + q_t^d + q_t^p + q_t^u - q_t^b - q_t^e - q_t^g, \qquad (3\text{-}13a)$$

$$V_t = V_{t-1} + q_t^r - q_t^u, \qquad (3\text{-}13b)$$

$$q_t^u = q^u(S_{t-1}, V_{t-1}), \qquad (3\text{-}13c)$$

wherein the correspondence of symbols of Eqs. 3-12 and 3-13 is obvious. This mathematical description of the system is depicted in Fig. 3-3. However, when the exchange flow cannot be adequately described by the system states at the beginning of a given time interval, the resulting states, S_t and V_t, must be used to evaluate q_t^u. Then, the mathematical formulation of the system becomes

$$S_t = S_{t-1} + q_t^c + q_t^d + q_t^p + q_t^u - q_t^b - q_t^e - q_t^g, \qquad (3\text{-}14a)$$

$$V_t = V_{t-1} + q_t^r - q_t^u, \qquad (3\text{-}14b)$$

$$q_t^u = q_t^u(S_{t-1}, V_{t-1}, S_t, V_t), \qquad (3\text{-}14c)$$

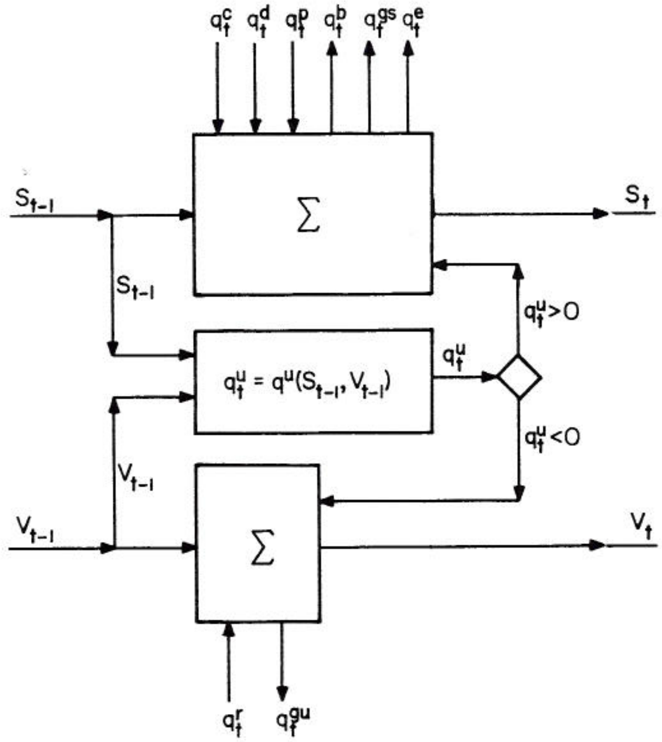

Fig. 3-3. System Diagram When the Exchange Flow q_t^u is a Function of the States at the Beginning of the Given Stage t.

which is represented graphically in Fig. 3-4. It should be noticed that an iterative procedure is required to evaluate q_t^u according to Eq. 3-14c. It should also be observed that in both cases, evaporation is regarded as independent of the surface subsystem state, S.

3-3 Brief Outline of the Problem Solution

Optimal operation of a water storage reservoir is based on some optimal policy of water release. Finding the optimal policy requires a hydrologic and a compatible optimization model. Each of these two models is further subdivided into two parts: the hydrologic model consisting of the surface flow model and the underground flow model, and the optimization model comprising the economic model and a computational algorithm.

The result of modelling as a totality depends, evidently, on many factors associated with complexity and compatibility of the models involved. For example, an oversimplified hydrologic model may offset the accuracy of results that can be achieved from a very

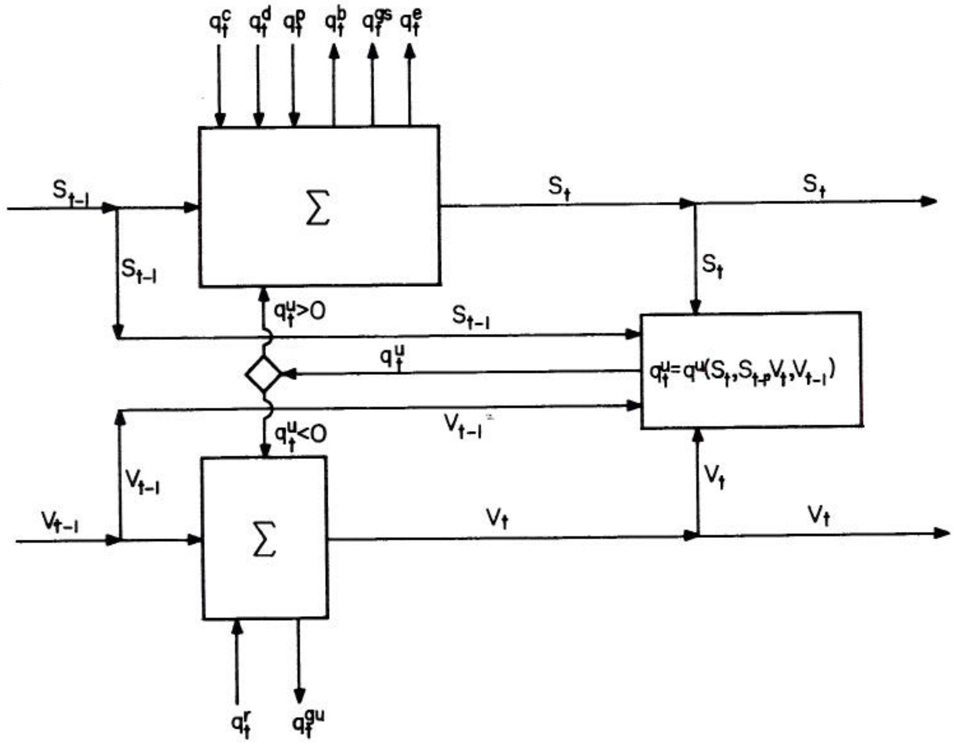

Fig. 3-4. System Diagram When the Exchange Flow q_t^u is a Function of the States at Both the Beginning and the End of the Given Stage t.

complex and computationally expensive optimization model, or conversely. On the other hand, if both models are too complex, computation may not be feasible.

The optimal policy can be based on evaluation of either benefits or losses resulting from the release of water in successive time intervals and the allocation of water to various economic activities. In this study, the optimal policy is founded on the gross benefits.

Reservoir optimization in many cases utilizes the year as the time increment. This treatment is only satisfactory for large surface reservoirs where the total annual inflow represents a relatively small fraction of the reservoir capacity. This approach may result in significant errors in optimal policy when applied to small or medium size reservoirs. The possible errors may be accentuated by large seasonal variations of river flows and/or demands. In addition, the optimal use of coupled surface and underground storage requires evaluation of the optimal policy over a finer grid of time points than a year. This is due to the fact that the water stored in the underground subsystem is not necessarily available for use at a given time. The delay in availability, as demonstrated in Appendix A, may in some karst areas be relatively small. In that case, points even a month apart would not properly account for the effects of underground storage capacity.

Out of the four models mentioned, only two will be dealt with in detail: the underground flow model and the computational algorithm. The economic model and the surface water model will be treated here only to the extent necessary to incorporate them into the more complex hydrologic and optimization models.

The economic model is treated only peripherally. However, in order to avoid ambiguities, the assumptions required to incorporate the economic into the optimization model are outlined. First, the economic projections over the project lifetime are assumed known. As a part of this assumption, it is further implied that values of water (x) to each potential user (i) such as irrigation, municipal and industrial supply, power generation, and low flow augmentation, are determined and given in terms of individual return functions, $g_i(x_i)$, for $i = 1,2,\ldots,n$, where n is the number of economic activities. The return functions, $g_i(x_i)$, may change with time in a periodic manner. The return functions of those economic activities where the return depends on the quantity of water delivered and the general state of the system (reservoir size, head, etc.) are given for different reservoir sizes. Thus, the effect of the system size is accounted for.

Dynamic programming, which is to be used in the computational algorithm, requires that selection of the best returns be independent of the previous decisions. The returns from agriculture, according to Hall and Dracup [1970], depends on the time of delivery. However, in a recent study, Twyford [1973] indicates that for some crops, in all but very drastic cases of water shortages, the timing of water delivery may not be as critical as previously thought. Based on these considerations, the dependency of water value on timing of delivery is not considered in the following optimization procedure with the exception of the known constraints imposed on the lower limit of irrigation demands.

The natural phenomena affecting water resources systems are stochastic in their nature. Furthermore, the reservoir outflow is subject to the uncertainty inherent in water demand and other economic factors. To cope with these problems within the framework of optimization techniques, two basic approaches were applied in the past: deterministic optimization and stochastic optimization. Even though deterministic optimization is inferior to stochastic, it is frequently applied because of its convenience.

Two types of stochastic optimization are in common use. Classical stochastic optimization methods, or, as called by Croley [1974], explicit stochastic optimization (ESO), are straightforward techniques. They consist of evaluating the expected optimal return from the known return function and inflow probabilities. In dynamic programming, which is the most frequently used technique in water resource problems, it was found that the computational requirements quickly increase [Roefs, 1968; Croley, 1974] and it becomes virtually inapplicable under some circumstances [Hall and Howell, 1963]. The ESO requires two state variables for each storage reservoir incorporated into the system [Roefs, 1968]. The number of state variables increases by the square of the number of storage reservoirs. The second difficulty is due to the serial dependency of hydrologic events. The time series must be described by a set of conditional distributions. This requires the manipulation of large matrices of conditional probabilities. The problem is accentuated as the time intervals become shorter since the dependency links between the variables discretized over short time periods grow stronger. Thus, larger probability matrices must be used to describe the processes adequately. Also, a large number of points may result in excessive computer time requirements.

While the first shortcoming of the method is not restrictive for the problem at hand, the difficulty resulting from the dependency of the time series is a serious disadvantage. The discretization must be done over relatively short time periods and probabilities must be conditioned not only on the past events of the same time series, but also on the past and present events of the other series.

To resolve the impasse into which ESO usually leads, an alternative method was proposed by Hall and Howell [1963]. The method was subsequently called generation-deterministic optimization-regression [Roefs, 1968] and also implicit stochastic optimization (ISO) [Croley, 1974]. In a particular example Croley [1974] showed by contradiction that the results obtained by the two methods do not agree, as they should not, because ESO gives the expected value of the return, while ISO gives the optimum return from a deterministic input.

Unlike the method due to Hall and Howell [1963], the research detailed herein is based on the generation of a precipitation series and thence streamflow and recharge. These are then used as deterministic input to ascertain an optimal policy. The resulting return is a random variable and can be used within a stochastic framework to assess its statistical properties. Further, these can be used in the manner of Hall and Howell to determine operational policy and optimal reservoir size.

Chapter IV
CONJUNCTIVE USE OF COUPLED SURFACE AND UNDERGROUND STORAGE CAPACITIES

4-1 Background Information

As given by Eq. 3-7, the underground subsystem can be described by the following mathematical relation

$$q_r(t) - q_u(t) = \frac{dV(t)}{dt} , \quad (4\text{-}1)$$

where $q_r(t)$ and $q_u(t)$ are the recharge to the subsystem caused by precipitation and the underground exchange flow between the two subsystems, respectively. $dV(t)/dt$ is the rate of change of the water content, $V(t)$, of the underground storage. This basic continuity equation combined with various assumptions concerning the functional relation of $q_u(t)$ and $V(t)$, has been extensively used to describe the river hydrograph. Hydrologic literature is abundant in practical applications of the model given by Eq. 4-1, but less so in theoretical considerations of the $q_u(t)$ to $V(t)$ relationship.

A very generalized analysis with regard to the type and method of solution of the basic differential equation associated with storage problems is given by Yevjevich [1959]. The study covers a wide range of situations that usually arise in practical problems. In the cited work the following assumptions were made

$$S_u = aH^m , \quad (4\text{-}2)$$

$$q_u = bH^r , \quad (4\text{-}3)$$

where $H = H(t)$ is the water table elevation from some reference point, q_u was defined previously, and a, b, m, and r are constants. In Eq. 4-2, S_u is some characteristic of the underground volume. Its relation to V and its precise definition is given later (see Eq. 4-39). Notice that for simplicity of notation, the argument t is dropped from Eqs. 4-2 and 4-3. Nevertheless, it is implied that q_u and S_u are continuous time functions unless specified otherwise. Equations 4-2 and 4-3 indicate that both the water content of a reservoir and the outflow from it are functions of a single variable H. Combining Eq. 4-2 and 4-3 gives

$$S_u = \frac{1}{c}q_u^n , \quad (4\text{-}4)$$

where $c = b^n/a$, and $n = m/r$. When Eq. 4-4 is differentiated with respect to time and introduced into Eq. 4-1,

$$nq_u^{n-1}dq_u + c(q_u - q_r)\, dt = 0 \quad (4\text{-}5)$$

is obtained. Equation 4-5 is derived under the assumption that $\frac{dS_u}{dt} = \frac{dV}{dt}$. The substitution $q_u^{-n} = y$ and $k = (2n-1)/n = 2 - 1/n = 2 - r/m$ yields

$$y' + cq_r y^2 - cy^k = 0, \quad (4\text{-}6)$$

which is the differential equation of the water storage. It should be remembered that y is a continuous function of time. The above differential equation was integrated analytically by Yevjevich [1959] for three basic cases: (a) the recharge equal to zero, $q_r = 0$; (b) the recharge a constant different from zero, $q_r = q_{ro} \neq 0$; and (c) the recharge a simple function, $q_r = f(t)$, of time t. A detailed solution of Eq. 4-6 for the three listed cases and a range of values of k can be found in the cited reference. A few selected solutions frequently used to analyze hydrologic systems are presented below.

(a) $q_r = 0$

$$\underline{k = 0 \rightarrow n = 1/2} \quad q_u = q_{uo}(1 + cq_{uo}^{1/2}t)^{-2} , \quad (4\text{-}7)$$

$$\underline{k = 1 \rightarrow n = 1} \quad q_u = q_{uo}e^{-ct} . \quad (4\text{-}8)$$

Both relationships have been used to analyze the recession part of the streamflow hydrograph or the spring discharge hydrograph, see for example Knisel [1972], and Burdon and Safadi [1963].

(b) $q_r = q_{ro} \neq 0$

$$\underline{k = 0 \rightarrow n = 1/2} \quad q_u = q_{ro}\left(\frac{e^{t/R} - \omega}{e^{t/R} + \omega}\right)^2 , \quad (4\text{-}9)$$

where $R = \frac{1}{c}q_{ro}^{-1/2}$ and $\omega = \dfrac{q_{ro}^{1/2} - q_{uo}^{1/2}}{q_{ro}^{1/2} + q_{uo}^{1/2}}$.

$$\underline{k = 1 \rightarrow n = 1} \quad q_u = q_{ro} + (q_{uo} - q_{ro})e^{-ct}, \quad (4\text{-}10)$$

where q_{ro} and q_{uo} are the initial values of q_u and q_r, respectively. The form of Eq. 4-10 was used by Dooge [1973] and others. Caution should be exercised when using the above expressions; the time origin $t = 0$ must always coincide with the time when the uniform recharge $q_r = q_{ro}$ commences.

It should be noticed that only in the two presented cases is it possible to obtain q_u as an explicit function of q_{uo}, q_{ro}, and t. In all other solutions to the differential equation 4-6, a numerical procedure is required to evaluate q_u, for the solution is given in terms of the functional relation $t = T(q_u, q_{uo}, q_{ro})$. This, perhaps, explains the reason hydrologists have resorted to a relatively limited number of computationally convenient models (mostly linear), despite decreased accuracy.

4-2 Basic Assumptions in Model Development

In the further model development, a proper definition of the underground storages is required. This definition is based on several assumptions that are listed below. They may create difficulties of a practical nature in model implementation. However, from a theoretical point of view these assumptions are not restrictive. The assumptions are:

(1) The total volume of water stored in the

underground storage subsystem up to the horizontal water level of the surface storage subsystem can be described by a function of effective porosity, γ, and the water level of the surface storage subsystem, $h(t)$, or by a set of discrete values in a tabular form as

$$W(t) = W[\gamma,\ h(t)]\ . \qquad (4\text{-}11)$$

(2) The state of the underground storage subsystem, i.e., the total water content in the subsystem can be described by a mathematical relation as a function of effective porosity, γ, the water level in the surface storage subsystem, $h(t)$, and an $m \times 1$ vector of observations of the water table levels, $\underline{H}(t)$, at m points of the underground storage subsystem (or by a set of discrete values in tabular form) as

$$V(t) = V[\gamma,\ \underline{H}\ (t),\ h(t)]. \qquad (4\text{-}12)$$

(3) The total water content of the surface storage subsystem can be described by a function of the water level, $h(t)$, in the subsystem (or by a set of discrete values in a tabular form) as

$$S(t) = S[h(t)]. \qquad (4\text{-}13)$$

Equations 4-11 through 4-13 describing the states of the two interconnected subsystems as continuous functions of time are based on the working scheme presented in Fig. 4-1. It should be observed that Eqs. 4-11 and 4-13 provide for a unique correspondence between the $W(\cdot)$ and $S(\cdot)$, so that $W(t) = W[S(t)]$. For simplicity of notation, the variables defined in Eqs. 4-11 through 4-13 will be continuous functions of time unless specified otherwise, so that the argument, t, can be omitted.

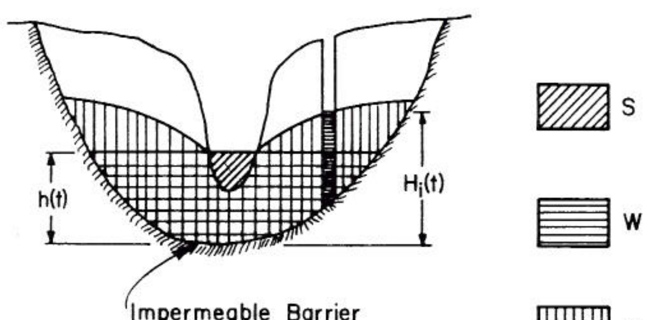

Fig. 4-1. Definition of Basic Variables of the Subsystems.

From the preceding description and Fig. 4-1, it is clear that Eqs. 4-11 and 4-12 imply an impermeable barrier separating the underground storage subsystem from the remaining portion of its own and surrounding drainage basins. It seems justified to emphasize also that the water table of the underground storage subsystem does not necessarily have to be assumed a smooth surface. Additional necessary assumptions are:

(4) The effective rock porosity, γ, of Eqs. 4-11 and 4-12 can be assumed either uniform over the whole underground subsystem or it can vary from one part of the subsystem to another. In the latter case, the rock prosity, γ, may be described by an $m \times 1$ vector of values, corresponding to the vector $\underline{H}(t)$ of Eq. 4-12. It is also assumed that γ can be determined either by a geophysical method, which is more desirable, or by means of the system identification.

(5) An area around the surface storage reservoir is covered by a number of observation boreholes in the permeable underground storage and a vector of observations, $\underline{H}(t)$, exists over a period of time.

(6) From the observed sequences of $q_c(t)$, $q_d(t)$, $q_p(t)$, $q_b(t)$, $q_e(t)$, $q_g(t)$, and $dS(t)/dt$, the time series of $q_u(t)$ may be determined from the basic budget equation (see Eq. 6-8).

(7) Observations of the precipitation series, $p(t)$, at the set of rainfall gauging stations covering the drainage basin are available.

(8) Observations of the state of the surface subsystem, $S(t)$, are recorded as a function of the elevation, $h(t)$, during both a time period prior to the reservoir construction and an initial time period of reservoir operation. It should be clear that depending upon the availability of the data concerning the surface subsystem, two different situations can arise. First, if the records prior to the reservoir construction (natural conditions) are available, all the conclusions--whatever their accuracy--may be extended up to and including selection of the reservoir size. When only the records of the modified system exist, it is only possible to determine the optimal policy and operational rules.

4-3 Mathematical Model of the Underground Subsystem

The first part of this section deals with the derivation of the mathematical expressions needed to formulate the proposed model. The second part uses the relations derived to establish the final form of the model. Assumptions concerning the interrelationships of the two subsystems are outlined whenever appropriate.

Preliminary Mathematical Derivations. In subsequent considerations of the model development, the knowledge of the rate of change of volumes described by Eqs. 4-11, 4-12, and 4-13, with respect to both the time and the elevation, is required. They can be expressed in the following forms:

$$1)\quad \frac{dW}{dt} = \frac{\partial W}{\partial h}\frac{dh}{dt} = D_{wh}\frac{dh}{dt}\ , \qquad (4\text{-}14)$$

in which $D_{wh} = \partial W[\gamma, h(t)]/\partial h$;

$$2)\quad \frac{dV}{dt} = \left[\frac{\partial V}{\partial \underline{H}}\right]^T \frac{d\underline{H}}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$= [\underline{D}_{VH}]^T \frac{d\underline{H}}{dt} + D_{Vh}\frac{dh}{dt}\ , \qquad (4\text{-}15)$$

with $[\underline{D}_{VH}]^T = \{\partial V[\gamma, \underline{H}(t),\ h(t)]/\partial \underline{H}\}^T$ and

$$D_{Vh} = \partial V[\gamma,\ \underline{H}(t),\ h(t)]/\partial h;$$

and

$$3)\quad \frac{dS}{dt} = \frac{\partial S}{\partial h}\frac{dh}{dt} = D_{sh}\frac{dh}{dt}\ , \qquad (4\text{-}16)$$

where $D_{sh} = \partial S[h(t)]/\partial h$.

Evidently, D_{sh} of Eq. 4-16 represents the horizontal area of the surface storage sybsystem for the given elevation $h(t)$. Similarly, D_{wh} of Eq. 4-14 is the effective horizontal area of the underground storage subsystem at the given elevation $h(t)$. These areas multiplied by the rate of change of elevation with respect to time, $dh(t)/dt$, give the rate of change of respective storage subsystem volumes with respect to time.

Values of $\underline{D}_{VH}$ and D_{Vh} of Eq. 4-15 are more difficult to visualize. Nevertheless, they have the same physical interpretation as D_{wh} and D_{sh}, that is to say that $\underline{D}_{VH}$ represents the rate of change of the volume $V(t)$ when the vector $\underline{H}(t)$ is changed for a unit vector and $h(t)$ is kept a constant. By the same token, D_{Vh} is the rate of change of the volume $V(t)$ when $h(t)$ is changed by unity while the vector $\underline{H}(t)$ is kept constant. Clearly, from this discussion, the total rate of change of the volume $V(t)$ is as given by Eq. 4-15.

To assure the usefulness of expressions given by Eqs. 4-14 through 4-16, the following remark is needed. From Eq. 4-16 dh/dt can be written as

$$\frac{dh}{dt} = \frac{1}{D_{sh}} \frac{dS}{dt} = \frac{q_u(t) - Q(t)}{D_{sh}} , \tag{4-17}$$

in which $Q(t)$ is (see Eq. 3-8)

$$Q(t) = q_b(t) + q_e(t) + q_g(t) - q_c(t) - q_d(t) - q_p(t) , \tag{4-18}$$

with all variables defined in Section 3-1. Introducing the right-hand side of Eq. 4-17 into Eq. 4-14 gives

$$\frac{dW}{dt} = \frac{D_{wh}}{D_{sh}}[q_u(t) - Q(t)] = R_{ws}[q_u(t) - Q(t)] , \tag{4-19}$$

with $R_{ws} = D_{wh}/D_{sh}$.

From another point of view, the total water content of the underground storage subsystem, $V(t)$, changes with time by reflecting the prevailing hydraulic conditions. As deduced by simple reasoning, the total change of the volume $V(t)$ is comprised of two factors: (1) flow from the underground storage subsystem to the surface storage subsystem (or conversely), denoted by

$$\frac{dV_u}{dt} = -q_u(t) , \tag{4-20}$$

and (2) the total recharge to the underground storage subsystem, namely

$$\frac{dV_r}{dt} = q_r(t) . \tag{4-21}$$

The sum of Eqs. 4-20 and 4-21 yields

$$\frac{dV}{dt} = \frac{dV_u}{dt} + \frac{dV_r}{dt} = -q_u(t) + q_r(t) = -[q_u(t) - q_r(t)] , \tag{4-22}$$

which is exactly Eq. 4-1. Its solution was discussed in Section 4-1. However, in general, it will no longer be possible to assume that the exchange flow, q_u, depends only on the total content of the underground storage subsystem. Instead, as outlined in the following sections, q_u depends on both the $V(t)$, and the $W(t)$.

Derivation of Mathematical Model. The basic equations of the model of conjunctive use of surface storage and underground storage are derived in this subsection. Let the exchange flow, q_u, be written as

$$q_u = F[V(t), W(t)] , \tag{4-23}$$

where $V(t)$ and $W(t)$ were defined by Eqs. 4-11 and 4-12. Differentiation of Eq. 4-23 with respect to time gives

$$\frac{dq_u}{dt} = \frac{dF[V(t), W(t)]}{dt} = \frac{\partial F}{\partial V}\frac{dV}{dt} + \frac{\partial F}{\partial W}\frac{dW}{dt} . \tag{4-24}$$

From Eq. 4-24, after rearrangement, dV/dt becomes

$$\frac{dV}{dt} = \frac{\frac{dq_u}{dt} - \frac{\partial F}{\partial W}\frac{dW}{dt}}{\frac{\partial F}{\partial V}} = \frac{\frac{dq_u}{dt} - D_{FW}\frac{dW}{dt}}{D_{FV}} , \tag{4-25}$$

with the abbreviated symbols in Eqs. 4-24 and 4-25 being

$$D_{FV} = \frac{\partial F}{\partial V} = \frac{\partial F[V(t), W(t)]}{\partial V} \tag{4-26}$$

and

$$D_{FW} = \frac{\partial F}{\partial W} = \frac{\partial F[V(t), W(t)]}{\partial W} . \tag{4-27}$$

Substituting the right-hand side of Eq. 4-19 for dW/dt in Eq. 4-25 yields

$$\frac{dV}{dt} = \frac{\frac{dq_u}{dt} - D_{FW}[R_{ws}\, q_u - R_{ws}\, Q]}{D_{FV}} . \tag{4-28}$$

Equating the right-hand side of Eq. 4-22 to the right-hand side of Eq. 4-28 gives

$$\frac{dq_u}{dt} - D_{FV} R_{ws} q_u + D_{FV} R_{ws} Q = -D_{FV}[q_u - q_r] , \tag{4-29}$$

from which it follows that

$$\frac{dq_u}{dt} = -D_{FV}\{[1 - \frac{D_{FW}}{D_{FV}} R_{ws}]q_u + \frac{D_{FW}}{D_{FV}} R_{ws} Q - q_r\} . \tag{4-30}$$

Denote D_{FW}/D_{FV} by R_{wv}, and $R_{wv}R_{ws}$ by R, then

$$\frac{dq_u}{dt} = -D_{FV}[(1-R)q_u + R_Q - q_r] , \tag{4-31}$$

or with $(1-R)$ taken out of the brackets,

$$\frac{dq_u}{dt} = -(1-R)D_{FV}[q_u + \frac{R}{1-R} Q - \frac{1}{1-R} q_r] . \tag{4-32}$$

Further simplification of Eq. 4-32 is possible by denoting

$$\psi = \frac{1}{1-R} , \tag{4-33}$$

$$\Phi = \frac{R}{1-R} = \psi R , \tag{4-34}$$

and

$$\Theta = D_{FV}(1-R) = \frac{D_{FV}}{\psi} \tag{4-35}$$

so that Eq. 4-32 finally becomes

$$\frac{dq_u}{dt} = -\Theta[q_u + \Phi Q - \psi q_r] \ , \qquad (4\text{-}36)$$

in which, it should be remembered, q_u, Q, and q_r, as well as Θ, Φ, and ψ, are continuous functions of time.

The differential equation of the form given by Eq. 4-36 can, under certain conditions, be integrated analytically. The number of cases in which this is possible is limited. These specific conditions are examined in the following section.

4-4 Integration of Basic Differential Equation

It is clear that Eq. 4-36 describing the flow between two interconnected storages is much more complex than its counterpart Eq. 4-1 that describes the outflow from a single reservoir. It should be observed that Eq. 4-1 is derived for modelling a natural hydrologic system.

In general, the exchange flow, $q_u(t)$, is a continuous time function, hence it is expressible in terms of an infinite series of polynomials of the form $f_j(V,W) = [f(V,W)]^j$, namely

$$q_u(t) = F(V,W) = \sum_{j=1}^{\infty} c_j f_j(V,W)$$

$$= \sum_{j=1}^{\infty} c_j [f(V,W)]^j \ . \qquad (4\text{-}37)$$

The use of only the first few terms is needed to achieve a reasonable approximation of $q_u(t)$, say $j = 1,\ldots,m$, so that Eq. 4-37 can be written as

$$q_u(t) = \sum_{j=1}^{m} c_j [f(V,W)]^j \ . \qquad (4\text{-}38)$$

Prior to attempting the integration of Eq. 4-36, it is shown that Eq. 4-5 represents a special case of Eq. 4-36. This exercise also helps to select the proper and physically justifiable form of $f(V,W)$. To that end let

$$S_u(t) = (V - W) \ , \qquad (4\text{-}39)$$

where, for the natural conditions of the hydrologic system, W = constant, so that $dW/dt = 0$. Combining the derivative with respect to time of Eq. 4-39 with Eq. 4-22 gives

$$\frac{dS_u}{dt} = \frac{dV}{dt} = -[q_u(t) - q_r(t)] \ . \qquad (4\text{-}40)$$

Furthermore, let

$$q_u(t) = (cS_u)^{1/n} \ , \qquad (4\text{-}41)$$

from which, after differentiating with respect to time and inserting into the right-hand side of Eq. 4-40 in place of dS_u/dt,

$$\frac{dq_u}{dt} = \frac{1}{n} c^{1/n} S_u^{1/n-1} \frac{dS_u}{dt}$$

$$= -\frac{1}{n} c^{1/n} S_u^{(1-n)/n} (q_u - q_r) \qquad (4\text{-}42)$$

is obtained. Eliminating S_u from Eqs. 4-42 and 4-41 yields

$$nq_u^{n-1} \, dq_u + c(q_u - q_r)dt = 0 \ , \qquad (4\text{-}43)$$

which is exactly Eq. 4-5.

Equation 4-43 can also be obtained from Eq. 4-36 directly. Remembering that W = constant, it follows from

$$q_u = F(V,W) = c^{1/n} S_u^{1/n} = c^{1/n} [V - W]^{1/n} \ , \qquad (4\text{-}44)$$

that

$$\frac{\partial F}{\partial V} = \frac{1}{n} c^{1/n} [V - W]^{(1-n)/n} = \frac{1}{n} c^{1/n} S_u^{(1-n)/n} \ , \qquad (4\text{-}45)$$

and

$$\frac{\partial F}{\partial W} = 0 \ . \qquad (4\text{-}46)$$

With $\partial F/\partial W = D_{FW} = 0$, $R_{WV} = D_{FW}/D_{FV} = 0$, so that $R = R_{WV} = 0$, it follows that

$$\Theta = \partial F/\partial V = \frac{1}{n} c^{1/n} S_u^{(1-n)/n} \ , \qquad (4\text{-}47)$$

$$\Phi = 0 \ , \qquad (4\text{-}48)$$

and

$$\psi = 1 \ . \qquad (4\text{-}49)$$

When Eqs. 4-47 through 4-49 are substituted into Eq. 4-36

$$\frac{dq_u}{dt} = -\frac{1}{n} c^{1/n} S_u^{(1-n)/n} (q_u - q_r) \qquad (4\text{-}50)$$

is obtained. This is Eq. 4-42 from which Eq. 4-43 was derived.

From the preceding discussion it appears that the function $f_j(V,W)$ which is to be used for the description of the exchange flow, q_u, under modified conditions ought to contain the difference term $(V - W)$ found in Eqs. 4-39 and 4-44 which are used to describe the underground flow under natural conditions. Indeed, this is an appealing and physically sound form since, remembering that both V and W are nonnegative time functions, it satisfies the following three conditions: (a) $q_u = 0$ when $V = W$; (b) $q_u > 0$ when $V > W$; and (c) $q_u < 0$ when $V < W$.

Since the water stage oscillation of a vast majority of rivers is relatively small, the assumption that the volume W = constant adopted for hydrologic systems under natural conditions is a reasonable approximation of hydraulic conditions. Except in special cases outlined later, this assumption might not be generally valid for a modified system where the water level of a surface reservoir can vary significantly enough to affect the exchange flow. This leads to the conclusion that an additional factor should be included in the function $f_j(V,W)$. For that reason, consider the form $W(V-W)^j$ which seems to be able to satisfy the requirements. The above stated conditions (a) through (c) still hold for this form. Furthermore, for a constant value of the difference, say $(V - W)$ = constant, it gives an increased exchange flow, q_u, as W increases and conversely, which is a desirable property of the function $f_j(V,W)$. In addition, the proposed form bears some resemblance to the formula describing the underground discharge into a gallery from an unconfined aquifer (for example see V. T. Chow [1964], pages 13-13 and 13-14). Hence, for the purpose of further discussion, the function $f(V,W)$ is assumed to be

$$f(V,W) = W^{\lambda}(V-W)^{\nu} , \qquad (4\text{-}51)$$

with λ and ν being constant parameters to be determined for each particular system. With Eq. 4-51, the exchange flow, q_u, can be expressed as

$$q_u = F(V,W) = \sum_{j=1}^{m} c_j [f(V,W)]^j = \sum_{j=1}^{m} c_j [W^{\lambda}(V-W)^{\nu}]^j , \qquad (4\text{-}52)$$

where c_j $(j = 1,\ldots,m)$ is a set of constant parameters satisfying the mathematical conditions for convergence of the series Eq. 4-52.

Since a particular case of the integration of Eq. 4-36 will involve the relation between the volumes W and S defined by Eqs. 4-11 and 4-13, reference is made to the statement after Eq. 4-13, namely that $W(t) = W[S(t)]$. Based on this, some modification of Eq. 4-14 is needed. Combining Eqs. 4-14 and 4-16

$$\frac{dW}{dt} = \frac{\partial W}{\partial S}\frac{dS}{dt} = \frac{\partial W}{\partial S}\frac{\partial S}{\partial h}\frac{dh}{dt} \qquad (4\text{-}53)$$

is obtained, which implies $D_{wh} = \frac{\partial W}{\partial S}\frac{\partial S}{\partial h}$ so that

$$R_{ws} = \frac{D_{wh}}{D_{sh}} = \left(\frac{\partial W}{\partial S}\frac{\partial S}{\partial h}\right)/\left(\frac{\partial S}{\partial h}\right) = \frac{\partial W}{\partial S} . \qquad (4\text{-}54)$$

With this in mind, several possibilities are analyzed as follows.

Case A. The following conditions are satisfied: (1) $m = 1$; (2) $\lambda = 0$; (3) $\nu = 1$; and (4) there is a linear functional relationship between W and S. The conditions (1) through (3) yield

$$q_u = F(V,W) = c_1(V-W) , \qquad (4\text{-}55)$$

and the condition (4) requires that

$$W = L(S) = a + b\,S . \qquad (4\text{-}56)$$

According to Eq. 4-54 and taking the partial derivative of Eq. 4-56 with respect to S, it follows that $R_{ws} = \partial W/\partial S = b$. Furthermore, since $D_{FW} = \partial F/\partial W = -c_1$ and $D_{FV} = \partial F/\partial V = c_1$, it follows that $R_{WV} = D_{FW}/D_{FV} = -1$, from which

$$R = R_{WV}\,R_{ws} = -1\,b = -b . \qquad (4\text{-}57)$$

Substituting Eq. 4-57 into Eqs. 4-33 through 4-35 gives

$$\psi = \frac{1}{1+b};\ \Phi = \frac{-b}{1+b};\ \text{and}\ \ \theta = c_1(1+b) . \qquad (4\text{-}58)$$

Thus, with the values of ψ, Φ, and θ as given by Eq. 4-58, Eq. 4-36 becomes a differential equation with constant coefficients, namely,

$$\frac{dq_u}{dt} = -c_1\ (1+b)\ \left[q_u - \frac{1}{1+b}(bQ+q_r)\right] . \qquad (4\text{-}59)$$

Solution to Eq. 4-59 pertaining to two particular cases was obtained by Yevjevich [1959]; (a) when $bQ + q_r = 0$, the solution is given by Eq. 4-8 in which c should be replaced by $c_1(1+b)$ of Eq. 4-59; and (b) when $bQ + q_r = \text{constant} \neq 0$ the solution is given by Eq. 4-10, where $c = c_1(1+b)$ and $q_{ro} = \frac{1}{1+b}(bQ + q_r)$.

Case B. Let the conditions of Case A hold, except for the condition (4). It is readily seen that when W and S are arbitrary functions of the water table elevation, h, the R_{ws} is a function of h, regardless of the fact that q_u is expressed by a very simple function $F(V,W)$, namely

$$R_{ws} = \frac{\partial W/\partial h}{\partial S_s/\partial h} = R_{ws}(h) . \qquad (4\text{-}60)$$

Thus, ψ, Φ and θ are all functions of h, that is

$$\psi = \psi(h);\ \Phi = \Phi(h),\ \text{and}\ \ \theta = \theta(h) .$$

When these values are substituted into Eq. 4-36

$$\frac{dq_u}{dt} = -\theta(h)[q_u + \phi(h)Q - \psi(h)q_r] \qquad (4\text{-}61)$$

is obtained. This is, even under the assumption that Q and q_r are known, a differential equation with two variables, q_u and h, both functions of time. Hence, in order to obtain a solution to the above equation, it will be necessary to couple Eq. 4-17 and Eq. 4-61. Thus, a solution to a system of two differential equations is required, namely:

$$\frac{dq_u}{dt} = -\theta(h)[q_u + \phi(h)\ Q - \psi(h)\ q_r] , \qquad (4\text{-}62a)$$

$$\frac{dh}{dt} = \frac{1}{D_{sh}}\ [q_u - Q] , \qquad (4\text{-}62b)$$

which, of course, can be converted into a single second-order differential equation of h.

From the theory of differential equations, it is known that only a relatively small number of differential equations with nonconstant coefficients can be solved analytically, i.e., solutions are obtainable only for the equations satisfying fairly rigid requirements concerning the functional relationships contained in the differential equation. In addition, the description of Q and q_r must be carried out piecewise. This creates further difficulties in obtaining an analytic solution. For that reason it is proposed that the solution to the set of Eq. 4-62 be found by a numerical method which will be examined in detail later.

Case C. This pertains to the most generalized approach to solving Eq. 4-36. As demonstrated in Case B, a relaxation of the condition (4) of Case A introduced some difficulties which could not be overcome when attempting to obtain an analytical solution. It can be expected that the further relaxation of the conditions stated in Case A will create additional difficulties. This is illustrated by a simple example. To that end, relax the condition (3), say that $\nu \neq 1$, while retaining the conditions (1) and (2). Then

$$q_u = F(V,W) = c_1(V-W)^{\nu} , \qquad (4\text{-}63)$$

from which it immediately follows that

$$\frac{\partial F}{\partial V} = D_{FV}(V,W) \quad \text{and} \quad \frac{\partial F}{\partial W} = D_{FW}(V,W) . \qquad (4\text{-}64)$$

Since $V = V[\gamma, \underline{H}(t), h(t)]$, it follows that in Eq. 4-36 the term $D_{\overline{FV}} = D_{FV}[\underline{H}(t), h]$, which leads to a system of three differential equations

$$\frac{dq_u}{dt} = -\Theta[q_u + \Phi Q - \psi\, q_r] , \qquad (4\text{-}65a)$$

$$\frac{dh}{dt} = \frac{1}{D_{sh}}(q_u - Q) , \qquad (4\text{-}65b)$$

and

$$\frac{dV}{dt} = - (q_u - q_r) , \qquad (4\text{-}65c)$$

which can be solved numerically.

4-5 Numerical Integration of the Storage Equation

The hydraulic conditions resulting in the simplified flow equations, such as those given by Eq. 4-59 for Case A and Eq. 4-62 for Case B, are infrequently found in the field. Furthermore, hydrologic data are almost exclusively discrete time series. Finally, as already demonstrated, it is necessary to simulate the system states, S and V, in such a manner that will in turn facilitate simulation of the flow components, $q_u(t)$ and $q_e(t)$. This is a compelling reason to devise a numerical approach capable of handling a relatively wide range of situations that may be encountered.

To that end, the set of differential equations 4-65 is rewritten in the finite difference form, for $\Delta t = 1$, as

$$S_t - S_{t-1} = \frac{q_t^u + q_{t-1}^u}{2} - \frac{Q_t + Q_{t-1}}{2}, \qquad (4\text{-}66a)$$

$$V_t - V_{t-1} = - \frac{q_t^u + q_{t-1}^u}{2} + \frac{q_t^r + q_{t-1}^r}{2}, \qquad (4\text{-}66b)$$

and

$$q_t^u - q_{t-1}^u = -\Theta\left[\frac{q_t^u + q_{t-1}^u}{2} + \Phi\,\frac{Q_t + Q_{t-1}}{2} - \psi\,\frac{q_t^r + q_{t-1}^r}{2}\right], \qquad (4\text{-}66c)$$

where Eq. 4-16 was incorporated into Eq. 4-66a, and where the correspondence of the symbols in Eq. 4-66 to those previously defined is obvious. Replacing $\overline{Q}_t = (Q_t + Q_{t-1})/2$, $\overline{q_t^u} = (q_t^u + q_{t-1}^u)/2$ and $\overline{q_t^r} = (q_t^r + q_{t-1}^r)/2$, Eq. 4-66, after rearrangement, becomes

$$S_t = S_{t-1} + \overline{q_t^u} - \overline{Q_t} , \qquad (4\text{-}67a)$$

$$V_t = V_{t-1} - \overline{q_t^u} + \overline{q_t^r} , \qquad (4\text{-}67b)$$

and

$$q_t^u = \frac{1 - \Theta/2}{1 + \Theta/2}\, q_{t-1}^u - \frac{\Theta}{1 + \Theta/2}\, \Phi\, \overline{Q}_t + \frac{\Theta}{1 + \Theta/2}\, \psi \overline{q_t^r} . \qquad (4\text{-}67c)$$

It is, however, customary to take time units of a length over which the flow components can be considered unchanged. Thus, the values $\overline{q_t^u}$, $\overline{q_t^r}$ and $\overline{Q_t}$ can be replaced by q_t^u, q_t^r and Q_t. In this case Eq. 4-67c will be slightly modified. With these modifications, the working set of equations is

$$S_t = S_{t-1} + q_t^u - Q_t , \qquad (4\text{-}68a)$$

$$V_t = V_{t-1} - q_t^u + q_t^r , \qquad (4\text{-}68b)$$

and

$$q_t^u = \frac{1}{1 + \Theta} q_{t-1}^u - \frac{\Theta}{1 + \Theta} \Phi Q_t + \frac{\Theta}{1 + \Theta} \psi q_t^r . \qquad (4\text{-}68c)$$

Thus, a system of nonlinear differential equations was converted into a system of nonlinear algebraic equations which are more convenient to work with. It is to be solved for all $t = 1,2,\ldots,N_t$, where solution is always based on the knowledge of the system at time $t - 1$, $\forall\, t = 1,2,\ldots,N_t$. Consequently, the system can be solved when the three initial conditions, S_o, V_o and q_o^u, are known.

Two points should be emphasized. (1) The value of Q_t consists of several flow components (Eq. 4-18). Under modified conditions, some of these components may be dependent on the system states, S_t and V_t. It appears that the evaporation, $q_e(t)$, has the most significant dependency on S_t. Inaccuracy resulting from the lumped Q_t can, of course, be reduced by splitting it into components. This, however, would introduce additional equations into the system 4-68, which, needless to say, increases the computational burden. (2) The flow components comprising Q_t and q_t^r are always assumed known, they are either observed or generated. Hence, if a simulation of the modified system is desired according to (1), some assumptions concerning q_t^e, q_t^p, q_t^d and q_t^r, as explained in Section 3-2, have to be made.

The above system of equations can be solved by utilizing various computing techniques. The remaining part of this section is devoted to this aspect of the problem. Nevertheless, it is impossible to prescribe a method for each particular case that can be found in the field, since these particular conditions determine preferability of one technique over another. It should be reiterated that, in general, Θ, Φ and ψ are functions of the volumes, V_t and $W_t = W(S_t)$. Thus, Eq. 4-68c can be written as

$$q_t^u = \Gamma(V,W)\, q_{t-1}^u - \Lambda(V,W)\, Q_t + \Pi(V,W) q_t^r , \qquad (4\text{-}69)$$

where $\Gamma(\cdot) = \frac{1}{1+\Theta}$, $\Lambda(\cdot) = \frac{\Theta}{1+\Theta}\Phi = \Gamma(\cdot)\Theta\Phi$, and $\Pi(\cdot) = \frac{\Theta}{1+\Theta}\psi = \Gamma(\cdot)\Theta\psi$. With this in mind, two possibilities are outlined.

(a) When the time intervals are short enough so that the following holds (see Fig. 3-3).

$$\Gamma(V_{t-1}, W_{t-1}) \doteq \Gamma\left(\frac{V_t + V_{t-1}}{2}, \frac{W_t + W_{t-1}}{2}\right),$$

$$\Lambda(V_{t-1}, W_{t-1}) \doteq \Lambda\left(\frac{V_t + V_{t-1}}{2}, \frac{W_t + W_{t-1}}{2}\right),$$

and

$$\Pi(V_{t-1}, W_{t-1}) \doteq \Pi\left(\frac{V_t + V_{t-1}}{2}, \frac{W_t + W_{t-1}}{2}\right).$$

This is a relatively simple computational problem consisting of the following steps.

(1) For the known V_{t-1} and S_{t-1}, find $W_{t-1}(S_{t-1})$ and evaluate the functions $\Gamma(\cdot)$, $\Lambda(\cdot)$, and $\Pi(\cdot)$.

(2) With q_{t-1}^u, Q_t and q_t^r and the evaluated functions of (1), compute q_t^u.

(3) Use the values of q_t^r, q_t^u and Q_t in Eqs. 4-68a and 4-68b to find S_t and V_t, respectively.

(4) Repeat steps (1) through (3) for every $t = 1,2,\ldots,N_t$.

(b) When the above conditions are not satisfied, it will be more difficult to obtain the solution because some iteration procedure is needed (see Fig. 3-4). Computation can be described by several steps as follows:

(1) Assume the values of V_t^a and S_t^a, and find W_t^a.

(2) With the assumed values of V_t^a and W_t^a, compute

$$\Gamma\left(\frac{V_t^a + V_{t-1}}{2}, \frac{W_t^a + W_{t-1}}{2}\right),$$

$$\Lambda\left(\frac{V_t^a + V_{t-1}}{2}, \frac{W_t^a + W_{t-1}}{2}\right),$$

and

$$\Pi\left(\frac{V_t^a + V_{t-1}}{2}, \frac{W_t^a + W_{t-1}}{2}\right).$$

(3) Compute q_t^u according to Eq. 4-68c.

(4) Insert q_t^u into Eqs. 4-68a and 4-68b to find V_t^c and S_t^c.

(5) Repeat steps (1) to (4) until $V_t^c \doteq V_t^a$ and $S_t^c \doteq S_t^a$ (each time redefine $S_t^a = S_t^c$ and $V_t^a = V_t^c$).

(6) Repeat steps (1) through (5) for every $t = 1,2,\ldots,N_t$.

Evidently, an initial guess will have to be made for every $t = 1,2,\ldots,N_t$. However, it does not appear to be a problem in practical computation. That is, the initial guesses of V_t and W_t are obtained by the steps (1) through (3) of possibility (a).

The symbols V_t^a, W_t^a and S_t^a stand for the assumed values and V_t^c, W_t^c and S_t^c stand for the computed values of the volumes V_t, W_t and S_t, respectively.

If a simple mathematical relation between the two subsystems such as that given in the Case A of the preceding section holds, and when finite difference equations are used instead of differential equations, some computational simplifications can be achieved. Under these conditions the parameters given by Eq. 4-58 are constants, namely

$$\psi = \frac{1}{1+b}, \quad \Phi = \frac{-b}{1+b}, \text{ and } \quad \Theta = c_1(1+b) ,$$

which, when inserted into Eq. 4-68c, give

$$\alpha_1 = \frac{1}{1+\Theta} = \frac{1}{1+c_1(1+b)} , \qquad (4\text{-}70)$$

$$\alpha_2 = \frac{\Theta}{1+\Theta}\,\phi = \frac{c_1(1+b)}{1+c_1(1+b)}\,\frac{-b}{(1+b)} = \frac{-c_1 b}{1+c_1(1+b)} , \qquad (4\text{-}71)$$

and

$$\alpha_3 = \frac{\Theta}{1+\Theta}\,\Psi = \frac{c_1(1+b)}{1+c_1(1+b)} \cdot \frac{1}{1+b} = \frac{c_1}{1+c_1(1+b)} . \qquad (4\text{-}72)$$

Since b and c_1 are constants, it implies that α_1, α_2 and α_3 are also constants. Thus, Eq. 4-68c can be written as

$$q_t^u = \alpha_1 q_{t-1}^u - \alpha_2\, Q_t + \alpha_3\, q_t^r , \qquad (4\text{-}73)$$

which is the autoregressive-moving average (ARMA) model. It is very appealing, for it describes the exchange flow, q_t^u, at time t, as a linear function of the same flow at the preceding time $t - 1$, the lumped surface flow components of Eq. 4-18, that is,

$$Q_t = q_t^b + q_t^e + q_t^g - q_t^c - q_t^d - q_t^p ,$$

and the recharge, q_t^r. Furthermore, since α_1, α_2, and α_3 are constants, thus giving the exchange flow independent of the volumes V and W, the former does have to be simulated. Hence, the equation 4-68b may be dropped out of computation so that only two equations must be solved simultaneously for every time interval. In order to be able to solve the equations at time $t = 1$, one must know the intial conditions, q_0^u.

The above discussion demonstrates two points. First, it shows what conditions are implied by assuming the ARMA model for the exchange flow between two reservoirs with significantly varied contents. Second, the ARMA model can be used when there are no data to identify the physical components, such as W and V, and the exchange flow, q^u, as described in Section 4-4, Eq. 4-38.

Chapter V
MATHEMATICAL MODEL FOR RECHARGE

5-1 Introductory Remarks

Recharge to an underground aquifer is a complex hydrologic process. It depends on a large number of atmospheric factors and on the properties of the drainage basin. These usually vary in both time and space. Classical hydrologic literature [Mainzer, 1942; Linsley et al., 1958; Chow, 1964] describes the infiltration as a decreasing function of time measured from the moment when precipitation starts. A frequently used expression is

$$q_{rp} = q_{rc} + (q_{ro} - q_{rc})e^{-kt}, \qquad (5\text{-}1)$$

where q_{rp} = the infiltration capacity, q_{ro} = the maximum infiltration capacity, q_{rc} = the infiltration rate approached asymptotically by the infiltration capacity, q_{rp}, as the soil becomes saturated, and k = constant river basin characteristic. Clearly, the infiltration capacity, q_{rp}, equals the maximum infiltration rate, q_{ro}, at time t = 0. The above expression is based on the assumption that the supply rate, p, is greater than or equal to the infiltration capacity ($p \geqq q_{rp}$). In that case the actual infiltration, q_r, equals the infiltration capacity q_{rp}. If this is not the case, namely when the supply rate, p, is smaller than the infiltration rate, q_{rp}, then $q_r = p$.

From examples given in the cited references it can be seen that the infiltration rate capacity, q_{rp}, rapidly approaches the constant value q_{rc}. On the other hand, a study of reservoir design and operation involving optimization can at best be based on daily observations. More frequently, the data are discretized over much longer time intervals. The following consideration assumes a day as the basic time unit. The rainfall events are assumed uniform over the selected time unit.

The conditions which determine the recharge, usually found in areas of karstified limestone, can be regarded as distinct from those prevailing in the more frequent river basins. Several factors affecting recharge to karstic aquifers are important [Le Grand, 1973] such as: (a) water movement occurs through privileged routes such as fissures, cracks, faults, and even large underground river channels; (b) vegetative cover is poor or nonexistant; and (c) geomorphological features allow for accumulation of large quantities of water over small areas, from which concentrated recharge of high intensity takes place.

All these factors, combined with the inevitable river basin heterogeneity and often the underground communication between adjacent catchments, make the physical recharge models speculative, if not unrealistic. The application of a simpler mathematical model for the description of the recharge to a karst aquifer is unavoidable. This approach seems particularly appealing when the recharge model was used in a process seeking optimal use of a surface reservoir physically connected to a natural underground storage.

The modelling of recharge to karst aquifers is, as is karst hydrology itself, a relatively recent development. Artificial recharge to a karst aquifer was described by Green [1967], while natural recharge to several small drainage basins in the United States was analyzed by Knisel [1972], with the recharge model based on daily precipitation given by

$$R = \frac{a\,b\,p}{a^2 + p^2}, \qquad (5\text{-}2)$$

where R = the fraction of the daily precipitation, p, which infiltrates, and a and b are constant characteristic parameters of the drainage basin. Two important conclusions can be drawn from the Knisel study: (1) description of karst hydrologic systems by linear models is a satisfactory approximation, and (2) when the recharge given by Eq. 5-2 is used in simulation of springflows a better fit to the observed data is obtained than when the precipitation events are used for the same purpose. In addition, it is readily seen from Eq. 5-2 that the recharge model is an appropriate nonlinear transformation of the daily rainfall.

Based on the above conclusions, an attempt is made herein to develop a mathematical model for recharge to karst aquifers from the data collected under the natural hydrologic conditions. By relaxing the assumption implied by Eq. 5-2 and by utilizing a model which is able to relate the recharge to both the daily precipitation and conditions in the river basin created by preceding hydrologic events, high quality simulation models were expected.

5-2 Model Formulation

It was demonstrated in Chapter IV (Eq. 4-40) that, under natural hydrologic conditions, the continuity equation of an underground subsystem can be written as

$$\frac{dS_u(t)}{dt} = -[q_u(t) - q_r(t)], \qquad (5\text{-}3)$$

where $S_u(t)$ is given by Eq. 4-39, and $q_u(t)$ and $q_r(t)$ are defined as the underground exchange flow and the recharge to the underground subsystem, respectively (see Eq. 3-6).

When the hydrologic system is assumed linear, then

$$\frac{dq_u(t)}{dt} = -c[q_u(t) - q_r(t)] \qquad 5\text{-}4)$$

is obtained according to Eq. 4-5 for n = 1. The variables of Eq. 5-4 are continuous time functions. Since the observations of the hydrologic time series are almost always given as discrete values over specified time intervals, and since treatment of the discretized data usually gives satisfactory results, Eq. 5-4 is rewritten in a finite difference form for two consecutive time points as

$$\frac{q_t^u - q_{t-1}^u}{t - (t-1)} = -c(q_t^u - q_t^r) \ . \qquad (5\text{-}5)$$

The notation was changed slightly to reflect the use of discrete observations. The correspondence to continuous variable notation is evident. Depending on what form of the right-hand side of Eq. 5-5 is used, two slightly different cases are presented:

Case 1

$$q_t^u - q_{t-1}^u = -c\left[\frac{q_t^u + q_{t-1}^u}{2} - q_t^r\right] , \qquad (5\text{-}6)$$

which, after a re-arrangement, gives

$$q_t^u = \frac{2-c}{2+c} q_{t-1}^u + \frac{2c}{2+c} q_t^r \ . \qquad (5\text{-}7)$$

Case 2

$$q_t^u - q_{t-1}^u = -c(q_t^u - q_t^r) , \qquad (5\text{-}8)$$

from which it follows that

$$q_t^u = \frac{1}{1+c} q_{t-1}^u + \frac{c}{1+c} q_t^r \ . \qquad (5\text{-}9)$$

Regardless of the expression used, it is readily seen that Eqs. 5-7 and 5-9 are mathematical formulations of the familiar autoregressive-moving average linear model of size (1,1). Detailed treatment of this type of model is described by Jenkins and Watts [1968], and by Box and Jenkins [1972], from which the abbreviated notation, ARMA (1,1), was adopted.

Let the recharge, q_t^r, of Eqs. 5-7 and 5-9 be expressed in terms of a linear combination of some functions of daily precipitation, $f_j(p)$, the form of which is elaborated upon in the subsequent text. Namely, let

$$q_t^r = \beta_o + \beta_1 f_1(p) + \ldots + \beta_m f_m(p) , \qquad (5\text{-}10)$$

with $\beta_o, \beta_1, \ldots, \beta_m$ being constant parameters, referred to in the further text as the vector $\underline{\beta}$. Since the recharge is a function of the daily rainfall at time t and is also affected by the conditions in the drainage basin created by the precipitation of the several days preceding the day t, the linear combination on the right-hand side of Eq. 5-10 should reflect these facts. The vector $\underline{\beta}$ of Eq. 5-10 cannot be identified directly. However, the identification can be done in an indirect manner as follows.

Inserting the right-hand side of Eq. 5-10 into Eqs. 5-7 and 5-9 yields

$$q_t^u = \phi_o + \phi_1 q_{t-1}^u + \phi_2 f_1(p) + \ldots + \phi_{m+1} f_m(p) , \qquad (5\text{-}11)$$

where the coefficients $\phi_o, \phi_1, \ldots, \phi_{m+1}$ for Case 1 are given by

$$\phi_o = \beta_o \frac{2c}{2+c} \ ; \ \phi_1 = \frac{2-c}{2+c} \ ; \ \phi_2 = \beta_1 \frac{2c}{2+c}; \ldots ;$$

$$\phi_{m+1} = \beta_m \frac{2c}{2+c} \ . \qquad (5\text{-}12)$$

The vector $\underline{\phi}$ for Case 2 is described by

$$\phi_o = \beta_o \frac{c}{1+c} \ ; \ \phi_1 = \frac{1}{1+c} \ ; \ \phi_2 = \beta_1 \frac{c}{1+c}; \ldots ;$$

$$\phi_{m+1} = \beta_m \frac{c}{1+c} \ . \qquad (5\text{-}13)$$

Equation 5-11 has retained the basic characteristics of Eqs. 5-7 and 5-9, i.e., it is still a mathematical formulation of the ARMA linear system. However, the order of the model is changed. Instead of only one term of the moving average part of Eqs. 5-7 and 5-9, Eq. 5-11 has m terms, thus it is defined as an ARMA (1,m) model.

Application of this type of linear model to hydrologic processes was described by Yevjevich [1972c] The model is usually convenient from the computational point of view because the order of the MA part, m, is usually a small number; it is very seldom greater than three. In many instances, m = 1 appears to be a sufficiently accurate approximation of the system. This can be explained by the fact that the information contained in the terms j = 2, j = 3, ... is already in the autoregressive (AR) part of the model. In the present model, the AR part consists of q_{t-1}^u. These properties of the model make it attractive for use under the conditions that the system response depends on the previously realized events, as is the case with infiltration.

From Eq. 5-11 the system can be identified. First, the vector $\underline{\phi}$ is estimated. From that, the vector $\underline{\beta}$ is found for either Case 1 or Case 2. The vectors $\underline{\beta}$ of the two cases given by Eqs. 5-7 and 5-9 are not significantly different from each other. Nevertheless, the form given by Eq. 5-7 seems more appealing and is recommended for use. However, both cases should be investigated and the one which gives a better approximation to the observed set of data should be selected. The measure of goodness of fit is discussed in the next section.

5-3 Calibration of the ARMA (1,m) Model

It is evident that the vector $\underline{\phi}$ can be estimated from Eq. 5-11 when a sequence of observations of the exchange flow, q_t^u, and of precipitation, p_t, are available. The estimation can be done by the least squares estimation method. It should be noticed that the coefficient ϕ_o of Eq. 5-11 can easily be forced to zero by the computational algorithm, which results in $\beta_o = 0$ in Eqs. 5-10, 5-12, and 5-13.

In the course of the model calibration, several aspects are of interest: observation of a sequence of the underground outflow, q_t^u, the form of the functions of precipitation, $f_j(p)$, and the size of the moving average part, m. There are no precise answers to the last two questions. However, some generally accepted rules in approaching the problem do exist. They are outlined below.

A set of data for the recharge model consists of daily observations of precipitation events at least one rainfall gauging station and of daily observations of the underground exchange flow into a reach of the streamflow channel, q_t^u, both collected while the system is under natural conditions. If there are more than one observation station, a weighted average according to the area covered by each station is recommended. The exchange flow series can be estimated from the observations of input and output series of the surface subsystem from Eqs. 3-5 and 3-7. When these equations are combined, they give the discretized form of the continuity equation of the system as

$$q_t^u = q_t^b + q_t^e + q_t^g - q_t^c - q_t^d - q_t^p + (S_t - S_{t-1}), \quad (5\text{-}14)$$

with all the variables as defined in Chapter III.

Selection of the form of the functions $f_j(p)$. This aspect of the system identification is probably the most important. Yet, there is no method for selecting the form of the $f_j(p)$. Several alternatives must be considered and the one which appears to be most suitable is chosen. Nevertheless, there is no guarantee that some other form of the $f_j(p)$ would not give a better fit. The following is only a guideline in selecting the form of the $f_j(p)$ functions.

It seems justified to attempt selection of the function $f_j(p)$ which is able to reflect the existing conditions affecting recharge. For that reason, some polynomial transformation involving precipitation at times t, t - 1, t - 2,..., seems suitable. Let the generalized expression of the $f_j(p)$ be of the form

$$f_j(p) = (p_t)^{z_1} (p_{t-j+k})^{z_2}, \quad (5\text{-}15)$$

where p_t and p_{t-j+k} are observations of daily rainfall on days t and t-j+k, respectively, and z_1 and z_2 are constants. From this, several particular choices can be derived.

(1) Let $z_1 = 0$, $z_2 = 1$, and $k = 1$, then

$$f_j(p) = p_{t-j+1}, \quad (5\text{-}16)$$

so that the recharge can be written as

$$q_t^r = \beta_1 p_t + \beta_2 p_{t-1} + \dots + \beta_m p_{t-m+1}. \quad (5\text{-}17)$$

This is the simplest form, where $f_j(p) = p_{t-j+1}$ is precipitation of j - 1 days prior to the time t. The form of the MA model of Eq. 5-17 is frequently used in hydrologic modelling and is considered to be a sufficently good approximation of the processes involved.

(2) Let $z_1 = 1$, $z_2 = 1$, $k = 1$, then

$$f_j(p) = p_t p_{t-j+1}, \quad (5\text{-}18)$$

from which it follows that

$$q_t^r = \beta_1 p_t^2 + \beta_2 p_t p_{t-1} + \dots + \beta_m p_t p_{t-m+1}. \quad (5\text{-}19)$$

In a similar manner some other forms can be derived and investigated. It should be noticed that Eqs. 5-17 and 5-19 represent the response function known as the linear unit hydrograph and the non-linear unit hydrograph, respectively.

Selection of the size of the MA part, m, can be performed by using the method outlined by Jenkins and Watts [1968]. To that end, let $\hat{q}_t^u$ represent the estimated value of the underground exchange flow, q_t^u. Thus,

$$\hat{q}_t^u = \phi_1 q_{t-1}^u + \phi_2 f_1(p) + \dots + \phi_{m+1} f_m(p). \quad (5\text{-}20)$$

From this, the vector $\underline{\phi}$ is estimated by solving a set of m + 1 linear equations which minimize the sum of the squares of the differences between the observed and the estimated values of the underground exchange flow. The sum of the squares, ε, is given by

$$\varepsilon = \sum_{t=1}^{N_t} (q_t^u - \hat{q}_t^u)^2$$

$$= \sum_{t=1}^{N_t} [q_t^u - \phi_1 q_{t-1}^u - \phi_2 f_1(p) - \dots - \phi_{m+1} f_m(p)]^2, \quad (5\text{-}21)$$

with N_t = the total number of observed data points. Evidently, the minimized value of ε depends on the number of terms included in the MA part of the model of Eq. 5-20. Let $\sigma_m^2 = \varepsilon/N_t$, where m denotes the size of the MA part of the model. When the values of σ_m^2 are computed and plotted against m, for m = 1, 2,..., an empirical function is obtained. This function can take one of the following two shapes: either it reaches the minimum for a certain value of m after which it increases with inclusion of more terms, or it is a nonincreasing function of m. In the former case, the size of the MA part is that m which gives the minimum value of σ_m^2. When the latter shape is found, the size m after which no significant reduction in σ_m^2 is accomplished should be selected. This makes the decision concerning m somewhat arbitrary.

After the function, $f_j(p)$, and the model size, m, are chosen, the estimated vector, $\underline{\phi}$, must be converted into the desired recharge model vector, $\underline{\beta}$. When the form of the model is that of Case 1 (Eq. 5-7), the parameters $\beta_o, \beta_1, \dots, \beta_m$ are given as follows:

$$\beta_o = \frac{2+c}{2c}\phi_o\ ;\ \beta_1 = \frac{2+c}{2c}\phi_2; \dots;\ B_m = \frac{2+c}{2c}\phi_{m+1}, \quad (5\text{-}22)$$

where in Eq. 5-22

$$c = \frac{2(1-\phi_1)}{1+\phi_1}. \quad (5\text{-}23)$$

When the Case 2 is chosen, the vector $\underline{\beta}$ is

$$\beta_o = \frac{1+c}{c}\phi_o;\ \beta_1 = \frac{1+c}{c}\phi_2; \dots;\ \beta_m = \frac{1+c}{c}\phi_{m+1}, \quad (5\text{-}24)$$

where in Eq. 5-24

$$c = \frac{1-\phi_1}{\phi_1} \quad (5\text{-}25)$$

5-4 Model Application

In order to demonstrate the validity of the recharge model developed above, the computational results of a case study are presented herein. The model was applied to the simulation of the flow at the large karst spring at the source of the Trebišnjica River in Yugoslavia. The drainage area was determined to be

926 km^2. The estimated statistical parameters of the springflow, assuming that the series is stationary, have been determined by Graupe et al., [1975] as: the mean daily discharge $\overline{q}_u = 45.36$ m^3/s., the variance of the daily flow $\sigma^2 = 2270.8$ resulting in the standard deviation $\sigma = 47.65$, the first serial correlation coefficient $r_1 = 0.937$.

Data availability. In addition to daily observations of the springflow from January 1, 1954 until October 31, 1967 (on which date filling of the Grančarevo Reservoir began, resulting in maximum water depth at the Spring of about 70 meters), the amount of daily precipitation was observed and recorded at 21 rainfall gauging stations unevenly distributed over the drainage basin. The rainfall observations are available from January 1, 1954 until December 31, 1967. However, almost every station had some discontinuities in observations.

Other data important in recharge modelling such as evaporation (or effective precipitation), snow pack and snow melt, and the area over which the observed precipitation record is representative were not available. For that reason no attempt has been made to incorporate these factors into the present study despite the fact that they may have an impact on the results.

Data preparation. For the reasons explained above, the precipitation data used in this study were prepared as follows:

$$p_t = \frac{1}{n_{pt}} \sum_{j=1}^{21} p_{t,j} I_{t,j} , \qquad (5\text{-}26)$$

where n_{pt} is number of observation stations at which the records exist for the day t, namely

$$n_{pt} = \sum_{j=1}^{21} I_{t,j} , \qquad (5\text{-}27)$$

with p_t = the arithmetic average of precipitation estimated from the available observations over the drainage basin during the day t, $p_{t,j}$ = the rainfall observation at station j on day t, and $I_{t,j}$ is the indicator function defined as $I_{t,j} = 1$ if observation of $p_{t,j}$ exists, and $I_{t,j} = 0$ otherwise.

The observed precipitation records were given in millimeters per day. These were converted to the units commensurable with the observed discharge (m^3/sec). The transformation factor for the given drainage area is: 1 mm/day = 0.001 926 10^6 (1/86,400) = 10.72 m^3/sec. It was found that the average rainfall is $\overline{p} = 53.73$ m^3/sec, so that the runoff coefficient is $C_r = \overline{q}_u/\overline{p} = 45.36/53.73 = 0.894$.

Based on the data described above, computations were performed for two distinct conditions of the drainage basin as follows:

A. System identification over dry periods. The definition of the dry period is somewhat arbitrary, but rather strict conditions have been imposed. For the purposes of this analysis, an event observed on day t was regarded as suitable for computation if no precipitation greater than 10m^3/sec (approximately 1 mm/day) occurred during that day and the two days preceding the day t. In other words, it was assumed that, when the above conditions were satisfied, there was no recharge to the underground aquifer. With $q_r(t) = 0$, Eq. 5-4, when transformed into discrete form and for a time unit $\Delta t = 1$, has a solution

$$q_t^u = q_{t-1}^u \, e^{-c} \qquad (5\text{-}28)$$

where the symbols are as defined previously. The purpose of the identification was to estimate parameter c. This was evaluated by several approaches.

The first approach was based on the average value of the coefficient $c_t = \ln(q_{t-1}^u/q_t^u)$ evaluated for all values of t for which the required conditions were satisfied. It was found that $c = 0.065129$, from which, for Case 1 of Section 5-2, the parameter ϕ_1 given by Eq. 5-12 is

$$\phi_1 = \frac{2 - 0.065129}{2 + 0.065129} = 0.9369 . \qquad (5\text{-}29)$$

For Case 2, the parameter ϕ_1 is given by Eq. 5-13 as

$$\phi_1 = \frac{1}{1 + 0.065129} = 0.9388. \qquad (5\text{-}30)$$

However, when the parameter ϕ_1 is evaluated by Eq. 5-28,

$$\phi_1 = e^{-c} = e^{-0.065129} = 0.9369 \qquad (5\text{-}31)$$

is obtained.

The second approach is based on the least squares estimate of the first serial correlation coefficient of the observations satisfying the required conditions (not the first correlation coefficient for the total springflow record). For the estimated values of r_1, following the reverse procedure of that given by Eqs. 5-29 through 5-31, the coefficient c is estimated as follows:

For Case 1, Eq. 5-23 gives

$$c = \frac{2(1 - r_1)}{1 + r_1} = \frac{2(1 - 0.985)}{1 + 0.985} = 0.042901 . \qquad (5\text{-}32)$$

For Case 2, Eq. 5-25 gives

$$c = \frac{1 - r_1}{r_1} = \frac{1 - 0.985}{0.985} = 0.043841 . \qquad (5\text{-}33)$$

From Eq. 5-31 it follows that

$$c = \ln(1/r_1) = \ln(1/0.958) = 0.042907. \qquad (5\text{-}34)$$

These results point to the fact that the estimated model parameters, ϕ_1 and c, are essentially the same for the two cases. However, the method of estimation affects the magnitude of the parameter c significantly, even though the parameter ϕ_1 remains almost unchanged.

B. Identification of the system under general conditions. It is inconvenient and mathematically incorrect to treat the system piecewise according to the hydrologic conditions prevailing at different time periods. Hence, estimation of the parameters to be used in the simulation of the system under rather general conditions is required. Thus, a complete set of the available observations should be used to estimate the parameter vector $\underline{\phi}$.

The model parameter vector $\underline{\phi}$ is evaluated for the precipitation functions given by Eq. 5-17. From the vector $\underline{\phi}$, the vector $\underline{\beta}$ is evaluated for Case 1 and Case 2. Aspects of the size of the model, m, are treated by simply evaluating the model parameters and the goodness of fit for various values of its size. The parameter vector $\underline{\phi}$ is estimated by the least squares method, whereas the goodness of fit is measured by the ratio of the mean square error and the estimated variance of the process being fitted. In the case of the springflow record,

$$\chi_m = \frac{\sigma_{em}^2}{\sigma_q^2} \tag{5-35}$$

where σ_{em}^2 is the mean square error (MSE) estimated by

$$\sigma_{em}^2 = \frac{1}{N_t-1} \sum_{t=1}^{N_t} e_m^2 , \tag{5-36}$$

and σ_q^2 is the estimate of the variance of the exchange flow

$$\sigma_q^2 = \frac{1}{N_t-1} \sum_{t=1}^{N_t} (q_t^u - \overline{q})^2 , \tag{5-37}$$

with $\overline{q}$ in Eq. 5-37 estimated by

$$\overline{q} = \frac{1}{N_t} \sum_{t=1}^{N_t} q_t^u . \tag{5-38}$$

In Eqs. 5-35 and 5-36, m denotes the size of the moving average model of Eq. 5-21.

Figure 5-1 gives the MSE, σ_{em}^2, for the precipitation function given by Eq. 5-17 when the model is regarded as stationary for m = 1, 3, 5. From this graph, it can be seen that the simulation does not improve significantly as m increases from m = 1 to m = 3 and m = 5. More importantly, this graph shows that the variation of the daily springflow that cannot be explained by the ARMA(1,1) is only less than 8.0 percent of the total variation of the daily springflow, which is estimated to be 2270. In addition, Fig. 5-1 shows the MSE, evaluated in an indirect manner, for the ARMA(1,0). The ARMA(1,0) is, in effect, the first order autoregressive model. Thus, from the estimated first serial correlation coefficient, $r_1 = 0.937$, and from the expression $\hat{\sigma}_e^2 = (1 - r_1^2)\hat{\sigma}_q^2$, $\hat{\sigma}_e^2 = 277$, or slightly over 12 percent. This result is plotted for m = 0.

Table 5-1 shows the parameters β_j (j = 1,...,m), where m = 1, 3, and 5. For each model size, m, the parameters are evaluated for Case 1 according to Eq. 5-22 (upper line) and for Case 2 according to Eq. 5-24 (lower line). Figure 5-2 depicts graphically the parameters of Case 1. Notice that the sum of the parameters, $\sum\beta$, for each particular model size is very close to the runoff coefficient, $C_r = 0.894$, as it should be.

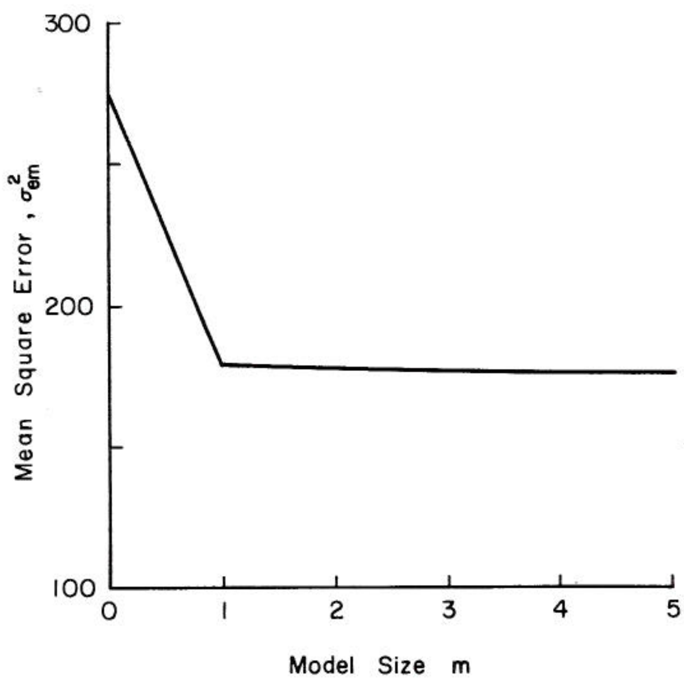

Fig. 5-1. Mean Square Error of the Runoff Prediction at the Trebišnjica River Spring (1954-1967) versus the Model Size.

Table 5-1. Parameters of the Stationary Recharge Model for Linear Precipitation Functions.

Model size m	j / Case	1	2	3	4	5	Σβ	Parameter c
		Parameter vector $\hat{\underline{\beta}}$						
1	1	0.8859					0.8859	0.1211
	2	0.8859					0.8859	0.1289
3	1	0.7906	0.1269	-0.0310			0.8865	0.1300
	2	0.7906	0.1269	-0.0310			0.8865	0.1390
5	1	0.7956	0.1298	-0.0398	0.0314	-0.1318	0.8852	0.1288
	2	0.7956	0.1298	-0.0398	0.0314	-0.0318	0.8852	0.1376

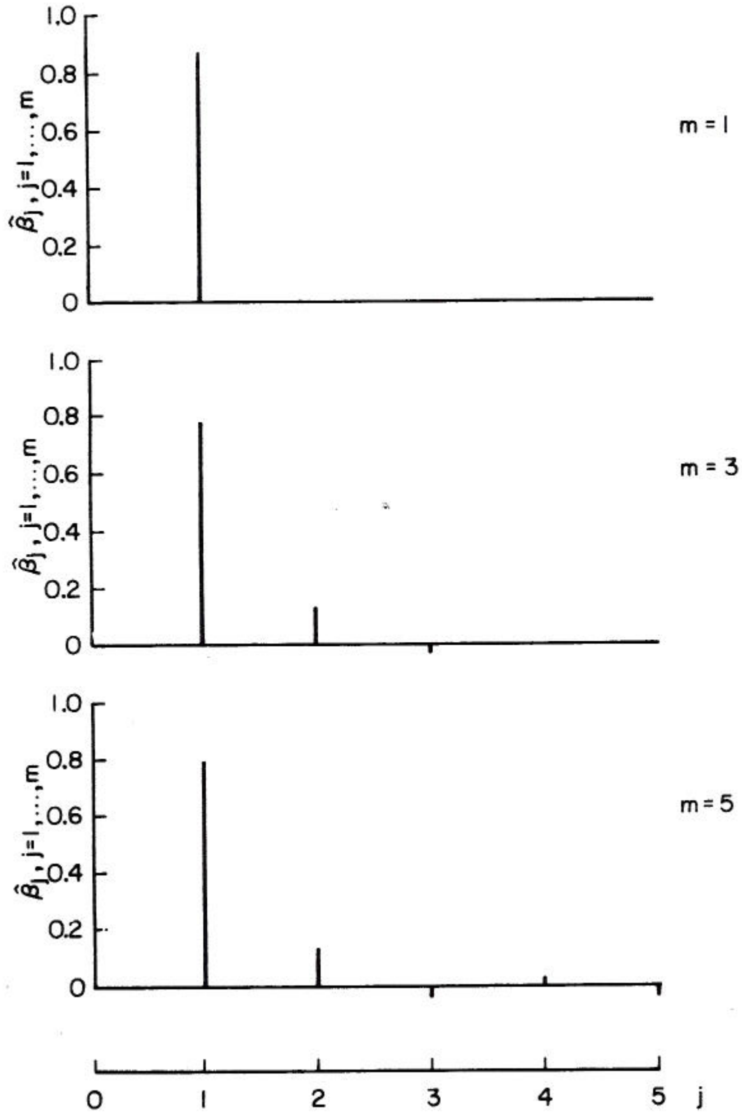

Fig. 5-2. Estimated Parameters, $\hat{\beta}$, of the MA Model for Recharge for Model Size m = 1, m = 3, and m = 5.

The preceding results confirm that, for a reasonable good prediction of the springflow based on precipitation, only a few parameters of the ARMA model must be estimated. Furthermore, since the springflow simulation by the ARMA model is satisfactorily good, acceptance of its part, i.e., the MA model for recharge, is justified. Using the parameters of this MA model and precipitation, the recharge, $q_r(t)$, of Eq. 4-36 can be estimated for either the natural or modified conditions of the hydrologic system.

Chapter VI
IDENTIFICATION OF THE SYSTEM

In attempting to identify the parameters of the hydrologic system described in previous chapters, the existance of two conditions must be kept in mind. Those conditions were defined as natural and modified conditions corresponding to the planning and operation stages of the storage, respectively.

Depending on the objective of the system identification, the system conditions, and a number of factors characterizing the drainage basin under consideration, various procedures of system identification can be applied. A precise outline of the procedure to be used in each particular case seems not feasible. Instead, the method of identification is chosen according to the field conditions.

The identification of the surface flow components is of no concern in this study. The following is a brief discussion of those system components which are used in the optimization scheme and an outline of the methods for the underground storage subsystem identification.

As already outlined, performance of the underground storage subsystem is usually affected by the system modification. However, it is assumed that the outside component entering the subsystem, namely the recharge $q_r(t)$, remains independent of the modification introduced. As a result of this assumption, the parameters of the recharge model identified under natural conditions are regarded valid after the reservoir has been constructed. A method to determine the parameters of the recharge model was described in Chapter V. Thus, the recharge $q_r(t)$, along with the surface flow components is assumed to be known.

6-1 Surface Reservoir Geometry

In order to carry out the proposed optimization, the surface reservoir geometry must be identified. Two components are of interest: (1) water content, S(h); and (2) surface area, $A_r(h)$. According to the assumptions made in Chapter IV, these two geometric components are easily defined from topographic maps and surveying data gathered prior to the system modification. The surface reservoir content, S(h), is then described as a function of the water table elevation measured either from sea level or from an arbitrary reference point [Yevjevich, 1959]. Some form of polynomial may be used to best fit the volume to level relationship. Likewise, the reservoir area, $A_r(h)$, may be described by a polynomial fit.

In addition to S(h) and $A_r(h)$, under some conditions, a functional relation, $A_r[S(t)]$, may be desired to help avoid lengthy computation of h from the known values of S(h). This relationship does not appear to be difficult to establish.

6-2 Underground Subsystem Geometry

In Chapter IV, magnitudes of W and V were assumed to be known. They were defined as underground water content up to the horizontal water level of the surface subsystem and the groundwater table, respectively, as shown in Fig. 4-1.

According to the assumption of the existance of an impermeable barrier that encompasses the underground subsystem, the magnitude of W can be defined as a function of the same elevation, h, that is used for identification of S(h). Additional requirements are that the impermeable barrier and the effective rock porosity, γ, be defined. While the former is beyond the scope of this study, some remarks concerning the latter are given below.

The volume V is not defined by purely geometric components of the system. It can also be regarded as a hydraulic characteristic of the underground storage subsystem for the following reasons. First, besides being dependent on the effective porosity, γ, and the surface subsystem elevation, h, the value V also depends on the shape of the groundwater table and the complex porosity structure. For that reason, system identification may require a large amount of data to cover a wide range of conditions that may exist. Second, as pointed out, the hydraulic conditions of the underground subsystem after construction of the surface reservoir may be distinctly different from the ones that exist under natural conditions. Thus, the system identification under the latter conditions may not hold under the former. From this it can be concluded that the process of identifying the $V(\gamma,\underline{H},h)$ relationship may be expensive and somewhat time consuming.

For identification of V, it is assumed that prior to the reservoir construction, a number, say m, boreholes were drilled into the underground subsystem and that observations of groundwater levels were recorded. Let the observations be given in the form of vectors $\underline{H}(t)$, t = 1,2,...,N, where

$$\underline{H}(t) = [H_1(t),\ldots,H_m(t)]^T. \tag{6-1}$$

With available observations of the boundary conditions, i.e., the water table elevation h(t), t = 1,2,...,N, at the contact of the surface and underground subsystems, the vector of the level difference can be established as

$$\underline{D}(t) = [\delta_1(t),\ldots,\delta_m(t)]^T = \underline{H}(t)-h(t)$$

$$= [H_1(t)-h(t),\ldots,H_m(t)-h(t)]^T. \tag{6-2}$$

With the knowledge of H(t) and h(t) the hypersurface of the groundwater table can be approximated. When the effective porosity, γ, is known, the volumes can be determined and recorded as a time sequence V(t), t = 1,2,...,N. Next, a relationship between V(t) and $\underline{D}(t)$ is postulated and its parameters evaluated.

A possible model is examined by means of an example. Let the desired relation be assumed as

$$\hat{V} = \underline{A}^T\,\underline{D} + \underline{D}^T\,\underline{B}\,\underline{D}, \tag{6-3}$$

where $\hat{V}$ denotes volume estimated by means of the observed vector $\underline{D}$, and $\underline{A}$ and $\underline{B}$ are matrices of constants of sizes m x 1 and m x m, respectively. It is desired to define $\underline{A}$ and $\underline{B}$ so as to minimize the sum of squares of errors over all t = 1,2,...,N.

The problem is solved by solving a set of equations representing partial derivatives with respect to the unknown parameters $\underline{A}$ and $\underline{B}$, namely

$$\sum_t \frac{\partial}{\partial \underline{A}} \{V - \underline{A}^T \underline{D} + \underline{D}^T \underline{B} \underline{D}\} = \underline{0}, \tag{6-4a}$$

and

$$\sum_t \frac{\partial}{\partial \underline{B}} \{V - \underline{A}^T \underline{D} + \underline{D}^T \underline{B} \underline{D}\} = \underline{0}. \tag{6-4b}$$

It can be easily shown that, even with such a simple relation as that assumed by Eq. 6-3, a large number of parameters is needed $(m + m^2)$. It can also be shown that the above set of equations involves complete polynomials of order four.

To avoid the drawbacks stemming from a large amount of computation, a special mathematical technique described by Ivakhnenko [1970, 1971] can be used. The technique is called the Group Method of Data Handling (GMDH). Its essence is solving a number of small problems instead of a large one, thus avoiding inversion of large matrices. For example, if $m = 8$, the size of the matrix formed by Eq. 6-4 is $(m + m^2) = 8 + 64 = 72$. However, computation can be reduced by the GMDH to solving seven problems with three parameters each. Thus, instead of inverting a 72 x 72 matrix, seven matrices of size 3 x 3 are to be processed.

In brief, the GMDH can be described as follows: Let there be eight variables, say $x_1,\ldots,x_8$. It is postulated that every two variables can be modelled into another variable say, y_1, y_2, y_3, and y_4, as

$$y_1 = a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_1x_2 ,$$

$$y_2 = a_{2,1}x_3 + a_{2,2}x_4 + a_{2,3}x_3x_4 ,$$

$$y_3 = a_{3,1}x_5 + a_{3,2}x_6 + a_{3,3}x_5x_6 ,$$

and

$$y_4 = a_{4,1}x_7 + a_{4,2}x_8 + a_{4,3}x_7x_8 . \tag{6-5}$$

From Eq. 6-5 the parameters $a_{i,j}$, $(i = 1,\ldots,4;$ $j = 1,\ldots,3)$ are estimated by solving four sets of three equations. With the parameters $a_{i,j}$, estimates of y_i values are obtained, which are then modelled into z variables, say by

$$z_1 = b_{1,1}y_1 + b_{1,2}\, y_2 + b_{1,3}\, y_1\, y_2$$

and

$$z_2 = b_{2,1}y_3 + b_{2,2}\, y_3 + b_{2,3}\, y_3\, y_4 . \tag{6-6}$$

Next, parameters $b_{1,1},\ldots,b_{2,3}$ as well as z_1 and z_2 are estimated. Finally, z_1 and z_2 are described by another variable, say V, as

$$V = c_1z_1 + c_2z_2 + c_3z_1z_2, \tag{6-7}$$

which is the desired mathematical description of the volume of the underground storage.

6-3 Identification of the Exchange Flow Model

Some appealing forms of mathematical relationships for modeling the exchange flow, $q_u(t)$, were outlined in Section 4-4. Also, special cases were discussed in Sections 4-4 and 4-5. Once a form of the model is chosen, it is necessary to find: (a) the satisfactory model size m of Eq. 4-38, and (b) to evaluate the model parameters.

The actual procedure for obtaining the answers to the above consists of evaluating the parameters of models of various sizes and comparing the goodness of fit of each model size. Selection of the proper size, being a matter of judgement, is somewhat arbitrary.

In addition to the $V(t)$ and $W(t)$ series, identification of the underground storage subsystem requires that a sequence of observed exchange flow, $q_u(t)$, be available. This is obtained when the value of $dV(t)/dt$ of Eq. 3-12b is substituted into Eq. 3-12a from which it follows that

$$q_u(t) = \frac{dS(t)}{dt} - q_c(t) - q_d(t) - q_p(t) + q_b(t) + q_e(t) + q_g(t) , \tag{6-8}$$

where the symbols are as defined in Chapter III.

If GMDH is used as a procedure to identify the V-model, then the q_u-model can be thought of as an additional level of computation, where q_u is a function of a transform of V and W instead of variable V alone.

6-4 Porosity of the Underground Subsystem

Effective porosity of an aquifer varies greatly over a drainage area. Usually, an estimate of the average effective porosity of the subsystem is sufficient to evaluate the effects of the underground storage on the surface reservoir. The estimation is deduced from hydrologic observations taken over representative time periods characterized by specific hydrologic conditions. A method to determine the effective porosity, γ, over large areas is described by Torbarov [1975]. It is based on the evaluation of the amount of outflow from an aquifer using the method described in Appendix A, or a similar procedure. Under the circumstances that a number of boreholes over the subsurface basin exist, it is possible to compare the average water table drawdown with the volume of water outflow from the subsystem. To that end, let the decrease of volume, ΔV, be defined as

$$\Delta V = V_o - V_t \tag{6-9}$$

where V_o is the initial aquifer content, and V_t is its content at time t. Furthermore, let the average drawdown be defined as

$$\Delta H = [(H_{1,0} - H_{1,t}) + \ldots + (H_{m,0} - H_{m,t})]/m, \tag{6-10}$$

where $H_{i,0}$ is piezometric level at borehole i at time $t = 0$, and $H_{i,t}$ is the observation at time t, for $i = 1,\ldots,m$. Then the average effective porosity, γ, can be computed as

$$\gamma = \frac{\Delta V}{A.\Delta H} \tag{6-11}$$

where A is the horizontal area of the aquifer.

Chapter VII
OPTIMIZATION AND OPTIMAL DECISION POLICY

7-1 Preliminary Remarks

It appears that the most suitable optimization method for solving problems associated with reservoir storage planning and operation is dynamic programming, for it handles discrete sets of numbers rather than continuous mathematical functions. This technique has limitations which are encountered in solving large-scale multidimensional problems. Nevertheless, under certain conditions dynamic programming remains the best method because of an innovation called decomposition. Detailed discussion concerning various aspects of dynamic programming is beyond the scope of this study. It can be found in numerous publications.

The two basic classes of dynamic programming problems are: (1) the resource allocation problem, and (2) multistage dynamic optimization. The former class is self-explanatory, while the latter refers to the selection of an optimal policy in time. The multistage optimization can be carried out either forward or backward. In this study both classes of optimization described above are utilized. The multistage dynamic optimization described above are utilized. The multistage dynamic optimization will be carried out forward.

There are two types of variables associated with every dynamic programming (DP) problem: state variables, and decision variables. As applied in this study to water allocation, the state variables denote the total amount of water allocated, hence they describe the state of the system. The decision variables in this case are the amounts of water given to each economic activity. These variables transform the state of the system from one value to another. Constraints describe the limits imposed on these variables. The region from which variables can be chosen without violating the constraints is called the feasible region.

Problems that can be solved using dynamic programming must satisfy the following three conditions [Bellman, 1962]:

(1) The return from different activities can be measured in common units,
(2) The return from any activity is independent of the allocation to another activity, and
(3) The total return can be obtained as the sum of the individual returns.

In the resource allocation problem treated in this study, the above conditions are satisfied. However, in the multistage dynamic optimization, as indicated in Section 3-3, there is some doubt that this is true. Except for some marginal consideration [Hall and Dracup, 1970], the dependency of decisions has not been thoroughly treated in the literature.

The specific problem that concerns this study is the optimal policy of water release under the condition of joint surface-underground storage. The underground exchange flow is an input to the surface reservoir; therefore, it is a feedback process and it can be expected that some iterative procedure will be involved in finding the optimal policy. To cope with these difficulties, a multilevel optimization approach is proposed. The first level is performed by the resource allocation procedure. It can include several sublevels of optimization. The next section is devoted to a detailed discussion of the mathematical methods needed to perform one sublevel of optimization. Additional sublevels represent a repetition of the techniques described. The essence of the procedure in each level is *conditional allocation*. Conditional allocation can be thought of as a contigency plan; it gives the optimal allocation for any future decision, whatever the future decision happens to be.

The second level of optimization is performed by the multistage dynamic optimization scheme, which is based on the final results of the first levels used. This optimization level is outlined in Section 7-3.

7-2 Resource Allocation Problem

Assume that there are n economic activities competing for a total amount of a resource X. Let each activity generate the return of $g_i(x_i)$ monetary units for allocation of x_i units of the resource. For computational convenience, it is assumed that X is a number belonging to a discrete set $[\chi] = [\mathcal{x}_1,\ldots,\mathcal{x}_j,\ldots,\mathcal{x}_m]$, that is $X \varepsilon [\chi]$. This assures that the computation is performed only over a set of discrete values. It is also assumed that $\mathcal{x}_j - \mathcal{x}_{j-1} =$ constant, $\forall\ j = 2,\ldots,m$.

According to the assumption stated in the Section 7-1, the total return can be expressed as

$$R(x_1,\ldots,x_n) = \sum_{i=1}^{n} g_i(x_i) , \qquad (7\text{-}1a)$$

subject to the constraints

$$x_i \geq a_i , \qquad (7\text{-}1b)$$

$$x_i \leq X , \qquad (7\text{-}1c)$$

$$X \varepsilon [\chi] , \qquad (7\text{-}1d)$$

and

$$x_1 + \ldots + x_n \leq X. \qquad (7\text{-}1e)$$

Where $a_i \geq 0$ is the lower limit of the allocation to the i-th activity, $\forall i = 1,\ldots,n$.

The problem of optimization arises from the fact that there are many ways in which a limited amount of the resource X can be allocated to the n economic activities. Yet, only one or a few of them are optimal. This problem is easily solved using dynamic programming. Detailed descriptions of the procedure involved in obtaining the optimal solution of Eq. 7-1a can be found in most texts on optimization.

A generalized form of the recursive relation which evolves from the process of maximizing $R(x_1,\ldots,x_n)$ is

$$f_i(X) = \max_{\substack{x_i \\ i=1,\ldots,n}} \{g_i(x_i) + f_{i-1}(X - x_i)\} \qquad (7\text{-}2a)$$

$$x_i \geq a_i \ , \tag{7-2b}$$

$$x_i \leq X \ , \tag{7-2c}$$

$$X \ \varepsilon \ [\chi] \ , \tag{7-2d}$$

and

$$x_1 + \ldots + x_i \leq X \ , \tag{7-2e}$$

in which, when $i = 1$, by definition $f_{i-1}(\cdot) = f_{1-1}(\cdot) = f_o(\cdot) \equiv 0$.

The meaning of the inequality constraint (IC) of Eqs. 7-1e and 7-2e is that the total amount of the resource X need not be allocated. Instead, only that quantity of the resource which gives the maximum return is used, while the remaining portion remains unallocated.

However, in rare practical problems, the situation requiring equality constraint on the total resource allocation is encountered. In addition, understanding the physical meaning, computational procedure, and the obtained results of an equality constraint allocation problem (ECAP) is of a general academic interest. It will also prove very useful in the sections that follow. For that reason, a brief discussion concerning the ECAP is presented.

In the ECAP, the constraints of the type 7-1e and 7-2e are strict equalities. The total amount of the resource must be allocated, regardless of the effect on the return. For example, this occurs when the reservoir capacity is exceeded. The resulting spillover may adversely affect downstream areas. All the other expressions (Eqs. 7-1 and 7-2) associated with the previously considered inequality constraint allocation problem (ICAP) are also valid for the ECAP.

Clearly, the optimal policy of the ECAP will be different from that of the ICAP. In order to find the difference in computational procedure for solving the two problems, let the generalized expression of the allocation problem formulation be rewritten as:

$$f_i(X) = \max_{\substack{x_i \\ i=1,\ldots,n}} [g_i(x_i) + f_{i-1}(X - x_i)] \ , \tag{7-3a}$$

$$x_i \geq a_i \tag{7-3b}$$

$$x_i \leq X \ , \tag{7-3c}$$

$$X \ \varepsilon \ [\chi] \ , \tag{7-3d}$$

and

$$x_1 + \ldots + x_i = X \ . \tag{7-3e}$$

It can be seen that there is no difference in computational procedure between the ECAP and the ICAP for $i = 2,\ldots,n$ since Eq. 7-3a is exactly Eq. 7-2a. However, the difference exists for $i = 1$, and this situation warrants additional comment. Keeping in mind that when $i = 1$, the function $f_{i-1}(\cdot) = f_o(\cdot) \equiv 0$, the rewritten set of equations of the form 7-3 becomes

$$f_1(X) = \max_{x_1} [g_1(x_1)] \ , \tag{7-4a}$$

$$x_1 \geq a_1 \ , \tag{7-4b}$$

$$x_1 \leq X \ , \tag{7-4c}$$

$$X \ \varepsilon \ [\chi] \ , \tag{7-4d}$$

and

$$x_1 = X \ . \tag{7-4e}$$

The constraint 7-4c is redundant to the constraint 7-4e, hence the former can be dropped out. Furthermore, since $x_1 = X$, the right-hand side of Eq. 7-4a becomes

$$f_1(X) = \max_{x_1} [g_1(x_1)] = g_1(X) \ , \tag{7-5a}$$

with

$$x \geq a_1 \tag{7-5b}$$

and

$$X \ \varepsilon \ [\chi] \ . \tag{7-5c}$$

Therefore, the computational procedure of the ECAP is different from that required to solve the ICAP for $i = 1$ only. Having the best return, $f_n(X)$, generated by allocating the quantity X of the resource to n activities, the optimal policy can be found.

Consider now the optimal return, $f_n(X)$, obtained under the conditions of the ECAP of Eq. 7-3. For clarity of notation, define $f_n(\mathscr{x}_j) = f_n(X = \mathscr{x}_j)$ for all $j = 1,\ldots,m$, so that $X \ \varepsilon \ [\chi] = [\mathscr{x}_1,\ldots,\mathscr{x}_m]$. In this case, the optimal policy of any allocation $X = \mathscr{x}_j$ can easily be found from the computational algorithm. Let it be described by a vector of the optimal allocations,

$$\underline{x}^*[f_n(X = \mathscr{x}_j)] = \underline{x}^*(\mathscr{x}_j) = \begin{bmatrix} x_1^*(\mathscr{x}_j) \\ x_2^*(\mathscr{x}_j) \\ \cdot \\ \cdot \\ \cdot \\ x_n^*(\mathscr{x}_j) \end{bmatrix} , \ \forall \ j = 1,\ldots,m \ , \tag{7-6}$$

where $\mathscr{x}_j = x_1^*(\mathscr{x}_j) + \ldots + x_n^*(\mathscr{x}_j)$. The optimal policy of Eq. 7-6 can be regarded as a conditional allocation, that is, as the optimal allocation obtained by solving the ECAP under the condition that there is exactly an amount of the resource $X = \mathscr{x}_j$ to be allocated. When the problem is solved for all values of $j = 1,\ldots,m$, the function $f_n(X)$ is obtained. Furthermore, for every particular value of $X = x_j$ the optimal policy is evaluated by the vector 7-6.

Knowing the results of the ECAP, and the corresponding optimal policies of Eq. 7-6, the next level of a hierarchical dynamic optimization can be carried out. Hence, one operates with a single return function, $f_n(X)$, instead of n return functions, $g_1(x_1), \ldots, g_n(x_n)$. The optimization process in this level yields the optimal value, $x^* = x_j$, for which the optimal allocation to the individual activities, $\underline{x}^*(x_j)$, is defined by 7-6.

The preceding discussion has demonstrated the application of the ECAP in solving multilevel dynamic programming problems frequently encountered in reservoir operation. As already outlined, it is particularly useful for those systems which occasionally can become uncontrollable and the adverse effects of the uncontrolled output are known, as in the case of reservoir spillovers. Actual methods of embedding the results of the ECAP into a multistage dynamic optimization is demonstrated in the next section.

7-3 Multistage Dynamic Optimization

Optimal utilization of water resource from a reservoir is achieved if the amounts of the resource used during different time periods of the reservoir lifetime constitute the optimal policy. It is immediately evident that the optimal policy depends on many factors. These factors are classified into three groups: state variables described by a state vector $\underline{S}_t$, decision variables described by a decision vector $\underline{D}_t$, and input variables described by an input vector $\underline{I}_t$. It should be noticed that, for purposes of this discussion, the input vector, $\underline{I}_t$, consists of variables which are both positive and negative with respect to the system. It should not be confused with the input variable Q_i, defined in Chapter III.

The present section describes a method for obtaining the optimal policy of sequential water releases. Assume that the system lifetime can be divided into N_t discrete time units, $t = 1,2,\ldots,N_t$. There is a state vector, $\underline{S}_t$, associated with every stage t. There is a function $T_t(\cdot)$ which transforms the state vector $\underline{S}_{t-1}$ of the stage $t-1$ into the state vector $\underline{S}_t$ of the stage t. Furthermore, the decision vector is defined as a set of variables that can be controlled. Then, the state vector transformation can be described by

$$\underline{S}_t = T_t(\underline{S}_{t-1}, \underline{I}_t, \underline{D}_t) , \qquad (7\text{-}7)$$

where $\underline{I}_t$ is the input vector as described previously, and where

$$\underline{S}_t \in [\underline{S}_t] \text{ and } \underline{D}_t \in [\underline{D}_t] , \qquad (7\text{-}8)$$

with $[\underline{S}_t]$ and $[\underline{D}_t]$ being the discrete sets of admissible values. These sets of admissible values are termed feasible time regions of the respective vectors. In addition, these feasible time regions must satisfy the following conditions:

$$[\underline{S}_t] \in [\underline{S}] \text{ and } [\underline{D}_t] \in [D] , \qquad (7\text{-}9)$$

where $[\underline{S}]$ and $[\underline{D}]$ are physical constraints on the system. The computational scheme of dynamic programming requires that the set $[S]$ be a set of discrete values, namely $[\underline{S}] = [(\underline{s}_1,\ldots,\underline{s}_m)]$.

The system effectiveness at the stage t is measured by the monetary performance index, $R_t(\underline{D}_t)$. The objective is to maximize the sum of all returns generated for every stage $t = 1,\ldots,N_t$. This is accomplished by selecting a particular set of the vector $\underline{D}_t$, say $\underline{D}_t^*$, $\forall\ t = 1,\ldots,N_t$. Hence, in mathematical terms it can be written

$$F_{N_t}(\underline{S}_{N_t}) = \max_{\underline{D}_t} \sum_{t=1}^{N_t} R_t(\underline{D}_t) , \qquad (7\text{-}10a)$$

$$\underline{S}_t = T_t(\underline{S}_{t-1}, \underline{I}_t, \underline{D}_t) , \qquad (7\text{-}10b)$$

$$\underline{S}_t \in [\underline{S}_t] \in [\underline{S}] , \qquad (7\text{-}10c)$$

and

$$\underline{D}_t \in [\underline{D}_t] \in [\underline{D}] . \qquad (7\text{-}10d)$$

The objective function of Eq. 7-10a can be rewritten read

$$F_{N_t}(\underline{S}_{N_t}) = \max_{\underline{D}_{N_t}} \{R_{N_t}(\underline{D}_{N_t}) + \max_{\underline{D}_t} \sum_{t=1}^{N_t-1} R_t(\underline{D}_t)\} . \qquad (7\text{-}11)$$

By a simple analogy it is obvious that

$$\max_{\underline{D}_t} \sum_{t=1}^{N_t-1} R_t(\underline{D}_t) = F_{N_t-1}(S_{N_t-1}) . \qquad (7\text{-}12)$$

Thus, the recursive mathematical formulation of the multistage dynamic optimization becomes similar to the formulation of the allocation problem, notably,

$$F_t(\underline{S}_t) = \max_{\underline{D}_t} \{R_t(\underline{D}_t) + F_{t-1}(\underline{S}_{t-1})\} , \qquad (7\text{-}13a)$$

$$t = 1,\ldots,N_t$$

$$\underline{S}_t = T_t(\underline{S}_{t-1}, \underline{I}_t, \underline{D}_t) , \qquad (7\text{-}13b)$$

$$\underline{S}_t \in [\underline{S}_t] \in [\underline{S}] , \qquad (7\text{-}13c)$$

and

$$\underline{D}_t \in [\underline{D}_t] \in [\underline{D}] . \qquad (7\text{-}13d)$$

The function $F_t(\underline{S}_t)$ is sometimes called the state function. In this study the term state function is used interchangeably with the stage return. It should be pointed out that in the formulation of Eq. 7-13a, for $t = 1$, $F_{t-1}(\cdot) = F_0(\cdot) \equiv 0$, as in the allocation problem.

The remainder of this section incorporates the components of the hydrologic system described in

Chapter III into the above mathematical formulations. The system of Chapter III is first treated as a two-state system. A distinctive feature resulting from the existance of feedback processes and the methods for solving this type of problem are discussed later. The variables to be used in the further discussion and their relationship to the vectors $\underline{S}_t$, $\underline{I}_t$, and $\underline{D}_t$ are outlined. The state vector, $\underline{S}_t$, consists of the state of the surface storage subsystem, S_t, and the state of the underground storage subsystem, V_t, namely

$$\underline{S}_t = \begin{bmatrix} S_t \\ V_t \end{bmatrix} . \tag{7-14}$$

Here, $\underline{S}_t$ should not be confused with S_t.

The decision vector, $\underline{D}_t$, consists of allocations to each economic activity, which in the previous section were denoted by $x_1,\ldots,x_n$. Here, since the allocation in time is required as well as the allocation among the users, the decision vector becomes

$$\underline{D}_t = \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ \cdot \\ \cdot \\ \cdot \\ x_{n,t} \end{bmatrix} . \tag{7-15}$$

The input vector $\underline{I}_t$ consists of all the variables that affect both the surface storage subsystem and the underground storage subsystem. Note that the natural outflow from the surface storage subsystem, which now becomes controlled release, is not a component of $\underline{I}_t$. Thus,

$$\underline{I}_t = \left[q_t^c \; q_t^d \; q_t^p \; q_t^e \; q_t^g \; q_t^r \; q_t^u \right]^T . \tag{7-16}$$

Matrices of constants, $\underline{C}_D$ and $\underline{C}_I$ are denoted as follows:

$$\underline{C}_D = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{7-17}$$

a 2xn matrix with n the number of economic activities (as in Eqs. 7-1 and 7-15), and

$$\underline{C}_I = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} . \tag{7-18}$$

Using Eq. 7-14 through Eq. 7-18, the transform function of Eq. 7-7 becomes

$$\underline{S}_t = \mathcal{T}_t(\underline{S}_{t-1}, \underline{I}_t, \underline{D}_t) = \underline{S}_{t-1} + \underline{C}_I\, \underline{I}_t - \underline{C}_D\, \underline{D}_t , \tag{7-19}$$

or, in explicit form, after it was rearranged,

$$S_t = S_{t-1} + q_t^i + q_t^u - q_t^o \tag{7-20a}$$

and

$$V_t = V_{t-1} + q_t^r - q_t^u . \tag{7-20b}$$

In Eq. 7-20a, $q_t^i = q_t^c + q_t^d + q_t^p - q_t^e - q_t^g$, and the term $q_t^o = x_{1,t} + \ldots + x_{n,t}$.

With the variables defined as above, and with the developments of the preceding section, one can proceed with the discussion of the multistage dynamic optimization. Keeping in mind that the principal goal is to optimize the resource utilization over the project lifetime as well as among the potential users, the total benefit can be expressed as

$$\begin{aligned} B(x_{i,t}) &= [g_{1,1}(x_{1,1}) + \ldots + g_{n,1}(x_{n,1})] + \ldots \\ &\quad + [g_{1,t}(x_{1,t}) + \ldots + g_{n,t}(x_{n,t})] + \ldots \\ &\quad + [g_{1,N_t}(x_{1,N_t}) + \ldots + g_{n,N_t}(x_{n,N_t})] \\ &= \sum_{t=1}^{N_t} \sum_{i=1}^{n} g_{i,t}(x_{i,t}) . \end{aligned} \tag{7-21}$$

Here the function $g_{i,t}(\cdot)$ gives the return from the activity, i, during the time increment, t.

The optimum return is obtained by maximizing the sum of benefits given by Eq. 7-21, namely

$$\begin{aligned} F_{N_t}(S_{N_t}, V_{N_t}) &= \max_{x_{i,t}} \{B(x_{i,t})\} \\ &= \max_{\substack{x_{i,t} \\ i=1,\ldots,n \\ t=1,\ldots,N_t}} \left\{ \sum_{t=1}^{N_t} \sum_{i=1}^{n} g_{i,t}(x_{i,t}) \right\} , \end{aligned} \tag{7-22a}$$

$$S_t = S_{t-1} + q_t^i + q_t^u - (x_{1,t} + \ldots + x_{n,t}) , \tag{7-22b}$$

$$V_t = V_{t-1} + q_t^r - q_t^u , \tag{7-22c}$$

and

$$q_t^u = q_u(S_{t-1}, V_{t-1}, S_t, V_t) . \tag{7-22d}$$

Taking q_t^o as defined after Eq. 7-20, the expressions of Eq. 7-22 can be rewritten in the following form

$$F_{N_t}(S_{N_t}, V_{N_t}) = \max_{q_t^o} \left\{ \sum_{t=1}^{N_t} \left[\max_{x_{i,t}} \sum_{i=1}^{n} g_{i,t}(x_{i,t}) \right] \right\} , \tag{7-23a}$$

$$\text{s.t. } x_{1,t} + \ldots + x_{n,t} = q_t^o$$

$$S_t = S_{t-1} + q_t^i + q_t^u - q_t^o , \tag{7-23b}$$

$$V_t = V_{t-1} + q^r_t - q^u_t \,, \qquad (7\text{-}23c)$$

and

$$q^u_t = q_u(S_{t-1}, V_{t-1}, S_t, V_t) \,. \qquad (7\text{-}23d)$$

It is readily seen that the right-hand side of Eq. 7-23a can be written as

$$F_{N_t}(S_{N_t}, V_{N_t}) = \max \sum_{t=1}^{N_t} f_{n,t}(q^o_t) \,, \qquad (7\text{-}24)$$

where $f_{n,t}(\cdot)$ is the solution to the equality constraint allocation problem as outlined in the preceding section, with

$$f_{n,t}(q^o_t) = \max_{x_{i,t}} \sum_{i=1}^{n} g_{i,t}(x_{i,t}) \,, \qquad (7\text{-}25a)$$

$$x_{i,t} \geq a_i \,, \qquad (7\text{-}25b)$$

$$x_{i,t} \leq q^o_t \,, \qquad (7\text{-}25c)$$

$$q^o_t \in [X] \,, \qquad (7\text{-}25d)$$

and

$$x_{1,t} + \dots + x_{n,t} = q^o_t \,. \qquad (7\text{-}25e)$$

Finally, the mathematical formulation of the multistage dynamic optimization problem becomes:

$$F_{N_t}(S_{N_t}, V_{N_t}) = \max_{q^o_t} \sum_{t=1}^{N_t} G_t(q^o_t) \,, \qquad (7\text{-}26a)$$

$$S_t = S_{t-1} + q^i_t + q^u_t - q^o_t \,, \qquad (7\text{-}26b)$$

$$V_t = V_{t-1} + q^r_t - q^u_t \,, \qquad (7\text{-}26c)$$

and

$$q^u_t = q_u(S_{t-1}, V_{t-1}, S_t, V_t) \,, \qquad (7\text{-}26d)$$

where $G_t(\cdot)$ replaces $f_{n,t}(\cdot)$, since n, being the number of activities, has no significance in the further optimization process. The problem as formulated by the set of Eq. 7-26 fits into the general scheme of the multistage dynamic optimization given by the set of Eq. 7-10, for which a recursive relation is given by the set of Eq. 7-13. It should be noticed that the process described by the constraint set of Eq. 7-26 is a feedback process.

In general, there is no explicit relation, which gives q^u_t, satisfying the relations of Eqs. 7-26b through 7-26d unless the exchange flow can be expressed as a linear function in both S and V, or some other simplification is possible. For example, if q^u_t can be approximated by the system states at the stage $t - 1$ only, namely $q^u_t = q_u(S_{t-1}, V_{t-1})$, then iteration will not be necessary. In any other situation an iterative process will be required to find the solution to the three equations 7-26b, 7-26c, and 7-26d.

7-4 Reduction of the State Vector

So far, only the generalized aspects of the multistage dynamic optimization (MDO) were outlined. It was demonstrated that the MDO can be applied to a water resource project characterized by a feedback process, i.e., in which an underground storage subsystem is coupled with a surface storage subsystem. The mathematical formulation of the MDO as presented in Section 7-3 is known as two state variable (S and V) dynamic programming. The two state problem represents a time-consuming computational process as compar d to the one state problem. Also, the computer memory required is much smaller in the one state variable problem. For that reason, the simplification in mathematical formulation, with no loss of accuracy, may prove beneficial.

A remark was made previously concerning the functional representation of the underground exchange flow, q^u_t, by means of the system states at the stage $t - 1$ only. The form suggested would eliminate iteration, but it might also affect accuracy. Whether this simplification is feasible or not must be determined for each particular computational case.

It is necessary to outline further possibilities of simplifications which may exist because of the nature of the water resource system under consideration. To do so, the essence of the computational algorithm is examined in detail. To that end, the recurrence relation of the mathematical formulation given by Eq. 7-26 is written in its general form, namely,

$$F_t(S_t, V_t) = \max_{q^o_t} \{G_t(q^o_t) + F_{t-1}(S_{t-1}, V_{t-1})\} \,, \quad t=1,\dots,N_t \qquad (7\text{-}27a)$$

$$S_t = S_{t-1} + q^i_t + q^u_t - q^o_t \,, \qquad (7\text{-}27b)$$

$$V_t = V_{t-1} + q^r_t - q^u_t \,, \qquad (7\text{-}27c)$$

and

$$q^u_t = q_u(S_{t-1}, V_{t-1}, S_t, V_t) \,. \qquad (7\text{-}27d)$$

The stage return $F_t(.,.)$ is evaluated at a set of discrete values of both state variables, S and V, by a computational algorithm consisting of several steps which can be described as follows:

a) Take a particular value of the surface storage subsystem state S_t, say $s_{t,i} \in [S]$, and at the same time a particular value of the underground storage subsystem V_t, say $v_{t,j} \in [V]$;

b) Depending upon the observed values of q^i_t and q^r_t, there will be a set of feasible values of states of both the surface storage subsystem, S_{t-1}, and the underground storage subsystem, V_{t-1}, say $s_{t-1,k} \in [S_{t-1}]$, and $v_{t-1,l} \in [V_{t-1}]$, respectively. From this the system states can be transformed into $S_t = \hat{s}_{t,i}$, and $V_t = v_{t,j}$. Each pair of values of S_{t-1} and V_{t-1}, namely $(s_{t-1,k}, v_{t-1,l})$, will determine a value of $q_t^o = q_{k,l}$;

c) Evaluate the return for every feasible value of $q_t^o = q_{k,l}$, and its corresponding pair of values, $(s_{t-1,k}, v_{t-1,l}$, from the relation

$$R_t(q_{k,l}) = [G_t(q_{k,l}) + F_{t-1}(s_{t-1,k}, v_{t-1,l})]; \quad (7\text{-}28)$$

d) Find the maximum return over the whole set of pairs of the feasible values, $(s_{t-1,k}, v_{t-1,l})$, to obtain the best return corresponding to the chosen pair of values, $(s_{t,i}, v_{t,j})$, namely

$$F_t(s_{t,i}, v_{t,j}) = \max_{q_{k,l}} \{G_t(q_{k,l}) + F_{t-1}(s_{t-1,k}, v_{t-1,l})\}; \quad (7\text{-}29)$$

e) Repeat the steps (a) through (d) to cover the whole set of pairs of values of the states $S_t \varepsilon [S]$ and $V_t \varepsilon [V]$ to obtain the function $F_t(S_t, V_t)$.

Assume now that at the stage $t - 1$ there is only one value of the underground storage subsystem state, V_{t-1}, corresponding to each discrete value of the surface storage subsystem state, S_{t-1}, namely, that there exists the following relation:

$$V_{t-1} = V_{t-1}(S_{t-1}) , \quad (7\text{-}30)$$

from which the underground exchange flow can be expressed as

$$q_t^u = q_u[S_{t-1}, V_{t-1}(S_{t-1}), S_t, V_t] = q_u(S_{t-1}, S_t, V_t) . \quad (7\text{-}31)$$

Substituting Eq. 7-30 and Eq. 7-39 into the mathematical formulation of the multistage dynamic optimization of Eqs. 7-27 yields

$$F_t(S_t, V_t) = \max_{q_t^o} \{G_t(q_t^o) + F_{t-1}(S_{t-1})\} , \quad t=1,\ldots,N_t \quad (7\text{-}32a)$$

$$S_t = S_{t-1} + q_t^i + q_u(S_{t-1}, S_t, V_t) - q_t^o , \quad (7\text{-}32b)$$

and

$$V_t = V_{t-1}(S_{t-1}) + q_t^r - q_u(S_{t-1}, S_t, V_t) . \quad (7\text{-}32c)$$

Keeping in mind that the computational algorithm as described previously by the steps (a) through (e) assumes two feasible values of the surface subsystem state, s_{t-1} and s_t, at the stages $t - 1$ and t, respectively, it is readily seen that Eq. 7-32c has a unique solution for V_t. When this is inserted into Eq. 7-32b, a unique solution for q_t^o is obtained. With a value of q_t^o defined in this manner, the return is evaluated from Eq. 7-32a. Clearly, for the given value of the surface subsystem state, $S_t = s_{t,i}$, and for any value of $S_{t-1} = s_{t-1,i}$, there will be a value of the underground storage subsystem at the stage t, $V_t = v_{t,i}$. However, when the optimal value is chosen by step (d) of the above described procedure, there will be only one value of the underground storage subsystem uniquely corresponding to the value $S_t = s_{t,i}$. Hence, the state of the underground storage subsystem, $V_t = V_t(S_t)$, as assumed in Eq. 7-30.

The described procedure eventually leads to the stage $t = 1$, whereby the values of the state $V_o = V_o(S_o)$ should be known. The legitimate question can be asked: Does one know the initial state V_o corresponding to the initial value of the state S_o? The answer is no. However, several other factors should not be overlooked. The underground storage cannot be controlled directly, but rather it is controlled by controlling the surface storage subsystem, since the surface and the underground subsystems are directly related.

Therefore, the underground storage subsystem, V_t, cannot be held at any arbitrary level for a given value of the surface storage subsystem, S_t. Instead, there is only a limited range of feasible values of the state V_t for any given value of the state S_t. Thus, taking a convenient value of the state V_o to be associated with every value of the state S_o should not affect significantly the evaluated returns. Yet, it greatly reduces the computation time and computer memory required. If the optimum is expected to be affected by the assumed initial conditions of the state V, it is easier in many cases to evaluate the optimum with one state variable and several initial conditions than with the two state variables.

7-5 Dynamic Programming with Integrated Inputs and Outputs

Multistage dynamic optimization is essentially an allocation process in time, by which an optimal policy is obtained. This optimal policy depends on the return function and is subject to dynamic constraints. The stage returns are evaluated at discrete time points and specified discrete values of the system state. The system behavior between the two discrete time points is assumed known. Often it is linearly approximated.

The time intervals between the discrete points can be of various lengths. Computational effort for solving a dynamic programming problem depends on the number of time intervals. Reduction of computational difficulty by decreasing the number of time intervals often results in inadequate accuracy. When the time intervals are small, not only the number of stages increases, but the number of discrete points of the system state must also be increased to make them compatible with the system inputs and outputs over the given time interval. This may cause an increase in both the computer memory and computer time requirements.

As mentioned earlier, the time increments for medium and small reservoirs, particularly under the conditions of large variations in river flow, should be relatively small. This reduces errors caused by high variations of inputs. A hypothetical example illustrates the situation which is frequently encountered in dealing with a water resource project. Let the reservoir lifetime be 50 years; then $N_t = 50 \times 365 = 18{,}250$ days. Solving an optimization problem with 18,250 stages is out of the question in most practical cases. If, on the other hand, a three-month time interval is chosen, then, in the cited example, there will be $N_T = N_t/\tau = 18{,}250/90 = 200$; an acceptable number of stages. However, the accuracy of the computed optimal policy may be insufficient when the time interval is three months. It is not enough to know the three-month sum of the inputs and the outputs, since the realization of the time series should be known completely. In addition, the simulation of the behavior of the underground storage subsystem is based on the physical relation described by the states of the two storage subsystems. That is, it is based on the evaluation of the exchange flow, q_t^u, which in turn affects the subsystem states. Such a feedback process cannot be described adequately only by the initial and the final conditions of a large time increment.

A new method of computation is proposed here. It is assumed that simplifications of the preceding section (reduction of the state vector dimension) have been made. The proposed method is based on the assumption that the allocation, q_t^o, over the time interval of length τ can be expressed in terms of the total allocation during that interval, Q_T^o, namely

$$q_t^o = q_t^o(Q_T^o) \quad . \tag{7-33}$$

Here, q_t^o is the release of the resource during a time unit $t \in [(T-1)\tau, T\tau]$, with $T = 1,\ldots,N_T$ being an index associated with the time intervals of length τ. The continuity condition requires that the following condition be satisfied

$$Q_t^o = \sum_{t=t_{T-1}+1}^{t_T} q_t^o \ , \tag{7-34}$$

with $t_{T-1} = (T-1)\tau$, and $t_T = T\tau$. The simplest example of the previous assumption is a uniform allocation each day over a three-month time interval.

The objective is to decide what is the total amount of the resource to be used over the three-month time interval. According to the assumption described by Eq. 7-33, the return over the time interval of length τ is

$$\sum_{t=t_{T-1}+1}^{t_T} G_t(q_t^o) = \sum_t G_t[q_t^o(Q_T^o)] = H_T(Q_T^o) \ , \tag{7-35}$$

where Q_T^o is given by Eq. 7-34. Using the objective function with only one state variable, the optimal return can be expressed as

$$F_{N_T}(S_{N_T}) = F_{N_t}(S_{N_t}) = \max_{q_t^o} \sum_{t=1}^{N_t} G_t(q_t^o)$$

$$= \max_{q_t^o(Q_T^o)} \sum_{t=1}^{N_t} G_t[q_t^o(Q_T^o)]$$

$$= \max_{Q_t^o} \sum_{T=1}^{N_T} \max_{q_t^o}\{ \sum_{t=t_{T-1}+1}^{t_T} G_t[q_t^o(Q_T^o)]\}$$

$$\text{s.t.} \sum_t q_t^o = Q_T^o$$

$$= \max \sum_{T=1}^{N_T} H_T(Q_T^o) \ , \tag{7-36}$$

which is the new objective function to be solved jointly with the constraint set. The last transition in Eq. 7-36 was made possible by virtue of the fact that the inner maximization is an ECAP.

The constraints must also be transformed to accomo accommodate the new formulation. To that end, the constraint operating on the surface storage subsystem for each time unit, t, is rewritten for the interval of length τ. Summing up the set of equations

$$\begin{aligned} S_{k+1} &= S_k + q_{k+1}^i + q_{k+1}^u - q_{k+1}^o \\ S_{k+2} &= S_{k+1} + q_{k+2}^i + q_{k+2}^u - q_{k+2}^o \\ S_{k+3} &= S_{k+2} + q_{k+3}^i + q_{k+3}^u - q_{k+3}^o \\ &\vdots \\ S_{k+\tau} &= S_{k+\tau-1} + q_{k+\tau}^i + q_{k+\tau}^u - q_{k+\tau}^o \end{aligned} \tag{7-37}$$

yields

$$S_{k+\tau} = S_k + \sum_{t=k+1}^{k+\tau} (q_t^i + q_t^u - q_t^o) \ . \tag{7-38}$$

Let the beginning of the (k+1)-st time unit correspond to the beginning of T-th time interval. Then, $t = k = (T-1)\tau = t_{T-1}$, and $t = k+\tau = (T-1)\tau + \tau = T\tau = t_T$, so that Eq. 7-38 becomes

$$S_T = S_{T\tau} = S_{T\tau-\tau} + \sum_{t=t_{T-1}+1}^{t_T} (q_t^i + q_t^u - q_t^o) \ . \tag{7-39}$$

For convenience, the notation can be simplified as follows:

$$S_T = S_{T-1} + Q_T^i + Q_T^u - Q_T^o \ , \tag{7-40}$$

where the correspondence of Eq. 7-40 to Eq. 7-39 is obvious. Following the same reasoning, the second constraint can be rewritten as

$$V_T = V_{T-1} + Q_T^r - Q_T^u \ , \tag{7-41}$$

where V_{T-1} and V_T are the states of the underground storage subsystem at the beginning and at the end of T-th time interval of length τ. Q_T^r and Q_T^u are the summations of the recharge and the underground exchange flow, q_t^r and q_t^u, respectively, over the same time interval, with $q_t^u = q_u(S_{t-1}, S_t, V_{t-1}, V_t)$.

The specific nature of dynamic programming is reflected in the necessity to evaluate the stage return at a number of specified discrete values of the system state. The subsystems that are dealt with in this study are linked by a feedback process. For that reason it is not generally possible to find a value of Q_t^o that transforms the system state from one discrete value to another when integrated over the time interval T.

Let the state of the surface subsystem take discrete values $s_{T,1},\ldots,s_{T,m}$. For a specific value, $s_{T-1,j} = S^i$, of the system state at the beginning of time interval T, and an assumed specific value of the decision variable, $Q_T^o = Q_{T,k}$, the state of the system at the end of the interval T will be transformed into

$$S^f = s_{T-1,j} + Q_T^i + Q_T^u - Q_T^o = S_T(Q_{T,k}^o | s_{T-1,j}) \,. \quad (7\text{-}42)$$

The value of $S^f = S_T(\cdot)$ is not necessarily one of the discrete values for which the stage return is to be evaluated. Because of the dependence between V and S, the state of the underground storage subsystem at the end of the interval T can be described by

$$\dot{V}_{T,j,k} = V_T(Q_{T,k}^o | s_{T-1,j}) \,, \quad (7\text{-}43)$$

where

$Q_{T,k}^o | s_{T-1,j}$ means $Q_{T,k}^o$ given $s_{T-1,j}$.

Conceptual visualization of the above process can be given in graphical form as shown by Fig. 7-1, where the curves $S^f = S_T(\cdot)$ of Eq. 7-42 and $V_T(\cdot)$ of Eq. 7-43 are constructed as functions of Q_T^o so that each curve is parametrized by a discrete value of the state $S_{T-1} = s_{T-1,j} = S^i$.

Once the graph is constructed, it is possible to evaluate the stage return at a set of specified discrete values of the state S_T as follows:

(1) Select a specific value of the surface state, $S_T = s_{T,i}$, for which the stage return is desired;

(2) Drawing a vertical line through $S_T = s_{T,i}$, the intersections with the curves $s_{T-1,j}$ determine the values of the output $Q_{T,j}^o$;

(3) Maximize the return over all feasible values of S_{T-1}, i.e., find:

$$F_T(S_T = s_{T,i}) = \max_{Q_{T,j}^o} \{H_T(Q_{T,j}^o) + F_{T-1}(S_{T-1} = s_{T-1,j})\}$$

(4) Find the value of the underground storage subsystem state, V_T, corresponding to the pair of values $Q_{T,j}^o$ and $s_{T-1,j}$ which maximizes the stage return and denote this by $v_{T,i} = V_T(s_{T,i})$;

(5) Store the values S_{T-1}, S_T, V_{T-1}, V_T, Q_T^o, and Q_T^u; and,

(6) Repeat steps 1 through 5 for all values of $S_T = s_{T,i}$, $i = 1,\ldots,m$.

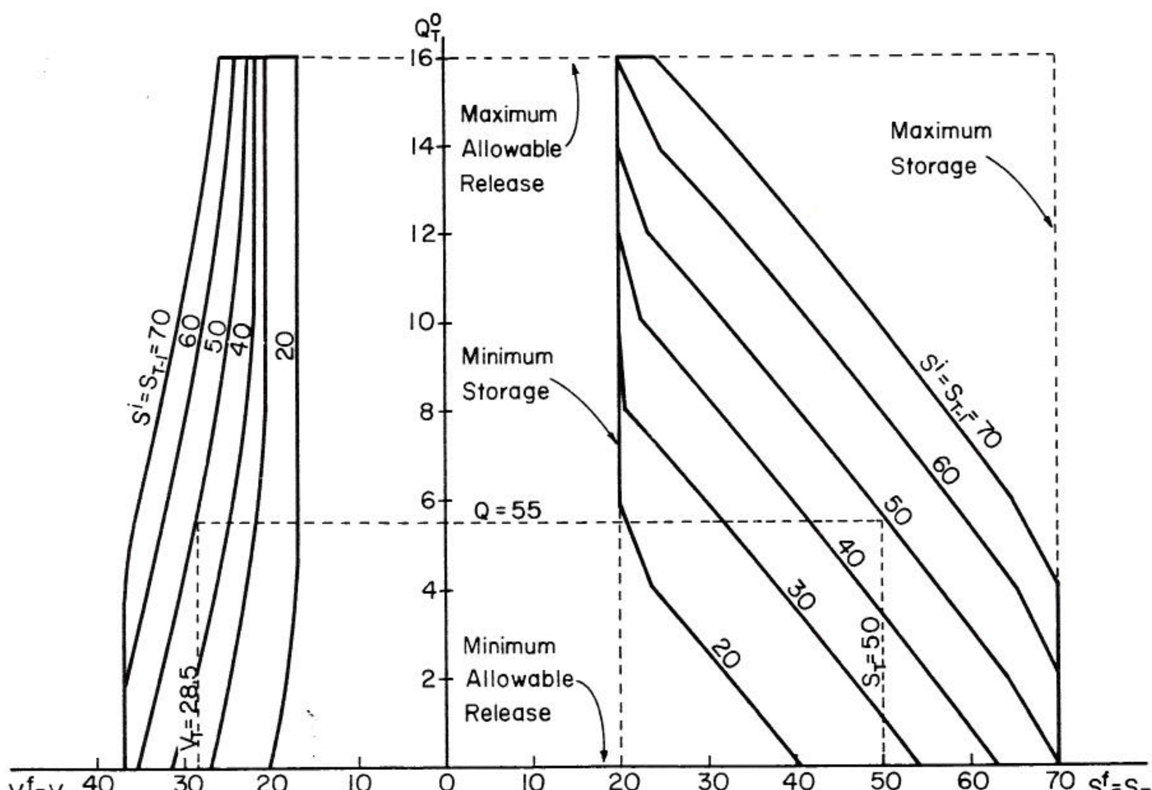

Fig. 7-1. Empirical Functions Relating S_T and V_T to S_{T-1} and Q_T^o for a Hypothetical Realization of Time Series q_t^c and q_t^r.

Chapter VIII
APPLICATION OF THE MODEL

The application of the model described in the previous chapters is demonstrated by two examples. The first example is based on a completely hypothetical set of data. It demonstrates the model application to a hydrologic system described in general terms. However, simplification is made to reduce the number of state variables from two to one.

The second example partially pertains to Lake Powell. It shows the procedure for finding the optimal policy under the specific conditions outlined in Section 4-5. In this example, data observed during actual reservoir operation were used to identify the system. The economic model was unavailable and it had to be hypothesized. For this reason and due to some other factors to be discussed later, no conclusion concerning the system's future operation should be drawn on the basis of results obtained by this computation.

Common features of the two examples are that the data are discrete sequences of monthly observations. Each year was subdivided into four periods, each of them three months long. Furthermore, it was assumed that the water release over each period is uniform. Nevertheless, it can vary from one period to another and from one year to another. Even though the return functions were hypothesized, they were constructed to reflect customary higher water demands during dry seasons than during wet seasons. Evaluation of the return is based on a five percent interest rate.

8-1 Example I

Maximization of the gross benefit was obtained by solving a multistage dynamic optimization problem. The objective function is given by Eq. 7-36 with the state constraints described by Eqs. 7-40 and 7-41. In addition, constraints reflecting the system size are incorporated.

The mathematical formulation of the problem is

$$F_{N_T}(S_{N_T}) = \max_{Q_T^o} \{ \sum_{T=1}^{N_T} H_T(Q_T^o) \}$$

$$= \max_{Q_T^o} \{H_T(Q_T^o) + F_{T-1}(S_{T-1})\} , \quad T = 1,\dots,N_T \tag{8-1a}$$

$$S_T = S_{T-1} + Q_T^i + Q_T^u - Q_T^o , \tag{8-1b}$$

$$V_T = V_{T-1} + Q_T^r - Q_T^u , \tag{8-1c}$$

$$20 \le S_T \le 120 , \tag{8-1d}$$

$$0 \le Q_t^o \le 24 , \tag{8-1e}$$

and

$$q_t^u = 0.047\ W_t(V_t - W_t) , \quad t = 1,2,\dots,N_t . \tag{8-1f}$$

The above formulation is computer processed for $N_t = 120$. This implies a sample size of ten years of monthly data. In addition, the number of stages of computation was $N_T = 40$, and $\tau = 3$. The terms Q_T^i, Q_T^u, Q_T^o and Q_T^r of Eq. 8-1 are defined in Eq. 7-39. They represent summations of monthly values of the corresponding quantities. The moving boundaries of summation over 40 three-month periods are defined as

$$\text{lower bound:} \quad t = t_{T-1} + 1 = (T-1)\ \tau + 1 \tag{8-2a}$$

and

$$\text{upper bound:} \quad t = t_T = T\ \tau . \tag{8-2b}$$

This yields the respective limits $(t_{T-1} + 1,\ t_T)$

$(1,3)$, $(4,6)$, $(7,9)$,..., $(118,120)$,

which correspond to the time periods

$T = 1, 2, 3,\dots,40.$

The surface flow components of Eq. 7-20 other than the channel inflow, q_t^c, were assumed to be zero, that is

$$q_t^d = q_t^p = q_t^e = q_t^g = 0 , \quad t = 1,\dots,N_t \tag{8-3}$$

so that

$$q_t^i = q_t^c . \tag{8-4}$$

In order to evaluate the exchange flow, q_t^u, of Eq. 8-1f, the following linear relationship was assumed:

$$W_t = 2.0 + 0.18\ S_t . \tag{8-5}$$

From Eq. 8-1a it is clear that the second state variable, V, was eliminated from the objective function according to the explanation given in Section 7-4. In that section it was stated that the initial value of the V-state at the beginning of the stage $T = 1$ is needed. This relationship was arbitrarily assumed as

$$V_o = 1.07\ W_o , \tag{8-6}$$

where W_o is obtained by Eq. 8-5. It should be observed that the condition required in Section 7-4, namely that V be a function of S, is satisfied.

Figure 8-1 shows the sequences of assumed discrete monthly observations of the channel inflow, q_t^c, and the recharge, q_t^r, $t = 1,\dots,N_t$. The value of water released for respective time periods is described by the four return functions, $H_T(\cdot)$, as given in table 8-1 and Fig. 8-2, where the time period each return function refers to is indicated. As already outlined, the water release during each month at the time period under consideration was assumed uniform. This fact is related to the requirements of Eq. 7-33 by

$$q_t^o = q_t^o(Q_t^o) = Q_T^o/\tau, \qquad (8\text{-}7)$$

where q_t^o is the quantity used in Eq. 8-1b to carry on the simultaneous process of integration of input and output variables.

Actual computation was performed over discrete sets of points. The sets are constrained by the reservoir size and the water release of Eq. 8-1d and Eq. 8-1e, respectively. The set of Q_T^o values consists of nine values with increments of three, namely

$$[Q_T^o] = [0, 3, 6, 9, 12, 15, 18, 21, 24] , \qquad (8\text{-}8)$$

while the set of S values is given by

$$[S_T] = [20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120]. \qquad (8\text{-}9)$$

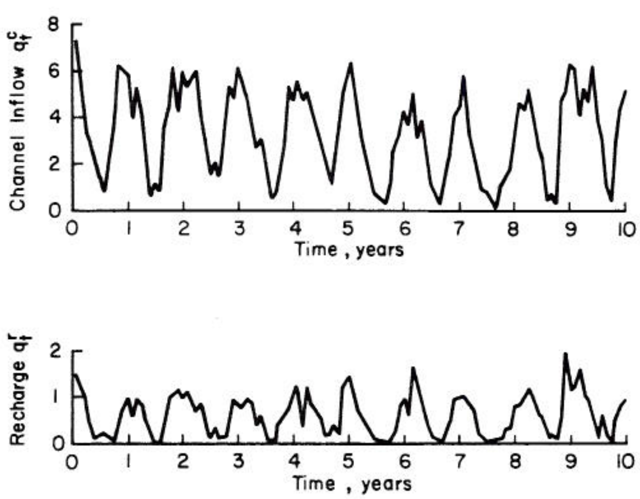

Fig. 8-1. Hypothetical Monthly Inflow (Upper Graph) and Monthly Groundwater Recharge (Lower Graph).

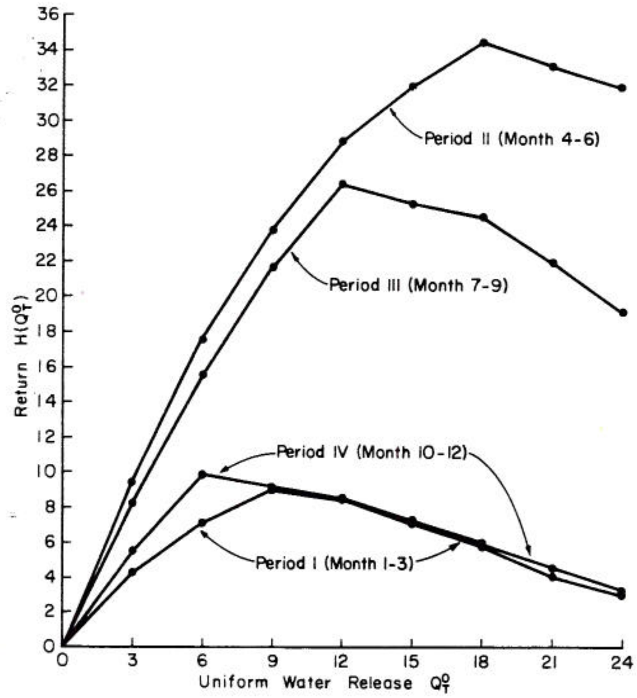

Fig. 8-2. Return Functions of a Uniform Water Release, Q_T^o, over Three-Month Periods.

The process of computation that is carried out for every stage T = 1,2,...,40 was described in Section 7-5. Typical results of this procedure are presented graphically in Figs. 8-3 through 8-7.

Figure 8-3 gives the final values of reservoir state, S^f, at the end of period T. These curves were obtained under the condition that the reservoir state at the beginning of the given time period was S^i, for various values of uniform water release Q_T^o, and for a particular sequence of input data realized during the given time period T. Similarly, Fig. 8-4 gives the final state of the underground subsystem, V^f, under the above condition. Fig. 8-5 gives the sum of the exchange flows, Q_t^u, obtained by evaluating q_t^u from Eq. 8-1f and summing them over the period (t_{T-1}, t_T) of Eqs. 8-2a and 8-2b.

Table 8-1. Return Functions of a Uniform Water Release, Q_T^o, over Three-Month Periods.

		Uniform Water Release Q_T^o								
Period	Month	0	3	6	9	12	15	18	21	24
1	1-3	0.0	4.3	7.2	9.1	8.5	7.2	5.8	4.2	3.0
2	4-6	0.0	8.3	15.6	21.7	26.4	25.3	24.6	22.0	19.2
3	7-9	0.0	9.5	17.6	23.8	28.9	32.0	34.5	33.2	32.0
4	10-12	0.0	5.6	9.9	9.1	8.6	7.5	6.0	4.6	3.2

Because of nature of the computation performed, it can happen that the assumed values of uniform water release, Q_T^o, deplete the reservoir below its minimum allowable value (20). Under some other conditions, the spillover from a full reservoir can occur. These facts are taken into account in the present scheme by virtue of the results depicted in Fig. 8-6 and Fig. 8-7. Figure 8-6 represents water deficit, Q_T^n, with respect to what was assumed to be a uniform release. That is, it gives that quantity of water that would have been released if the reservoir had been drained below the minimum state at the uniform rate Q_T^o. Since the reservoir must not be drained below its minimum allowable content, Q_T^n is nonexistant. By the same token, spillover, Q_T^s, results from having the reservoir full. Thus, actual release is obtained by

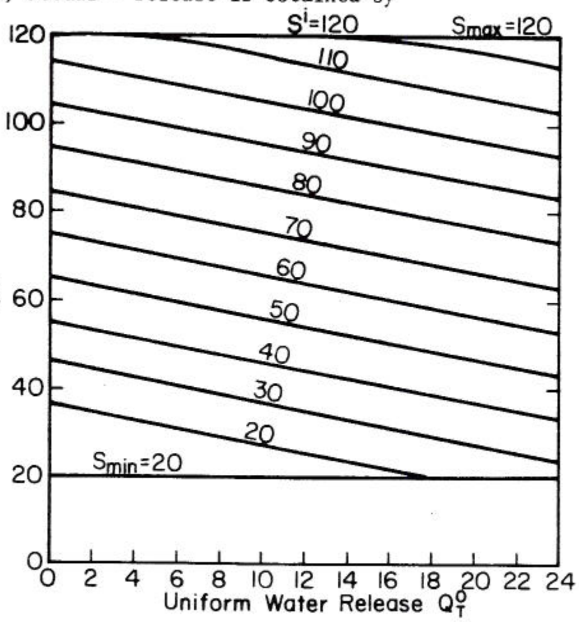

Fig. 8-3. Final Values of the Surface State, S^f, as a Function of Uniform Water Release, Q_T^o, and Initial State, S^i, at the Stage T.

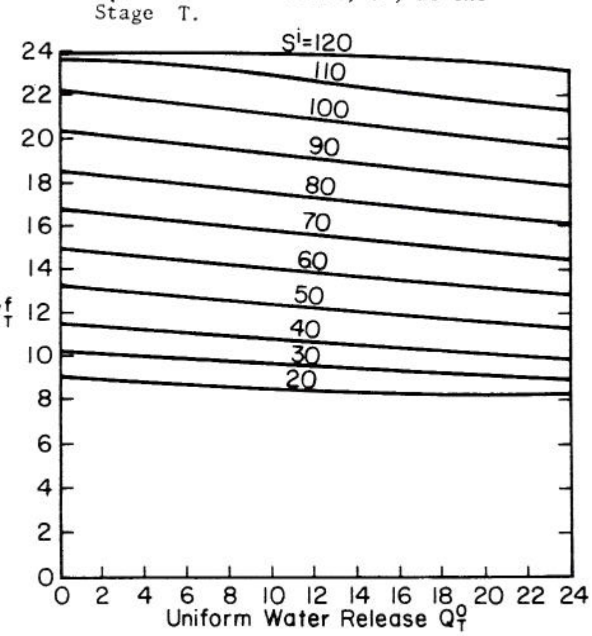

Fig. 8-4. Final Values of the Underground State, V^f, as a Function of Uniform Water Release, Q_T^o, and Initial State, S^i, at the Stage T.

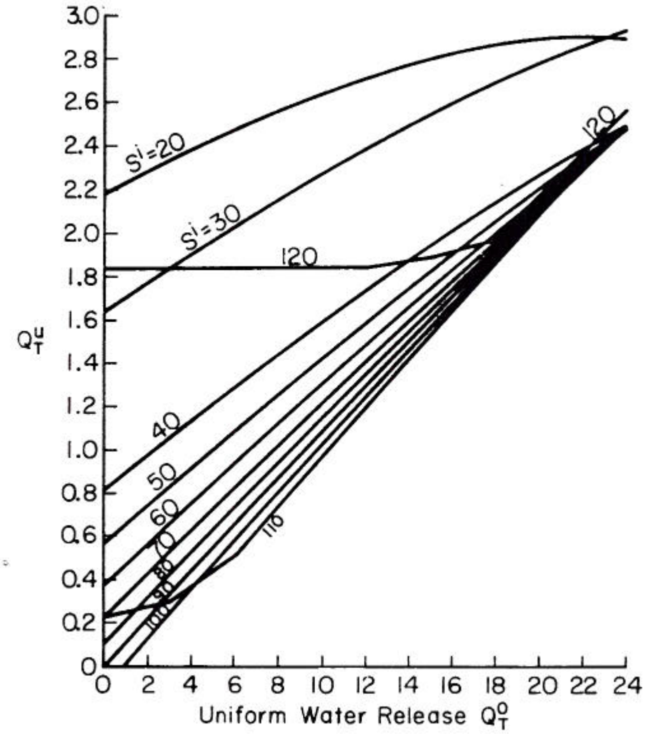

Fig. 8-5. Exchange Flow, Q_T^u, as a Function of Uniform Water Release, Q_T^o, and Initial State, S^i, at the Stage T.

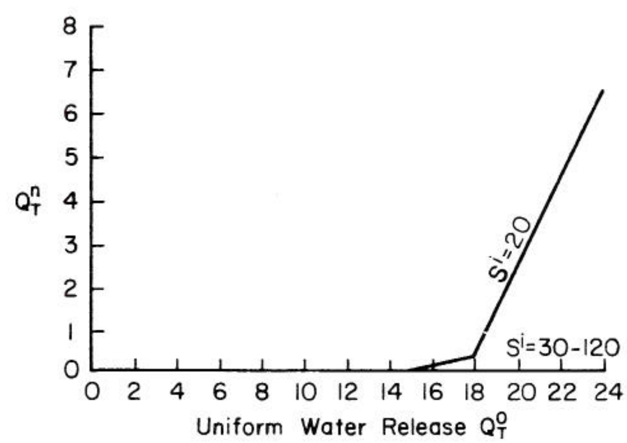

Fig. 8-6. Water Deficit, Q_T^n, with Respect to the Assumed Uniform Release, Q_T^o, as a Function of Q_T^o and Initial State, S^i, at the Stage T.

$$Q_T^{rel} = Q_T^o - Q_T^n + Q_T^s , \tag{8-10}$$

while the useful release is given by

$$Q_T^{ben} = Q_T^o - Q_T^n . \tag{8-11}$$

From Fig. 8-6 it can be seen that the deficit occurs when the initial reservoir state, s^i, is low and the release rate, Q_T^o, is relatively high. The spillovers result from relatively high initial state, S^i, and

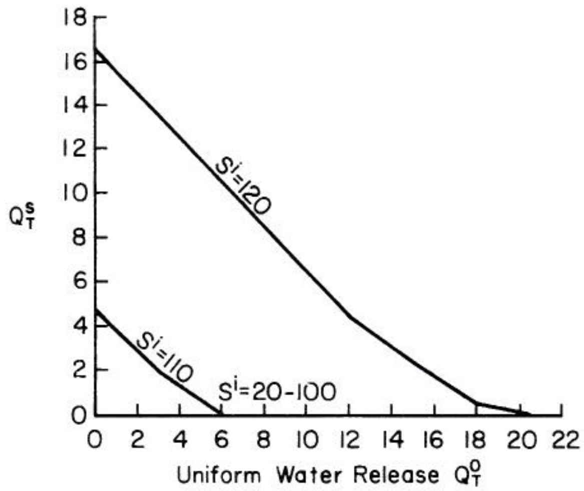

Fig. 8-7. Water Spillover, Q^s_T, as a Function of Uniform Release, Q^o_T, and Initial State, S^i, at the Stage T.

relatively low release rate, Q^o_T, as shown in Fig. 8-7. It is emphasized that the actual magnitude of the deficit, Q^n_T, depends not only on the initial state and rate of release, but also on the hydrologic sequence realized during the given interval. It should be noticed that the final reservoir states, S^f, of Fig. 8-3 never violate the boundary conditions described by Eqs. 8-1d and 8-1e. Instead, as soon as the S^f reaches the lower boundary (S_{min} = 20) it coincides with it for all higher release rates. Also, when it touches the upper boundary (S_{max} = 120) it is identical with it for all lower release rates.

From the results represented in Figs. 8-3 through 8-7 for every feasible value of the new reservoir state described by Eq. 8-9, the following is evaluated:

- the optimum return, $F_T(S_T)$
- the beneficial water release Q^{ben}_T
- the old reservoir state, S_{T-1}, from which the system was transformed into the new state, S_T
- the state of the underground subsystem $V_T(S_T) = V^f$, at the end of the time interval T, and
- the total value of the exchange flow, Q^u_T.

Table 8-2 illustrates a typical summary of the results obtained for every stage $T = 1,\dots,N_T$. In this particular case, the results of the last stage are shown. From Table 8-2 the maximum value is selected, that is $\max_{S_{40}} F_{40}(S_{40}) = 567.52 = F^*$. The corresponding reservoir state at the end of period 40 is S^*_{40}=30. Knowing the value of the reservoir state at the end of this time period, it is possible to find the value at the end of the preceding time period. The procedure is carried backward until the optimal policy is defined. The optimal policy is given in Table 8-3 and Fig. 8-8. The corresponding optimal releases constitute a sequence of a periodic nature as should be expected from the periodic return functions. Statistical properties of water release during each period of the year are given in Table 8-4. Fluctuations of the optimal release are relatively small as indicated by its standard deviation.

8-2 Example II

This example utilizes sequences of actual data observed at Lake Powell during twelve years of its operation. The reservoir characteristics are as follows: maximum reservoir capacity $S_{max} = 27\cdot10^6$ acre feet, of which the dead storage is $S_{min} = 2\cdot10^6$ acre feet. The reservoir content below approximately $6\cdot10^6$ acre-feet cannot be used for power generation. For reasons of obtaining rounded values of storage states, the minimum storage was assumed $3\cdot10^6$ acre feet.

Data collection. The process of filling the reservoir began on January 1, 1963, with a simultaneous observation of various data. The sequences of data available for this research represent the channel inflow, q^c_t, evaporation q^e_t, and release q^o_t. Also, surface reservoir states S_t, underground storage, V_t, and exchange flow, q^u_t, were observed for the years 1963-1974 inclusive. Monthly values are given for $t = 1,2,\dots,N_t$, where N_t = 144 (12 years of data). It is pointed out that the sequence of exchange flow, q^u_t, was obtained from the budget equation (see Eq. 6-8) and that the underground subsystem content, V_t, is the summation of the q^u_t, namely

$$V_t = \sum_{\nu=1}^{t} q^u_\nu. \tag{8-12}$$

Figure 8-9 represents monthly channel inflow, q^c_t (lower graph). The observed exchange flow, q^u_t, along with the values simulated, $\hat{q}^u_t$, are depicted on the upper graph. Observed states of the respective subsystems S and V are presented in Fig. 8-10. From that it is seen that the reservoir has not yet been filled. This leads to the conclusion that the system never reached the steady state.

It was already stated that the evaporation, q^e_t, is a function of the area from which evaporation takes place. Thus, observed values, q^e_t, which represent total evaporation from the lake surface, were modelled into e_t, that is, the evaporation from a unit of surface area. Average monthly values of e_t are given in Table 8-5 and graphically depicted in Fig. 8-11.

Reservoir volume and area as functions of water elevation are shown in Fig. 8-12, while their functional relationship (assumed to be linear) is shown in Fig. 8-13. This relationship is given by

$$A_r = 0.0185 + 0.00546448\ S\ , \tag{8-13}$$

where A_r is the reservoir surface area in millions of acres and S is the storage content in millions of acre feet. Expression 8-13 is needed in order to provide for a convenient evaluation of q^e_t at any state that the surface reservoir happens to be. Equation 3-11 is used to calculate the evaporation, q^e_t. In Fig. 8-14, the observed value of underground storage,

Table 8-2. Summarv of Results of Computation at the End of Stage T = 40.

J	$S_{T-1}(J)$	$S_T(J)$	$V_{T-1}(J)$	$V_T(J)$	$F_{T-1}(J)$	$F_T(J)$	$Q_T^{ben}(J)$
1	20.00	20.00	7.95	8.30	561.83	565.52	14.04
2	30.00	30.00	9.25	9.47	562.82	567.52*	14.17
3	30.00	40.00	10.77	10.04	561.25	566.79	3.60
4	40.00	50.00	12.36	11.61	559.94	565.17	3.55
5	60.00	60.00	14.01	14.05	559.18	563.83	14.35
6	60.00	70.00	15.68	14.97	558.31	562.99	3.43
7	70.00	80.00	17.38	16.70	556.33	562.08	3.37
8	80.00	90.00	19.09	18.45	554.00	560.05	3.32
9	90.00	100.00	20.82	20.21	550.99	557.67	3.27
10	100.00	110.00	22.55	21.99	545.98	554.63	3.22
11	110.00	120.00	23.67	23.78	525.45	549.42	3.17

Table 8-3. Optimal Policy of the State Transformation and Water Release.

T	J^*_{T-1}	S^*_{T-1}	J^*_T	S^*_T	Q^*_T	F^*_T
1	9	100.0	10	110.0	8.61	8.43
2	10	110.0	9	100.0	18.45	31.49
3	9	100.0	8	90.0	19.63	63.68
4	8	90.0	9	100.0	8.57	72.45
5	9	100.0	9	100.0	16.13	78.50
6	9	100.0	8	90.0	16.35	101.17
7	8	90.0	7	80.0	21.39	131.14
8	7	80.0	8	90.0	8.34	139.55
9	8	90.0	9	100.0	8.25	147.00
10	9	100.0	9	100.0	9.49	166.42
11	9	100.0	8	90.0	19.25	195.75
12	8	90.0	9	100.0	7.22	204.02
13	9	100.0	9	100.0	15.67	209.69
14	9	100.0	8	90.0	20.41	228.21
15	8	90.0	7	80.0	15.34	254.77
16	7	80.0	8	90.0	4.22	260.82
17	8	90.0	9	100.0	6.30	266.61
18	9	100.0	9	100.0	12.08	287.27
19	9	100.0	8	90.0	17.31	313.85
20	8	90.0	9	100.0	6.05	321.60
21	9	100.0	9	100.0	15.39	326.84
22	9	100.0	8	90.0	16.14	345.52
23	8	90.0	7	80.0	13.96	368.59
24	7	80.0	7	80.0	11.88	375.02
25	7	80.0	7	80.0	14.30	380.36
26	7	80.0	6	70.0	20.26	396.45
27	6	70.0	5	60.0	14.15	418.93
28	5	60.0	5	60.0	13.39	424.61
29	5	60.0	5	60.0	13.95	429.80
30	5	60.0	4	50.0	15.54	446.83
31	4	50.0	2	30.0	23.93	468.51
32	2	30.0	2	30.0	8.06	474.84
33	2	30.0	2	30.0	16.07	479.16
34	2	30.0	2	30.0	11.13	495.30
35	2	30.0	1	20.0	13.70	515.93
36	1	20.0	2	30.0	7.95	521.98
37	2	30.0	3	40.0	7.77	527.08
38	3	40.0	3	40.0	16.68	542.38
39	3	40.0	2	30.0	16.56	562.82
40	2	30.0	2	30.0	14.17	567.52

Table 8-4. Statistical Properties of Optimal Water Release

	PERIOD 1	2	3	4
$\bar{Q}^*$	12.24	15.65	17.52	8.98
σ	3.99	3.72	3.46	3.19
σ^2	15.92	13.85	11.98	10.18

V, was regressed on the corresponding values of the surface reservoir observed states, S.

Identification of the exchange flow, q_t^u, is done by Eq. 4-73, where recharge, q_t^r, is assumed to be zero. In addition, the zero intercept, α_o, was introduced because of an improvement of the model. The parameters are evaluated as

$$\alpha_o = -0.01952, \quad \alpha_1 = 0.41400, \quad \text{and} \quad \alpha_2 = -0.07181.$$

Return functions, $H_T(Q_T^o)$, are assumed since the monetary values of water releases were not available. These functions are given in Table 8-6 and are schematically shown in Fig. 8-13. As in the preceding example, four return functions are taken to represent four seasons during the year, where each season is of equal length, $\tau = 3$. The water releases are assumed uniform within any given period for any level of output, Q_T^o.

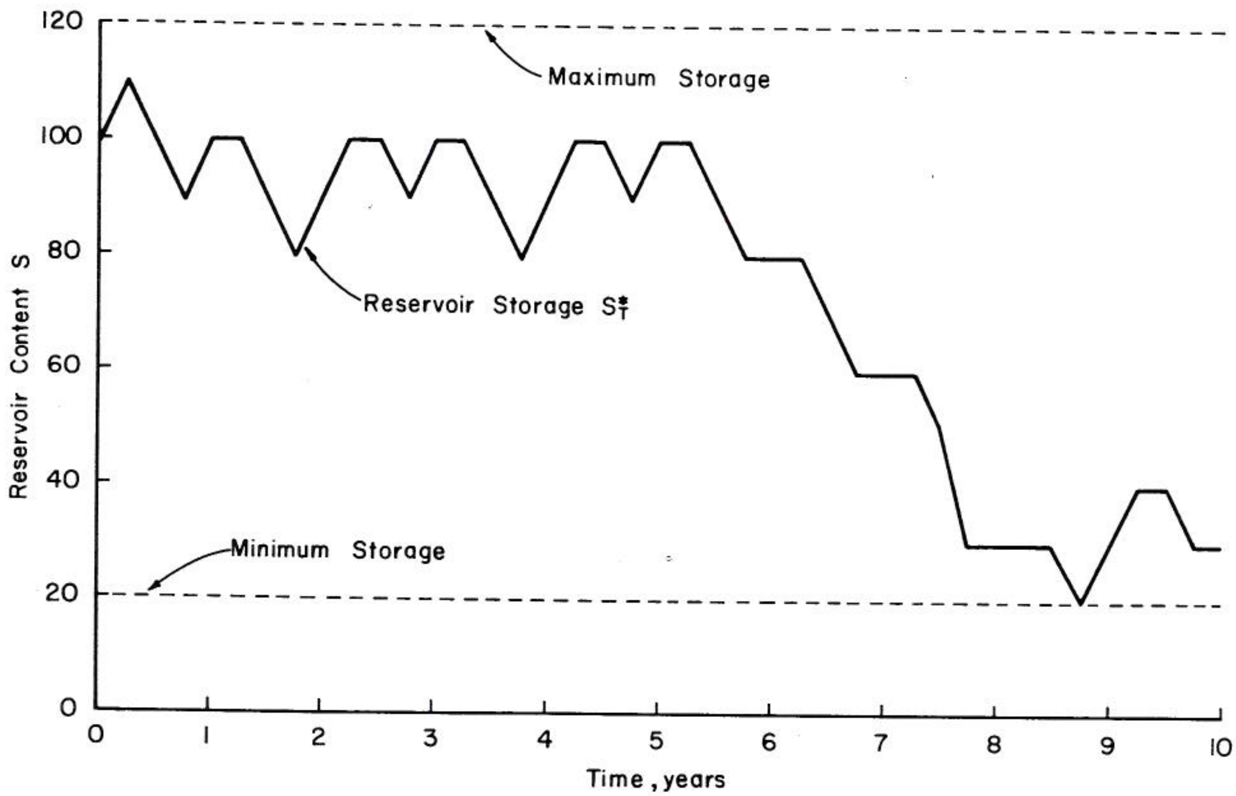

Fig. 8-8. Optimal Policy of the Reservoir State Transformation.

Table 8-5. Monthly Evaporation, e_t, from a Unit of Reservoir Surface Area.

Month	1	2	3	4	5	6	7	8	9	10	11	12
e_t[ft]	0.169	0.163	0.204	0.225	0.315	0.360	0.420	0.422	0.372	0.327	0.297	0.234

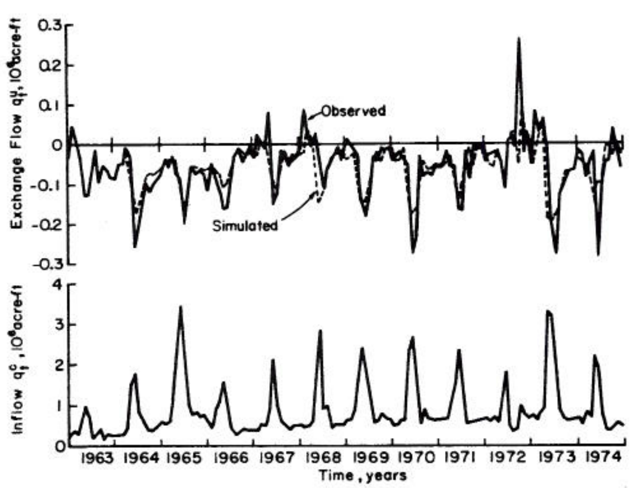

Fig. 8-9. Surface Inflow into Lake Powell (Lower Graph), and Exchange Flow (Upper Graph).

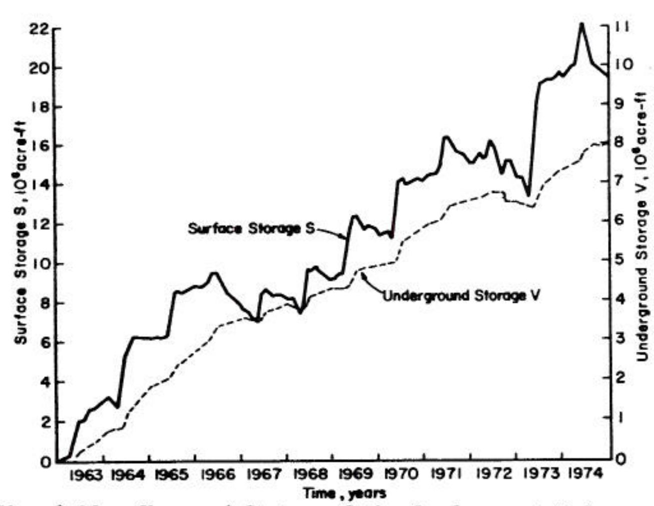

Fig. 8-10. Observed States of the Surface and Underground Subsystems at Lake Powell.

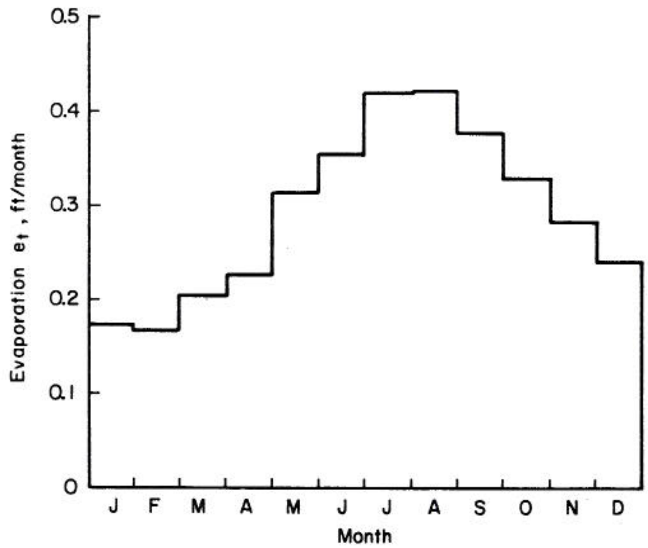

Fig. 8-11. Monthly Evaporation, e_t, from a Unit of Reservoir Surface Area.

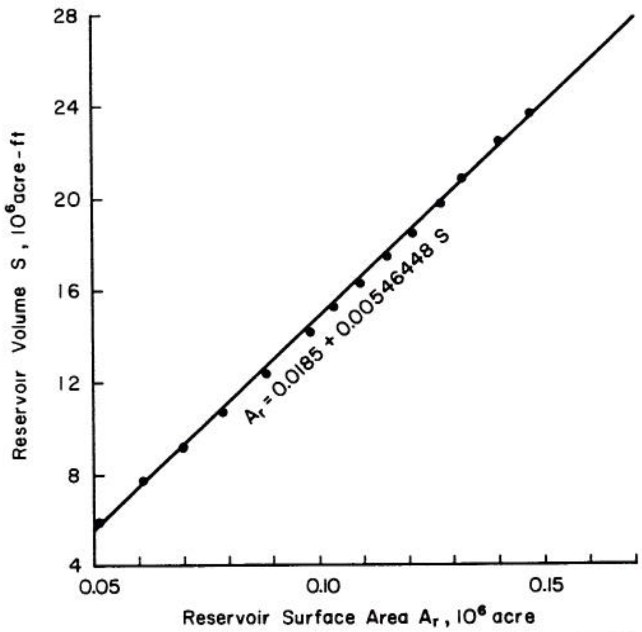

Fig. 8-13. Reservoir Surface Area, A_r, as a Function of Water Content, S.

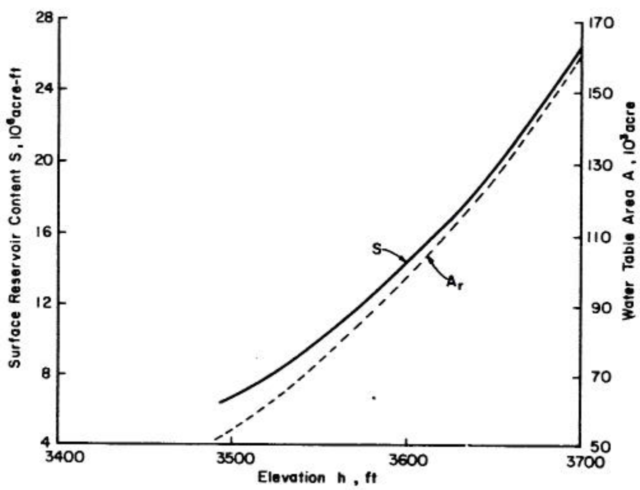

Fig. 8-12. Water Content, S, and Surface Area, A_r, as Functions of the Elevation.

Based on these return functions and the results of system identification, optimization of the reservoir operation was performed over a span of 12 years for which inflow data exists. The mathematical formulation of the optimization problem is

$$F_{N_T}(S_{N_T}) = \max_{Q_T^o} \{ \sum_{t=1}^{N_T} H_T(Q_T^o) \}$$

$$= \max_{\substack{Q_T^o \\ T=1,\ldots,N_T}} \{H_T(Q_T^o) + F_{T-1}(S_{T-1})\}, \qquad (8\text{-}13a)$$

$$S_T = S_{T-1} + Q_T^i + Q_T^u - Q_T^o, \qquad (8\text{-}13b)$$

$$Q_T^i = Q_T^c - Q_T^e, \qquad (8\text{-}13c)$$

Table 8-6. Return Function of a Uniform Water Release over Three-Month Periods.

		Uniform Water Release Q_T^o												
Period	Month	0	1	2	3	4	5	6	7	8	9	10	11	12
I	1-3	0.0	3.2	5.3	4.6	4.0	3.5	3.0	2.5	2.0	1.5	1.0	0.5	0.0
II	4-6	0.0	5.5	9.9	12.7	13.4	12.5	11.0	9.5	8.0	6.5	5.0	3.5	1.0
III	7-9	0.0	6.2	10.5	14.2	12.4	10.6	8.3	6.2	4.7	2.4	1.8	1.1	0.7
IV	10-12	0.0	2.8	4.6	3.9	3.0	2.7	2.4	2.1	1.8	1.5	1.2	0.9	0.6

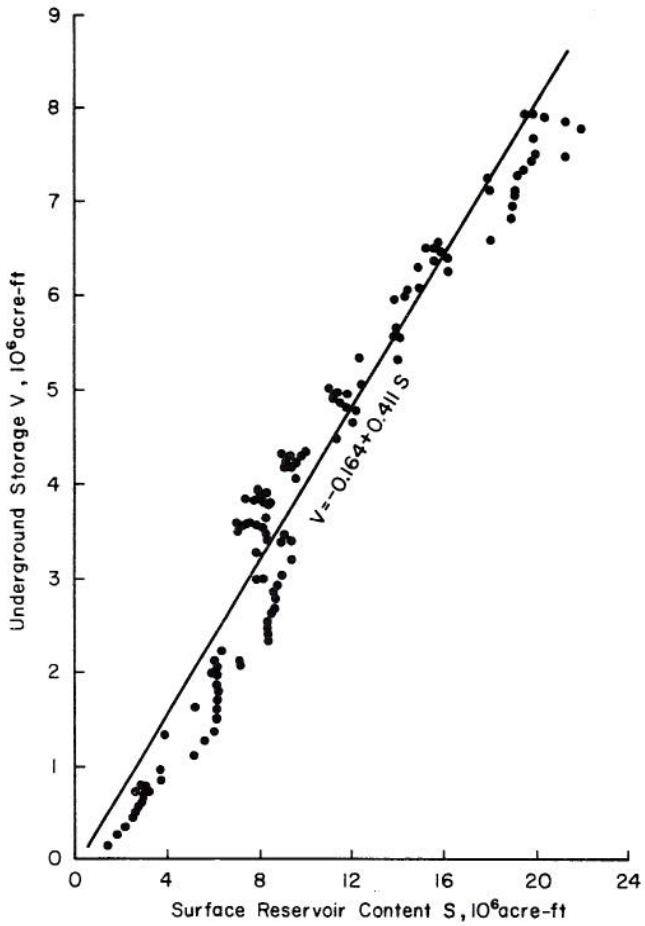

Fig. 8-14. Underground Storage, V, as a Function of the Surface Reservoir Content, S.

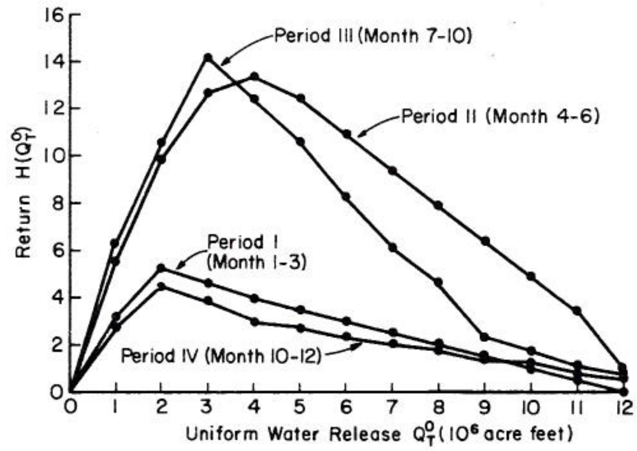

Fig. 8-15. Return Functions of a Uniform Water Release over Three-Month Periods.

$$q_t^e = A^r_{t-1} e_t = (0.0185 + 0.00546448)\ S_{t-1}\ e_t\ , \quad t = 1,\ldots,N_t \tag{8-13d}$$

$$q_t^u = \alpha_o + \alpha_1\ q_{t-1}^u - \alpha_2\ Q_t\ , \quad t = 1,\ldots,N_t \tag{8-13e}$$

$$Q_t = q_t^o + q_t^e - q_t^c\ , \quad t = 1,\ldots,N_t \tag{8-13f}$$

$$3.0 \le S_T \le 27.0\ , \tag{8-13g}$$

and

$$0.0 \le Q_T^o \le 12.0\ . \tag{8-13h}$$

Table 8-7. A Summary of Typical Results of Computation at the End of the Stage.

J	$S_{T-1}(J)$	$S_T(J)$	$Q^u_{T-1}(J)$	$Q^u_T(J)$	$F_{T-1}(J)$	$F_T(J)$	$Q_T^{ben}(J)$
1	11.00	3.00	-.03	.28	299.70	302.71	11.48
2	5.00	5.00	-.04	-.04	301.02	303.34	2.63
3	7.00	7.00	-.04	-.04	301.40	303.72	2.62
4	9.00	9.00	-.04	-.04	301.85	304.18	2.61
5	11.00	11.00	-.04	-.04	302.29	304.61	2.61
6	13.00	13.00	-.04	-.04	302.90	305.22*	2.60
7	15.00	15.00	-.04	-.04	302.86	305.19	2.59
8	17.00	17.00	-.04	-.04	302.66	305.00	2.58
9	19.00	19.00	-.04	-.04	302.40	304.74	2.57
10	21.00	21.00	-.04	-.04	302.02	304.36	2.56
11	23.00	23.00	-.04	-.04	301.52	303.86	2.55
12	25.00	25.00	-.04	-.04	300.31	303.16	2.54
13	27.00	27.00	-.04	-.04	298.37	300.93	2.53

The values of Q_T^i, Q_T^c, Q_T^e, and Q_T^u are obtained as given by Eq. 7-39. Observe that q_t^o of Eq. 8-13f is given by $q_t^o = Q_t^o/\tau$. In this example $N_T = 48$, $N_T = 144$, and $\tau = 3$, so that moving bounds of simulation are analogous to those of Example I.

In Eq. 8-13d, evaporation, q_t^e, was computed using only the reservoir state at the beginning of the given time interval t. Of course, it is possible to perform computations using the average value of the areas A_{t-1}^r and A_t^r. However, this would require that an iterative computing procedure be introduced. It is felt that improvements due to this alternative approach would not warrant its implementation.

The computational procedure is very much similar to that of Example I. The only difference is that the exchange flow, q_t^u, had to be carried on as a semi-state variable instead of V_t. The initial value of the exchange flow required to facilitate the computation of q_1^u, was assumed to be $q_o^u = 0$.

A summary of typical results obtained at every stage is given in Table 8-7. The final results of the optimization, that is, the optimal policy of reservoir transformation, is presented in Table 8-8 and in Fig. 8-16. Statistical properties of the water release are given in Table 8-9. Figure 8-16 indicates, as in Example I, that the optimal policy seemingly follows no regular pattern. However, analyzing the statistical properties of the water release leads to the conclusion that the water release is, to a large degree, regular and that it is periodic.

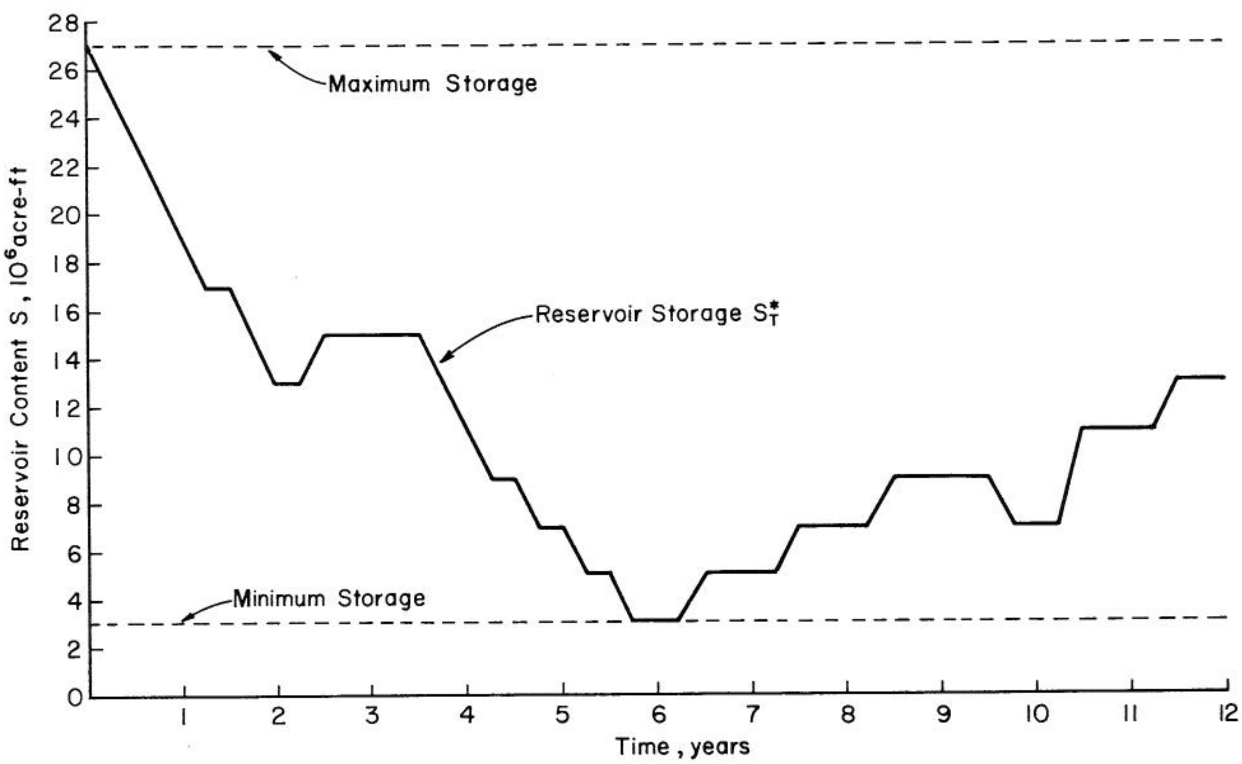

Fig. 8-16. Optimal Policy of the Reservoir State Transformation.

Table 8-8. Optimal Policy of the State Transformation and Water Release.

T	J^*_{T-1}	S^*_{T-1}	J^*_T	S^*_T	Q^*_T	F^*_T
1	13	27.0	12	25.0	3.03	4.36
2	12	25.0	11	23.0	4.31	16.86
3	11	23.0	10	21.0	2.93	30.14
4	10	21.0	9	19.0	2.88	33.93
5	9	19.0	8	17.0	2.99	38.11
6	8	17.0	8	17.0	3.62	50.03
7	8	17.0	7	15.0	3.71	61.75
8	7	15.0	6	13.0	3.46	64.91
9	6	13.0	6	13.0	1.72	68.98
10	6	13.0	7	15.0	4.76	79.97
11	7	15.0	7	15.0	3.58	91.33
12	7	15.0	7	15.0	2.16	95.21
13	7	15.0	7	15.0	2.63	99.21
14	7	15.0	7	15.0	3.41	109.90
15	7	15.0	6	13.0	3.08	121.46
16	6	13.0	5	11.0	3.37	124.40
17	5	11.0	4	9.0	3.46	127.79
18	4	9.0	4	9.0	3.47	137.99
19	4	9.0	3	7.0	4.11	147.55
20	3	7.0	3	7.0	1.40	150.31
21	3	7.0	2	5.0	3.55	153.50
22	2	5.0	2	5.0	4.82	162.95
23	2	5.0	1	3.0	4.34	171.75
24	1	3.0	1	3.0	1.50	174.51
25	1	3.0	1	3.0	1.95	178.20
26	1	3.0	2	5.0	3.77	187.61
27	2	5.0	2	5.0	2.46	196.29
28	2	5.0	2	5.0	2.30	199.41
29	2	5.0	2	5.0	1.86	202.80
30	2	5.0	3	7.0	3.60	211.68
31	3	7.0	3	7.0	2.69	220.52
32	3	7.0	3	7.0	3.03	223.14
33	3	7.0	3	7.0	3.08	226.07
34	3	7.0	4	9.0	3.94	234.68
35	4	9.0	4	9.0	3.20	243.60
36	4	9.0	4	9.0	2.94	246.14
37	4	9.0	4	9.0	3.16	248.90
38	4	9.0	4	9.0	4.61	256.80
39	4	9.0	3	7.0	4.46	263.90
40	3	7.0	3	7.0	3.51	266.02
41	3	7.0	3	7.0	3.32	268.59
42	3	7.0	5	11.0	4.11	276.37
43	5	11.0	5	11.0	4.31	283.30
44	5	11.0	5	11.0	3.03	285.56
45	5	11.0	5	11.0	3.11	288.09
46	5	11.0	6	13.0	3.98	295.54
47	6	13.0	6	13.0	2.73	302.90
48	6	13.0	6	13.0	2.60	305.22

Table 8-9. Statistical Properties of the Optimal Water Release.

	PERIOD			
	1	2	3	4
$\bar{Q}^*$	2.82	4.03	3.47	2.68
σ	0.637	0.496	0.714	0.712
σ^2	0.405	0.246	0.510	0.507

Chapter IX
SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

Performance of a surface reservoir can be significantly affected by a connected large natural underground storage. Finding the optimal policy of water used from such a coupled surface-underground storage was the subject of this dissertation. To develop the method of determining the optimal policy, two main problems had to be solved: (1) Modelling of the system; and, (2) Optimization of water use.

The hydrologic modelling of the system was approached from the point of view that a convenient hydrologic model is needed to facilitate the optimization without the use of excessive computer time, and particularly, without the requirement of huge computer memory. Within the framework of this study, only two hydrologic components of the river flow were treated in detail: (a) the exchange flow between the two storages, q_t^u; and, (b) the recharge to the aquifer under specific conditions usually associated with karst areas. The exchange flow was described using the theory of hydrologic systems which is frequently used to treat the practical problems of river hydrographs. The exchange flow was assumed to be a function of the states of the two interconnected storages. The development of the model was carried out in a generalized form. Then, the customary simplifications, which are usually made possible by the specific conditions found in river basins, were discussed. In addition, the implications of frequently made assumptions were examined.

Recharge modelling can be regarded as inseparable from the modelling of the exchange flow. The process of recharge in karstified catchments is somewhat specific. The model developed in this study was shown to be able to accurately simulate the springflow of a karstic river. Yet, the amount of computation to identify the system was held at a minimum, because the use of only two parameters of the model was sufficient to explain over 90 percent of the springflow variation.

The optimization process in this study was necessarily complex. It was solved by multilevel dynamic optimization. The problem was decomposed into the resource allocation problem and multistage dynamic optimization. The resource allocation problem was treated as an equality constraint allocation problem thought of as a conditional allocation.

Deterministic sequences were routed through the multistage dynamic optimization. The integration of the known components was carried out simultaneously with the simulation of the exchange flow. In this manner it was possible to perform the optimization at a number of stages considered to be not computationally excessive. Still, the variations of the hydrologic series within the time intervals represented by one stage are taken into account.

The fact that the optimal policy is found from deterministic data sequences should not be regarded as a drawback of the model. A similar technique was extensively used in the past under somewhat simpler conditions, that is, with no underground storage.

The study was limited by the availability of test data. Actual evaluation of the optimal policy of water release was performed on two examples. Except for the hydrologic sequences of Lake Powell, all the data had to be hypothesized. For that reason, the results presented herein should be viewed as a demonstration that the model is capable of fulfilling the stated objective. No conclusions concerning the actual operation of the existing system at Lake Powell should be based on these results.

The proposed model should be tested on actual coupled surface-underground storage systems in karst, from which some experience concerning these systems can be gained. The most important aspects to be investigated are:

1. To what extent the performance of a natural hydrologic system will be affected by the system modification.

2. What projections concerning the system response to the atmospheric processes can be made at the planning stage of the water resources project. In other words, what is the worth of data collected prior to the reservoir construction.

3. What is the effect of the assumption of the initial underground storage which was made to eliminate the second state variable from the computation.

4. The process of determining the value of water is complex. Economic and other factors that determine water value are often unpredictable. Sensitivity analysis should be performed to determine the possible effects of changes in economic factors.

APPENDIX A

Karstic springs are usually outlets of large natural underground storages. To demonstrate the significance of these storages, three illustrative examples, compiled from various references, are presented in Table A-1 and Fig. A-1. The underground storage capacities associated with (1) San Felipe Spring, (2) Goodenough Spring, and (3) the Trebišnjica River Spring are estimated assuming that the hydrologic systems are linear, i.e., that the discharge is proportional to the water content of the underground reservoir, so that the falling limb of the hydrograph is expressed by Eq. 4-8. That is,

$$q_u(t) = q_u(t_o) \exp\left[-c(t - t_o)\right] , \qquad \text{(A-1)}$$

where $q_u(\cdot)$ is the underground outflow at the respective times t and t_o, and c is a constant to be determined for every specific drainage basin under consideration. When Eq. A-1 is integrated from time $t_o = 0$ to time $t = \infty$ and multiplied by 86,400 to convert the discharge per second into the total daily flow,

$$V = \int_{t=0}^{\infty} 86{,}400\, q_u(t)\, dt = [86{,}400\, q_u(0)]/c \ , \qquad \text{(A-2)}$$

is obtained. In Eq. A-2, V represents the total water content of the underground reservoir at time $t = 0$.

The coefficients, c, for the springs numbered (1) and (2) are taken from Knisel [1972], while the coefficient of the Trebišnjica River Spring was estimated in this study, following a pattern similar to Knisel's. Time t_o was taken to correspond to the maximum daily spring discharge for the denoted time period.

The mean daily discharge time equivalent is defined as

$$t_e = \frac{V}{\bar{q}_u} \qquad \text{(A-3)}$$

with $\bar{q}_u$ being the average daily flow over the studied time period. Equation A-3 says that the mean daily discharge time equivalent is the time that the mean daily flow would need to discharge an amount of water equal to the underground storage determined by Eq. A-2.

Another interesting example is given by Burdon and Safadi [1963]. It shows that the Ras-el-Ain karstic spring in Syria issues from an underground storage which has a volume of approximately 7.0 10^9 m^3.

Figure A-1 depicts the mean daily discharge time equivalent in days versus percentage of the total outflow (denoted as "discharged storage percent" on the left-hand ordinate). From these curves it can be concluded that the system (3) corresponding to the Trebišnjica River is fast responding as compared to the system (2) of Goodenough Spring. The same conclusion could be drawn from a visual inspection of the respective hydrographs.

The Trebišnjica River Spring is presently submerged under the water of a surface reservoir created by a recently built dam. When compared with the capacity of the surface storage of approximately 1.3 10^9 m^3, the maximum recorded underground storage that occurred in the period prior to the reservoir construction (1954-1966), was about 13.5 percent to 25.0 percent, depending on what estimate of the coefficient c was taken. Since the preceding analysis reflects the hydraulic conditions associated with extreme events, it should not be expected that the underground capacity augments the existing surface storage by 13.5 percent to 25.0 percent. Instead, these figures should be taken as an indication of the existance of the underground storage. Accurate estimates of the actual underground storages for this example are unavailable at this time. However, they are believed to be somewhere between 5.0 percent and 10.0 percent.

Investigations of the bank storage of Lake Powell are based on an incomplete set of data, since the reservoir has yet to be filled. Nevertheless, the presently available observations show that the underground storage is about 20.0 percent to 25.0 percent of the surface storage.

Significant underground storage associated with the Libby Reservoir in northwestern Montana was reported by Coffin [1970]. A preliminary investigation by an electric analog model estimated the bank storage to be about 5.0 percent of the surface capacity of five million acre-feet (approximately 6.1 10^9 m^3).

Table A-1. An Illustrative Example of Underground Storage Associated with Karstic Springs.

Spring & Location	Average daily flow $\bar{q}_u$ [m^3/s]	Period	Maximum daily flow $\bar{q}_u(0)$ [m^3/s]	Date	Coefficient exhaustion c	Maximum underground storage V [m^3]	Mean daily discharge time equivalent t_e [days]
San Felipe Spring	2.25	1967	3.34	09.16.67	0.0121	23.8 10^6	123
Del Rio, Texas USA	2.27	1961-1967	+	+	+	+	+
Goodenough Spring	3.02	1967	9.08	09.03.67	0.00463	169.0 10^6	650
Comstock, Texas USA	3.80	1930-1967	18.4	10.10.58	0.00463	343.0 10^6	1050
The Trebišnjica River Spring near Biléca, Yugoslavia*	49.0	1966	218.0	10.10.66	A) 0.065	290.0 10^6	68
					B) 0.121	156.0 10^6	36
	44.4	1954-1966	247.0	03.03.65	A) 0.065	328.0 10^6	86
					B) 0.121	177.0 10^6	46

*Coefficient c refers to Case A of Section 5-4, Eq. 5-28 and Case B of the same section Eq. 5-23.

+Data unavailable.

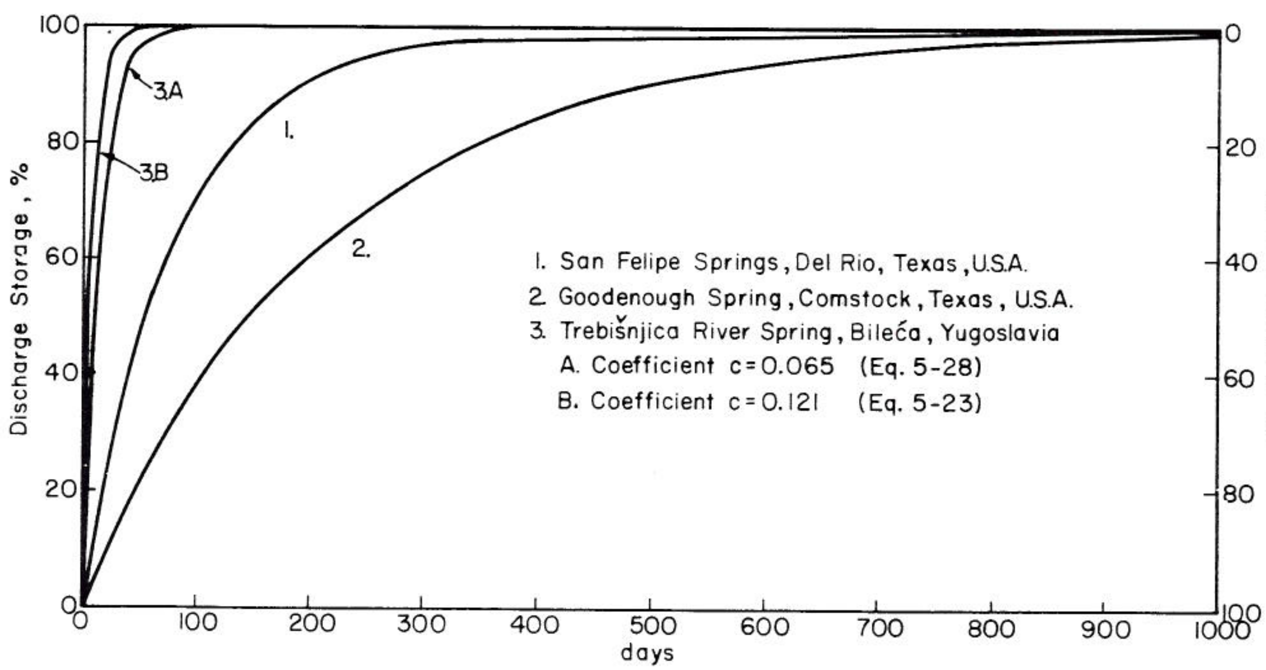

Fig. A-1. Mean Daily Discharge Time Equivalent versus Discharged Volume.

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