1. COUNTING & GAMES

Horse Racing and Committee Building

$$(n)_k = n(n-1)\cdots(n-k+1) \quad \text{permutations}$$

$$= n! \ \text{ for } k = n$$

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \qquad \text{(combinations or binomial)}$$

$$\binom{n}{k_1 \cdots k_r} = \frac{n!}{k_1! \cdots k_r!} \qquad \text{(multinomial)}$$

Keno and Lotto

$$\frac{\binom{n}{k}\binom{N-n}{K-k}}{\binom{N}{K}} \quad \text{typical} \quad \frac{\binom{20}{k}\binom{60-20}{10-k}}{\binom{60}{10}}$$

$$\frac{\binom{6}{k}\binom{42-6}{6-k}}{\binom{42}{6}}$$

Polling

$$\frac{\binom{n_1}{k_1}\cdots\binom{n_r}{k_r}}{\binom{N}{K}} \quad \text{hypergeometric}$$

Reliability Sampling $(N, n; K, k)$

$$\frac{\binom{n}{k}\binom{N-n}{K-k}}{\binom{N}{K}}$$

Poker

$$(xxyyz) \ : \ \frac{\binom{13}{3}\binom{3}{1}\binom{4}{2}\binom{4}{2}\binom{4}{1}}{\binom{52}{5}}$$

Fair Games : Odds, Payout, ...

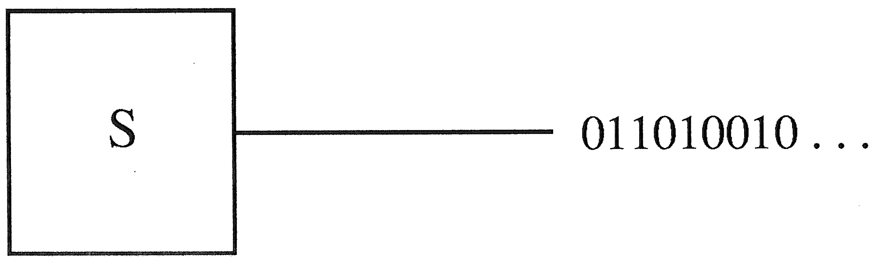
$$W = \frac{1}{P}B = (0+1)B; \ \ \frac{1}{P} = 0+1$$

MB, BE, FD

$$\binom{n}{n_1\cdots n_r}k^r, \ \prod_1^r\binom{k-1+n_i}{n_i}, \ \prod_1^r\binom{k}{n_i}$$

2. PROBABILITY $(\Omega, \mathcal{F}, P)$

Example: Binary Source



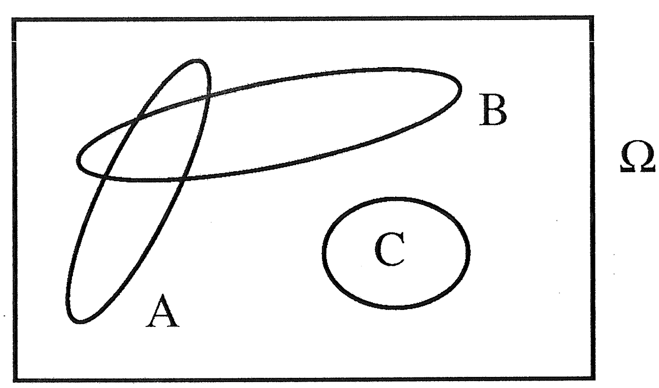
$\Omega = \{\omega_i,\ i = 0, 1, \ldots, 2^n - 1;\ \omega_i = \text{binary rep. of } i\}$

$\mathcal{F} = \{2^{2^n} \text{ sets constructed from } \Omega\}$

$P(\omega_i) = p^k(1-p)^{n-k};\ k = \#$ ones in ω_i

$P(A) \geq 0\ \ P(\Omega) = 1\ \ P(A \cup B) = P(A) + P(B)$ if (A, B) are disjoint

Venn Diagrams & Probability



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(B \cap C) = 0$$

3. DEPENDENCE & INDEPENDENCE

(Value of Side Information, Evidence, Measurements)

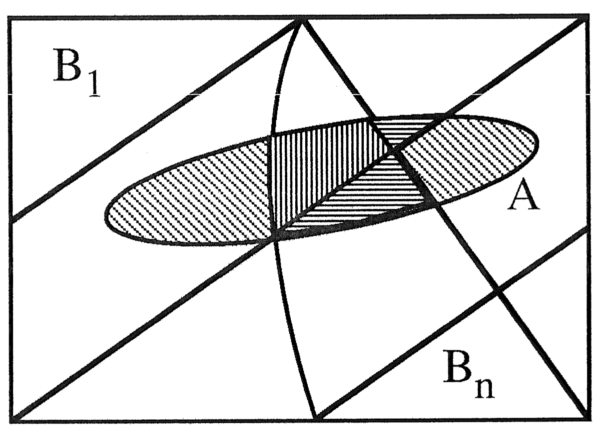
Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$
$$= P(A) \text{ if } P(A \cap B) = P(A)P(B)$$

(independence)

Total Probability

$$P(A \cap B) = \sum_i P(A \cap B_i) \quad \text{if} \quad \begin{matrix} \bigcup_i B_i = B \\ B_i \cap B_j = \phi \end{matrix}$$
$$= \sum_i P(A|B_i)P(B_i)$$



Craps, Modulo-Arithmetic, Quantizing Errors

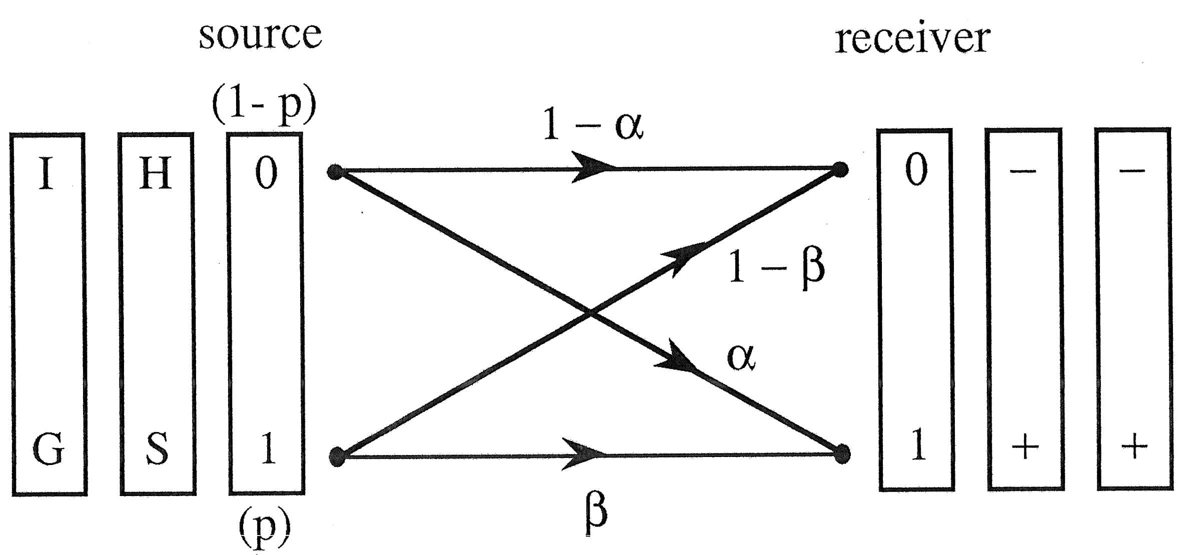
4. BAYES RULE

A_i : Source Symbol $\quad$ B_j : Received Symbol

Formula

$$P(A_i|B_j) = \frac{P(B_j|A_i)P(A_i)}{\sum_{i=1}^{r} P\left(B_j(A_i)P(A_i)\right)}$$

Bayes Diagram

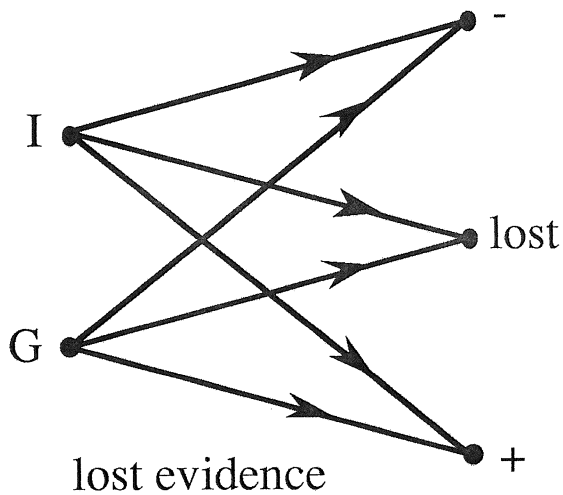


β = specificity $\qquad$ α = false pos.

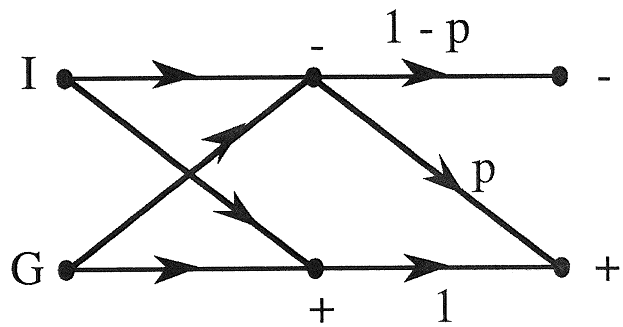
$$P[G|+] = \frac{1}{1 + \frac{\alpha}{\beta}\,\frac{1-p}{p}}; \quad P[G|+^r] = \frac{1}{1 + \left(\frac{\alpha}{\beta}\right)^r \frac{1-p}{p}}$$

$$\frac{1}{1 + \left(\frac{\alpha}{\beta}\right)^r \frac{1-p}{p}} = 1 - \epsilon \;:\; \text{solve for something}$$

Generalization



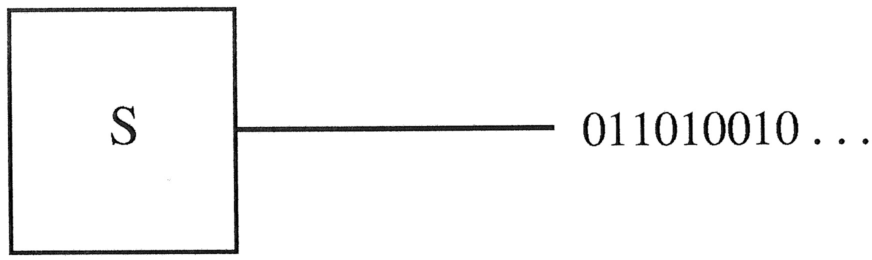
lost evidence



tampered evidence

5. RANDOM VARIABLES & CLASSICAL EXPERIMENTS

Bernoulli Experiment



berni : $P_X(k) = p^k(1-p)^{1-k}; \quad k = 0, 1$

bino : $P_X(k) = \binom{n}{k} p^k(1-p)^{n-k}; \quad k = 0, 1, \ldots, n$

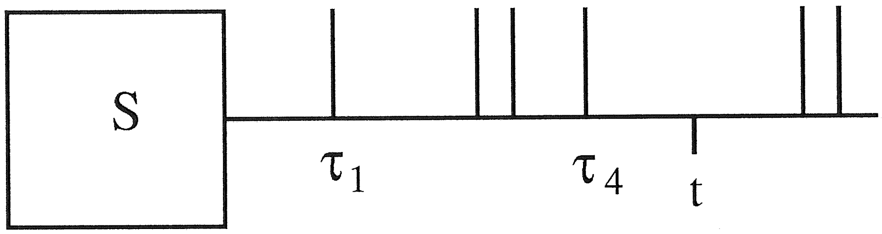
geo : $P_X(k) = (1-p)^{k-1}p; \quad k = 1, 2, \ldots$

pascal : $P_X(k) = \binom{k-1}{r-1} p^r(1-p)^{k-r}; \quad k = r, r+1, \ldots$

bino = n-fold convolution of berni

pascal = r-fold convolution of geo

Poisson Experiment



poisson : $p_{N(t)}(k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!};\quad k = 0, 1, \ldots$

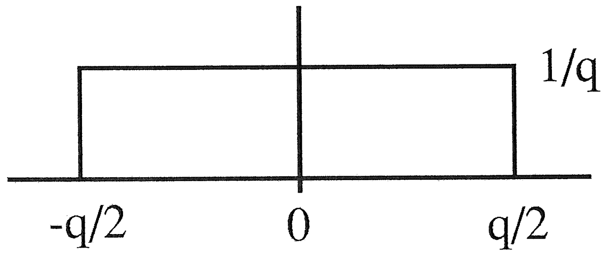
exponential : $f_X(t) = \lambda e^{-\lambda t};\quad t \geq 0$

erlang : $f_X(t) = \frac{(\lambda t)^{r-1}}{(r-1)!}\lambda e^{-\lambda t};\quad t \geq 0$

poisson = limit of Bernoulli as $n \uparrow \infty$, $p \downarrow 0$, $np = \lambda$

erlang = r-fold convolution of exponential

Quantization Noise



$q = 2^{-n}$

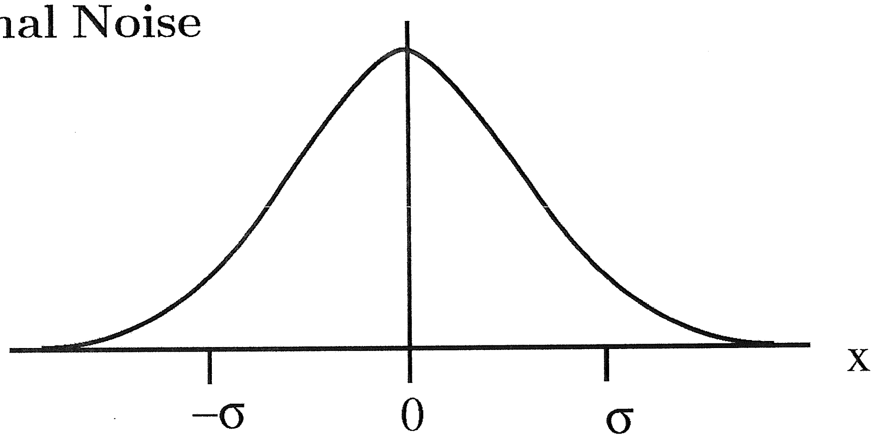
$$f_X(x) = \frac{1}{q}\psi_{[-q/2,q/2]}(x) \qquad \text{(uniform)}$$

$$\mu_X = 0$$

$$\sigma_X^2 = q^2/12 = 2^{-2n}/12$$

(solve for wordlength to get desired quantization noise power)

Thermal Noise



$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} \qquad \text{(Gaussian)}$$

$$\mu_X = 0; \qquad \sigma_X^2 = \sigma^2$$

6. MOMENTS & CHARACTERISTIC FUNCTIONS: LINEAR SYSTEMS FOR PROBABILITY

$$X = X_1 + X_2 + \cdots + X_n \quad X_i \perp X_j$$

$$\mu_X = \sum \mu_{X_i}$$

$$\sigma_X^2 = \sum \sigma_{X_i}^2$$

$$\Phi_X(\omega) = \prod \phi_{X_i}(\omega)$$

$$f_X(x) = f_{X_1} * \cdots * f_{X_n}(x)$$

$$1 = \phi_X(\omega)\Big|_{\omega=0} \text{ : “dc gain”}$$

$$\mu_{X_i} = j\,\tfrac{\partial}{\partial\omega}\,\phi_{X_i}(\omega)\Big|_{\omega=0} \text{ : “dc slope”}$$

$$\sigma_{X_i}^2 \sim (j)^2\,\tfrac{\partial^2}{\partial\omega^2}\,\phi_{X_i}(\omega)\Big|_{\omega=0}$$

$$F_X(x)$$

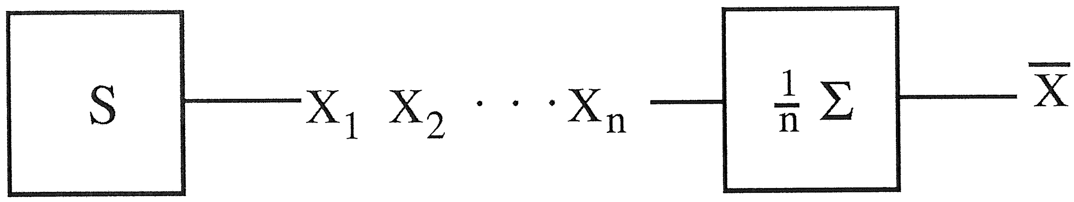
Experiment	Discrete RNG	Binary Transmission	Binary Sequence	1-Wait Time	r-Wait Time	Keno & Sampling
Distribution	Discrete Uniform	Bernoulli	Binomial	Geometric	Pascal	Hyper-geometric
Values	k=1,2, …,n	k=0,1	k=0,1,…,n	k=1,2,…	k=r,r+1,…	k=0,1,…,m
Pmf/Pdf	P(k)=1/n	$P(k)=p^k(1-p)^{1-k}$	$P(k)=\binom{n}{k}p^k(1-p)^{n-k}$	$P(k)=p(1-p)^{k-1}$	$P(k)=\binom{k-1}{r-1}p^r(1-p)^{k-r}$	$P(k)=\frac{\binom{np}{k}\binom{n(1-p)}{m-k}}{\binom{n}{m}}$
Charcteristic Function	$\frac{1}{n}\cdot e^{-j\omega}\frac{1-e^{-jn\omega}}{1-e^{-j\omega}}$	$(1-p)+pe^{-j\omega}$	$[(1-p)+pe^{-j\omega}]^n$	$\frac{pe^{-j\omega}}{1-(1-p)e^{-j\omega}}$	$\left[\frac{pe^{-j\omega}}{1-(1-p)e^{-j\omega}}\right]^r$	$\sum_{k=0}^{m}\frac{\binom{np}{k}\binom{n(1-p)}{m-k}}{\binom{n}{m}}e^{-j\omega k}$
Mean	(n+1)/2	p	np	1/p	r/p	mp
Variance	$(n^2-1)/12$	p(1-p)	np(1-p)	$(1-p)/p^2$	$r(1-p)/p^2$	$m\frac{n-m}{n-1}p(1-p)$

Experiment	Continuous RNG		Radioactivity	1-Wait Time	r-Wait Time	Sum of Effects
Distribution	Continuous Uniform	Laplace	Poisson	Exponential	Erlangen	Gaussian (Normal)
Values	a<x≤b	-∞<x<∞	k=0, 1, …	x≥0	x≥0	-∞<x<∞
Pmf/pdf	$f(x)=(b-a)^{-1}$	$f(x)=\frac{\alpha}{2}e^{-\alpha\|k\|},\alpha>0$	$P_X(k)=e^{-\lambda}\lambda^k/k!$	$f(x)=\lambda e^{-\lambda x},\lambda>0$	$f(x)=\frac{(\lambda x)^{r-1}}{(r-1)!}\lambda e^{-\lambda x}$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$
Characteristic Function	$e^{-j\omega\frac{b+a}{2}}\text{sinc}(\omega\frac{b-a}{2})$	$\frac{\alpha^2}{\alpha^2+\omega^2}$	$e^{\lambda(e^{-j\omega}-1)}$	$\frac{\lambda}{\lambda+j\omega}$	$\left(\frac{\lambda}{\lambda+j\omega}\right)^r$	$e^{-j\omega\mu}\cdot e^{-\frac{\omega^2\sigma^2}{2}}$
Mean	(a+b)/2	0	λ	1/λ	r/λ	μ
Variance	$(b-a)^2/12$	$2/\alpha^2$	λ	$1/\lambda^2$	r/λ^2	σ^2

7. CENTRAL LIMIT THEOREM & NORMAL

$$\sum_{i=1}^{n} \frac{(X_i - \mu)}{\sigma\sqrt{n}} \longrightarrow N[0,1]$$

8. AVERAGING & CHEBYSHEV

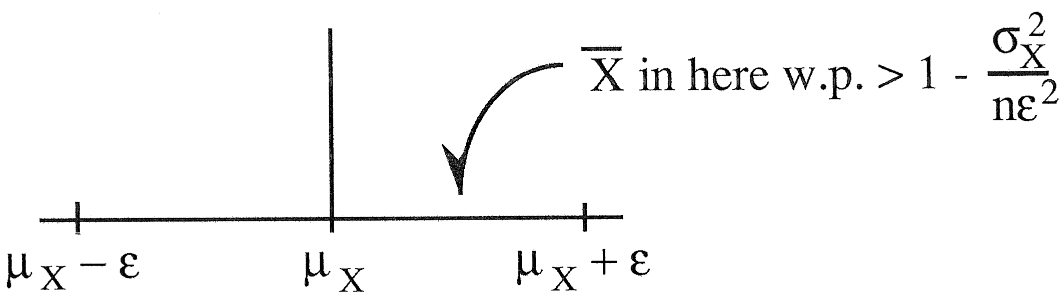


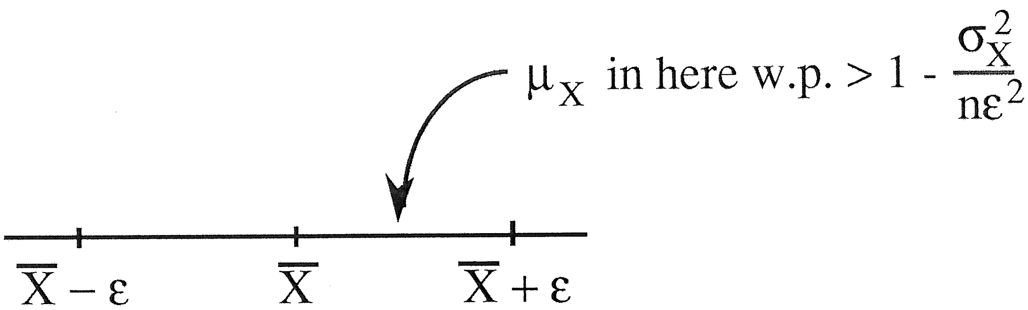
$$\mu_{\bar{X}} = \mu_X \qquad \sigma^2_{\bar{X}} = \frac{1}{n}\sigma^2_X$$

$$P\left[|\bar{X} - \mu_X| > \epsilon\right] \le \frac{\sigma^2_X}{n\epsilon^2} = p$$

$$P\left[|\bar{X} - \mu_X| < \epsilon\right] > 1 - p = 1 - \frac{\sigma^2_X}{n\epsilon^2}$$

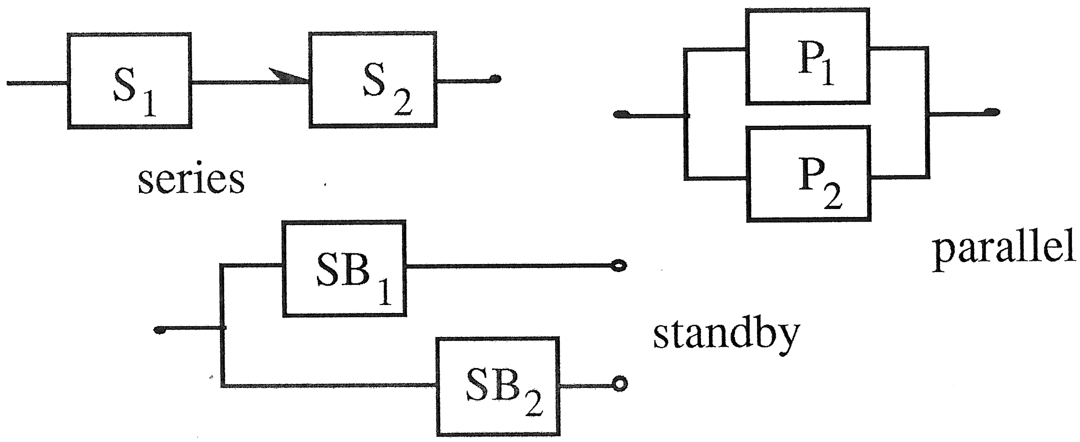
(choose n to get desired confidence $1 - p$)



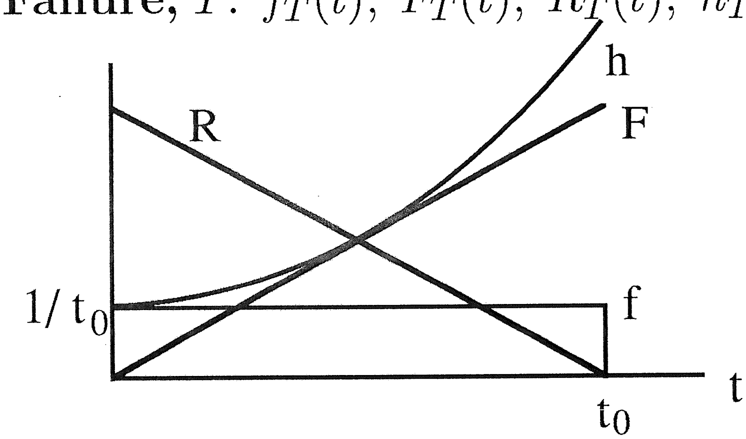


9. RELIABILITY OF SYSTEMS

Systems

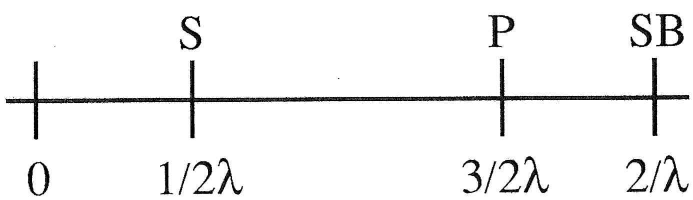


Time to Failure, T: $f_T(t)$, $F_T(t)$, $R_T(t)$, $h_T(t)$, μ_T



Exponential

	series	parallel	standby
μ_T	$\frac{1}{\lambda_1+\lambda_2}$	$\frac{1}{\lambda_1}+\frac{1}{\lambda_2}-\frac{1}{\lambda_1+\lambda_2}$	$\frac{1}{\lambda_1}+\frac{1}{\lambda_2}$



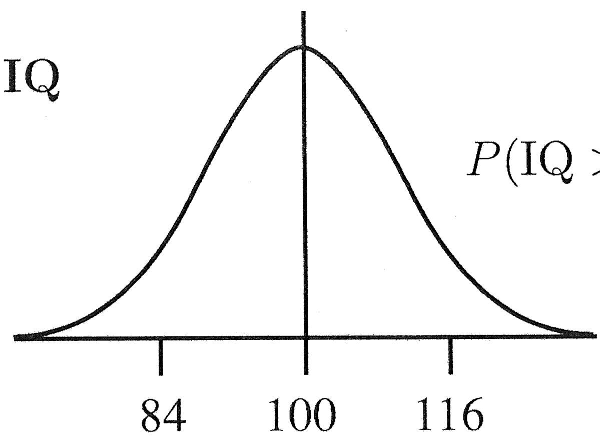
; $(\mu_T)_{\text{SB}} = (\mu_T)_{\text{S}} + (\mu_T)_{\text{P}}$

10. BIT ERROR RATES IN BINARY COMMUNICATION

Formulas

$$\begin{aligned} P(E) &= P[Y > t]; \quad Y : N[\mu, \sigma^2] \\ &= P\left[X > \frac{t-\mu}{\sigma}\right]; \quad X : N[0,1] \\ &= Q\left(\frac{t-\mu}{\sigma}\right) \quad : \text{ TABLES} \end{aligned}$$

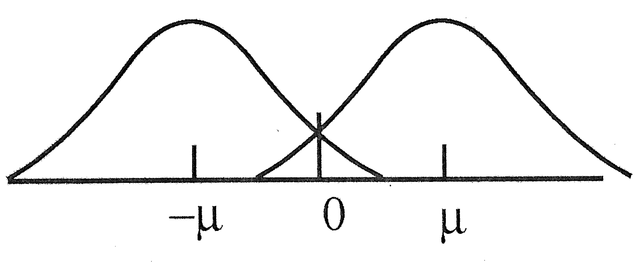
IQ



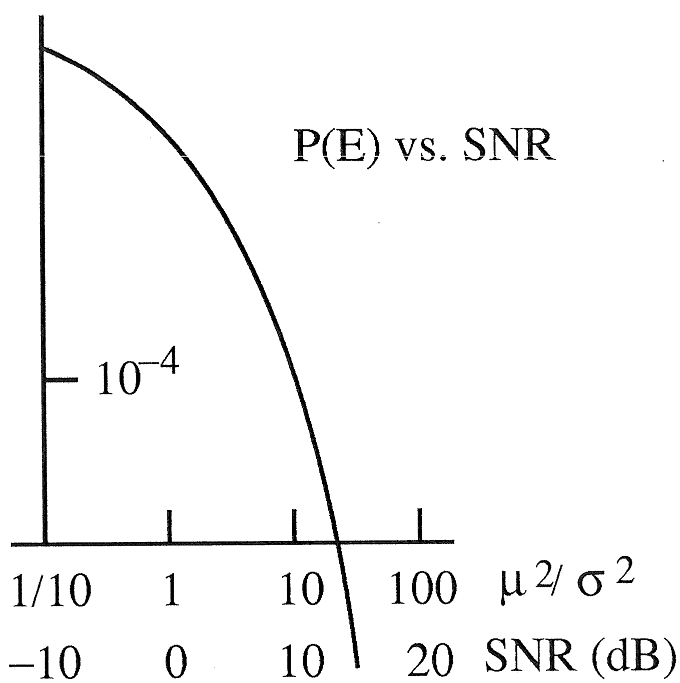
$$\begin{aligned} P(\text{IQ} > 132) &= Q\left(\frac{132 - 100}{16}\right) \\ &= Q(2) = 0.0228 \quad (98\%) \end{aligned}$$

$\mu_X = 100 \qquad \sigma_X = 16$

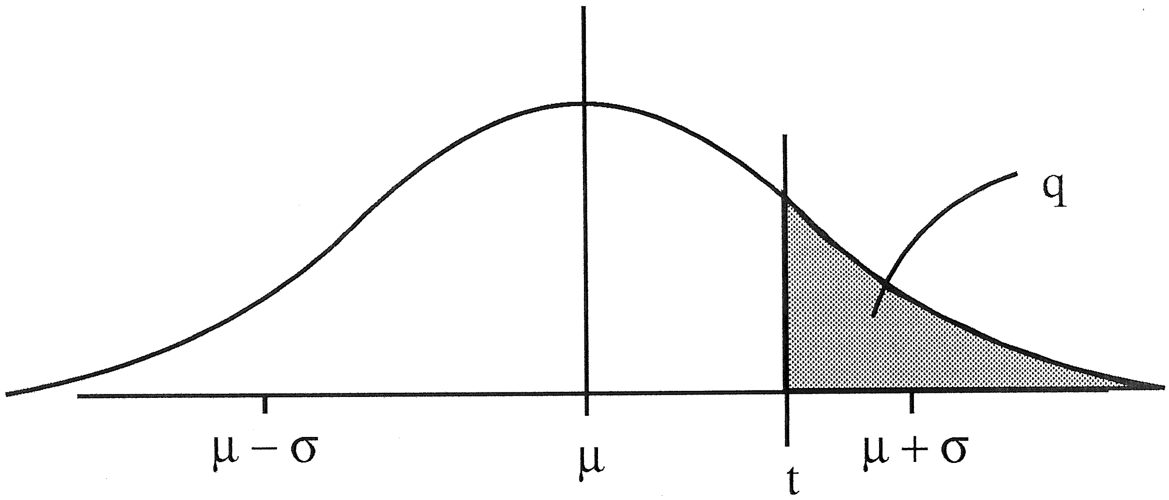
Binary Communication



$$P(E) = Q\left(\frac{\mu}{\sigma}\right)$$



General

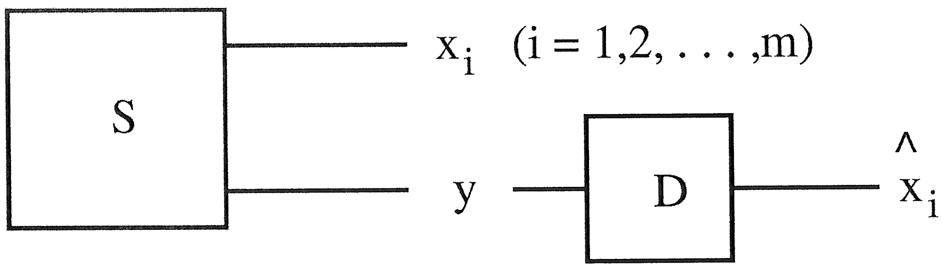


Given three of four (μ, σ, t, q), find the fourth.

$$Q(\text{tables}) \ : \ Q\left(\frac{t-\mu}{\sigma}\right) = q$$

11. THE MAP RULE FOR min $P(E)$ DETECTION

Experiment



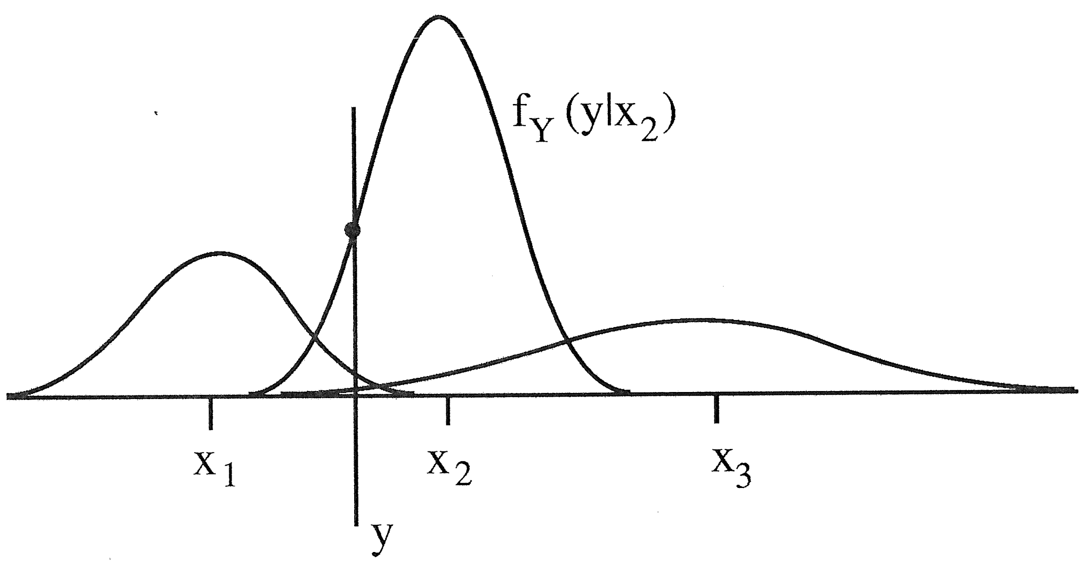
$$\min P[\hat{x}_i \neq x_i]$$

Detector

$$\max_{x_i} f_Y(y|x_i) \quad \text{ML}$$

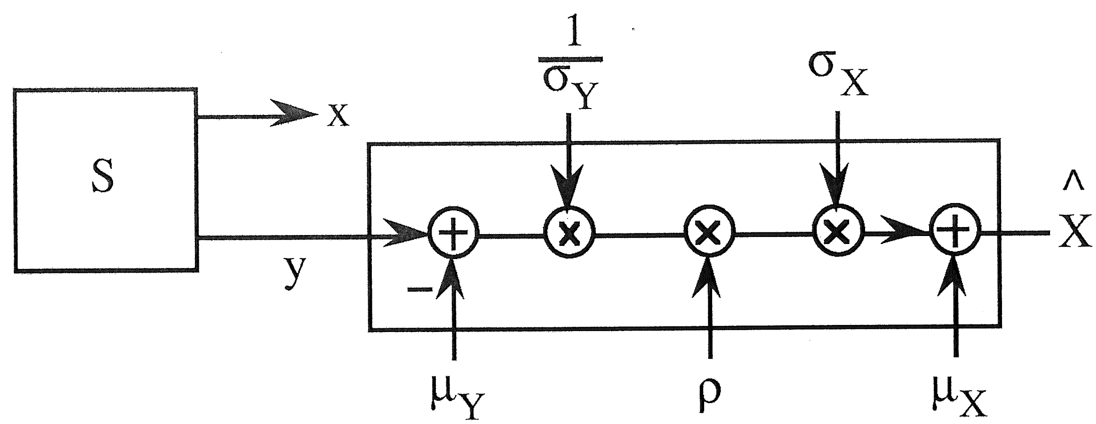
$$\max_{x_i} f_Y(y|x_i)p_i \quad \text{PML}$$

$$\max_{x_i} P_X(x|y) \quad \text{MAP}$$



12 & 13. LINEAR REGRESSION & MMSE TEST

Experiment



Linear

$$X : \mu_X, \sigma_X^2 \quad Y : \mu_Y,\ \sigma_Y^2$$

$$(X, Y) : \rho = \frac{E(X - \mu_X)(Y - \mu_Y)}{\sigma_X \sigma_Y} \ : \ \text{corr coef}$$

$$\frac{\hat{X} - \mu_X}{\sigma_X} = \rho \frac{Y - \mu_Y}{\sigma_Y}$$

MMSE

$$\hat{X}_0 = E[X|Y] \ : \ \text{conditional mean}$$

Bivariate Normal

$$\hat{X}_0 = \hat{X}$$