**Colorado State University, Ft Collins**

**ECE 652, Estimation and Filtering**
**Spring Semester 2010**

**Homework**

# 1 HW Set #1: due Tues, Feb 2, 2010

1. Read Ch 1 in text.

2. *Study* Ch 2 in the text, and lecture notes.

3. Read Wikipedia where reqd to fill in your understanding of linear and matrix algebra.

4. Let $\mathbf{X} \in \mathcal{R}^{3\times 2}$ and $\mathbf{A} \in \mathcal{R}^{3\times 1}$ be the following matrices:

$$\mathbf{X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{A} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

   Work out the following matrices and show how they operate on vectors $\mathbf{x} = [x_1, x_2, x_3]^T \in \mathcal{R}^3$:

   - projection, $\mathbf{P}_X = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$
   - rotation in subspace $< \mathbf{X} >$,

   $$\mathbf{Q} = \mathbf{X} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mathbf{X}^T + \mathbf{P}_A \tag{2}$$

     Explain the statement, "the matrix $\mathbf{Q}$ rotates vectors in the subspace $< \mathbf{X} >$."

   - pseudo-inverse, $\mathbf{X}^{\#} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$.

5. There are many problems in estimation and filtering where a Grammian or covariance matrix $\mathbf{R}_0$ is updated with a rank one update of the form $\gamma^2 \mathbf{u}\mathbf{u}^H$. Thus we are led to consider the matrix $\mathbf{R} = \mathbf{R}_0 + \gamma^2 \mathbf{u}\mathbf{u}^H \in \mathcal{C}^{n\times n}$ and its inverse. By direct evaluation of $\mathbf{R}\mathbf{R}^{-1} = \mathbf{I}$, verify that the inverse of $\mathbf{R}$ is

$$\mathbf{R}^{-1} = \mathbf{R}_0^{-1} - \frac{\gamma^2}{1 + \gamma^2 \mathbf{u}^H \mathbf{R}_0^{-1} \mathbf{u}} \mathbf{R}_0^{-1} \mathbf{u}\mathbf{u}^H \mathbf{R}_0^{-1}. \tag{3}$$

This is called Woodbury's Identity.

6. Consider the patterned matrix

$$\mathbf{H}_t = \begin{bmatrix} \mathbf{H}_{t-1} \\ \mathbf{c}_t^H \end{bmatrix} \in \mathcal{C}^{t\times n}, t \geq n. \tag{4}$$

- Write the Grammian of this matrix as $\mathbf{G}_t = \mathbf{H}_t^H \mathbf{H}_t = \mathbf{G}_{t-1} + \mathbf{c}_t \mathbf{c}_t^H$. Use Woodbury's identity to write down a recursion for the inverse of the Grammian $\mathbf{G}_t^{-1}$.
- In the theory of least squares (to come), the inverse of the Grammian is an error covariance matrix $\mathbf{P}_t$. That is $\mathbf{P}_t = \mathbf{G}_t^{-1}$. It is a triviality to write the recursion for this error covariance matrix by changing the notation in your recursion for $\mathbf{G}_t^{-1}$. But just for drill, and to fix the idea, do so.

7. Begin with the following basis for $\mathcal{R}^3$: $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]$:

$$\mathbf{X} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix} \tag{5}$$

Use Gram-Schmidt to orthogonalize this basis. Call your orthogonal basis $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ and write it as $\mathbf{U} = \mathbf{X}\mathbf{R}$, with $\mathbf{R}$ an upper triangular matrix. Fill in the entries in $\mathbf{R}$.

8. Define the cyclic shift matrix $\mathbf{S} \in \mathcal{R}^{n\times n}$ and the DFT matrix $\mathbf{V} \in \mathcal{C}^{n\times n}$:

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \mathbf{V} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{j2\pi/n} & e^{j(2)2\pi/n} & \cdots & e^{j(n-1)2\pi/n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & e^{j(n-1)2\pi/n} & e^{j(n-1)(2)2\pi/n} & \cdots & e^{j(n-1)(n-1)2\pi/n} \end{bmatrix} \tag{6}$$

Show the DFT matrix $\mathbf{V}$ diagonalizes $\mathbf{S}$ as $\mathbf{S} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$, by showing

- $\mathbf{V}^H\mathbf{V} = \mathbf{I} = \mathbf{V}\mathbf{V}^H$,
- $\mathbf{V}$ contains the eigenvectors of $\mathbf{S}$; that is, $\mathbf{S}\mathbf{V} = \mathbf{V}\mathbf{\Lambda}$, where $\mathbf{\Lambda} = diag[1, W_n^*, W_n^{*2}, \cdots, W_n^{*(n-1)}]$, and $W_n = e^{j2\pi/n}$.

9. Show that every circulant matrix $\mathbf{C} \in \mathcal{C}^{n\times n}$ may be written in terms of the cyclic shift matrix $\mathbf{S}$ as

$$\mathbf{C} = \begin{bmatrix} c_0 & c_{n-1} & c_{n-2} & \cdots & c_1 \\ c_1 & c_0 & c_{n-1} & \cdots & c_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix} = c_0\mathbf{I} + c_1\mathbf{S} + \cdots + c_{n-1}\mathbf{S}^{n-1} \tag{7}$$

Use the result of the previous problem to show that every circulant matrix may be diagonalized by the DFT matrix. That is (these are all equivalent statements), $\mathbf{C}\mathbf{V} = \mathbf{V}\mathbf{D}$, or $\mathbf{C} = \mathbf{V}\mathbf{D}\mathbf{V}^H$, or $\mathbf{D} = \mathbf{V}^H\mathbf{C}\mathbf{V}$, where $\mathbf{D} = c_0\mathbf{I} + c_1\mathbf{\Lambda} + \cdots + c_{n-1}\mathbf{\Lambda}^{(n-1)}$. Interpret the diagonal elements of the diagonal matrix $\mathbf{D}$ as DFT coefficients (or "the DFT") of the first column of the circulant matrix $\mathbf{C}$.

10. Begin with discrete-time sequence $\{x[k]\}$, $x[k] \in \mathcal{C}$, satisfying the homogeneous difference equation

$$x[k] + a_1x[k-1] + \cdots + a_nx[k-n] = 0 \text{ for all } k \tag{8}$$

Our aim is to write this ordinary difference equation in its state-space form and then use what we know about diagonalization to expose the natural modes of this state equation.

- Define the state vector $\mathbf{x}[k] \in \mathcal{C}^n : \mathbf{x}[k] = [x[k-(n-1)], x[k-(n-2], \cdots x[k]]^T$. Show that the difference equation may be written $\mathbf{x}[k] = \mathbf{A}\mathbf{x}[k-1], x[k] = [0, 0, \cdots, 1]\mathbf{x}[k]$, where the state matrix $\mathbf{A}$ is the companion matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{bmatrix} \tag{9}$$

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- Show that the eigenvectors of the companion matrix are the columns of the Vandermonde matrix

$$\mathbf{V} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & \vdots & \vdots \\ z_1^{n-1} & z_2^{n-1} & \cdots & z_n^{n-1} \end{bmatrix} \tag{10}$$

  Find the complex mode parameters $\{z_1, z_2, \cdots, z_n\}$ from the coefficients $\{a_1, a_2, \cdots, a_n\}$. (They will be zeros of a particular polynomial. And we assume them to be distinct.)

- What are the eigenvalues of $\mathbf{A}$?
- What is the determinant of $\mathbf{V}$? What is determinant of $\mathbf{A}$? Does this make sense?

11. Work out the previous problem for the second-order ordinary difference equation $x[n] + 2\xi\omega_0 x[n-1] + \omega_0^2 x[n-2] = 0$.

12. Which of the following matrices is normal?

- unitary $\mathbf{U}$,
- Hermitian $\mathbf{A} = \mathbf{A}^H$,
- skew-symmetric $\mathbf{A} = -\mathbf{A}^H$,
- Toeplitz $\mathbf{A}$,
- circulant $\mathbf{A}$,
- companion $\mathbf{A}$,

- lower triangular Toeplitz $\mathbf{A}$
- Find a $2 \times 2$ matrix that is normal, but does not fall into any of these categories.

# 2 HW Set #2: due Thurs, Feb 11, 2010

1. Read Ch. 2 of text.
2. Read course notes
3. In class I claimed the vector-valued function

$$\mathbf{x}(t) = \exp\{\mathbf{A}t\}\mathbf{x}_0 + \int_0^t \exp\{\mathbf{A}(t-\tau)\}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{11}$$

   solves the vector-valued first-order ode

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}(t) \tag{12}$$

   Be mindful of Leibnitz's rule for differentiation when proving this claim. When $\mathbf{u}(t) = 0$ for all $t$, then the first term on the RHS is the zero-input natural response, or homogeneous response. When $\mathbf{x}_0 = \mathbf{0}$, the second-term on the RHS gives the zero-initial condition forced response, or particular response. Show that there is a choice of $\mathbf{u}(t)$ that makes this particular response equal to the homogeneous response.

4. How mysterious is the claim $(s\mathbf{I} - \mathbf{A})^{-1} \longleftrightarrow \exp\{\mathbf{A}t\}$, where the notation indicates that the LHS and RHS of the double arrow are a Laplace transform pair? Not very, as your use of the factorization $\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}$, with $\mathbf{S}$ nonsingular and $\mathbf{\Lambda} = diag[\lambda_1, \lambda_2, \cdots, \lambda_n]$, together with the rudimentary transform pair $\frac{1}{s-\lambda} \longleftrightarrow \exp\{\lambda t\}$, will show.

5. Here is a powerful identity for the inverse of an invertible patterned matrix (see App A in KSH for this identity and its twin, eqns (A.1.7) and (A.1.8)):

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{I} & -\mathbf{A}^{-1}\mathbf{B} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Delta}_A^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{C}\mathbf{A}^{-1} & \mathbf{I} \end{bmatrix} \tag{13}$$

where $\mathbf{\Delta}_A = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ is called the Schur complement of $\mathbf{A}$. In the special case where this patterned matrix is a Grammian, which is to say

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{H}^H \\ \mathbf{S}^H \end{bmatrix} \begin{bmatrix} \mathbf{H} & \mathbf{S} \end{bmatrix} = \begin{bmatrix} \mathbf{H}^H\mathbf{H} & \mathbf{H}^H\mathbf{S} \\ \mathbf{S}^H\mathbf{H} & \mathbf{S}^H\mathbf{S} \end{bmatrix} \tag{14}$$

then this formula may be written

$$\begin{bmatrix} \mathbf{H}^H\mathbf{H} & \mathbf{H}^H\mathbf{S} \\ \mathbf{S}^H\mathbf{H} & \mathbf{S}^H\mathbf{S} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{I} & -\mathbf{H}^{\#}\mathbf{S} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} (\mathbf{H}^H\mathbf{H})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Delta}_A^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{S}^H\mathbf{H}^{\#H} & \mathbf{I} \end{bmatrix} \tag{15}$$

where $\mathbf{\Delta}_A = \mathbf{S}^H\mathbf{S} - \mathbf{S}^H\mathbf{H}(\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H\mathbf{S} = \mathbf{S}^H(\mathbf{I} - \mathbf{P}_H)\mathbf{S}$ and $\mathbf{H}^{\#} = (\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H$. Now specialize this to the case where $\mathbf{H} = \mathbf{H}_n$ and $\mathbf{S} = \mathbf{h}_{n+1}$ to re-derive the order recursive least squares solution from scratch.

6. Problems 2.1, 2.2, 2.9 in KSH.

7. (Extra credit for the truly inspired) 2.17

# 3 HW Set #3: due Tues, Mar 2, 2010

1. Conclude your reading of Ch. 3 of KSH text. Read Ch 4.

2. In class I derived the SVD for a system matrix $\mathbf{H} \in \mathcal{C}^{n\times m}$ with $m \leq n$, the overdetermined case. Re-do this derivation for the case $m \geq n$, the underdetermined case. Be explicit about the dimensions of $\mathbf{U}$, $\mathbf{\Lambda}$, and $\mathbf{V}$ in the SVD $\mathbf{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^H$.

3. For arbitrarily-shaped $\mathbf{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^H$, determine the pseudo-inverse of $\mathbf{H}$, in the SVD coordinates.

4. Using Wikipedia or some other source, determine the best matrix of rank $m-1$ in the overdetermined case, and of rank $n-1$ in the underdetermined case for approximating $\mathbf{H}$. Use this result to sharpen the argument given in class for the finding the TLS solution to a system $\mathbf{y} = \mathbf{H}\mathbf{x}$.

5. Begin with the measurement model $y[n] = x[n] + e[n]$, with $x[n] = na + b$. Choose your own values of $a$ and $b$ and generate a sequence of iid rvs $e[0], e[1], \cdots, e[N-1]$. Choose your own variance and your own measurement dimension, $N$. Use these to generate the sequence of measurements $y[0], y[1], \cdots, y[N-1]$. Construct the linear model $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{e}$. Find the LS and TLS solutions for $\mathbf{x}$. Re-do this experiment many times, changing $\mathbf{e}$, but not $\mathbf{x}$, and plot the solutions on a two-dimensional plane, illustrating the error concentration ellipse for the LS solution. Comment on your findings. You might add a slight perturbation of the form $p[n] = \epsilon n^2$ to $y[n]$, in which case the correct model $\mathbf{H}$ would have a third column. Solve for the new LS solution using the three column $\mathbf{H}$ and compare it with the TLS solution that ignores the third column but adjusts the 2-column $\mathbf{H}$.

6. Begin with two adjacent DFT modes $e^{j2\pi km/n}$, $k = 0, 1, \cdots, n-1, n, n+1, \cdots$ and $exp^{j2\pi k(m+1)/n}$, $k = 0, 1, \cdots, n-1, n, n+1, \cdots$. The value of $m$ can be anything between 0 and $n-1$. Note that the first mode is periodic, with period $n/m$ and frequency $m/n$, and the second is periodic, with period $n/(m+1)$ and frequency $(m+1)/n$. Call $\psi_m = [1, e^{j2\pi m/n}, \cdots, e^{j2\pi m(n-1)/n}, e^{j2\pi mn/n}, e^{j2\pi m(n+1)/n}, \cdots, e^{j2\pi m(N-1)/n}] \in \mathcal{C}^N$. Notice this mode vector has $2m$ periods for $N = 2n-1$, etc.

   Give yourself an arbitrary linear combination of these modes and add independent noise terms, along the same lines as the previous problem, with variance of your choosing, to produce the sequence of measurements $\{y[0], y[1], \cdots, y[N-1], y[N], \cdots\}$.

   Program RLS recursions for estimates of the linear coefficients and for their error covariance. Update your estimates of $\mathbf{x}$ and overplot the sequence of telescoping error concentration ellipses. Comment on your findings.

7. Derive the identities in eqns 5.39 and 5.40 in the hand-out on Estimation

8. Prove or disprove that the LMMSE estimator of the random vector $\mathbf{x}$ from the random vector $\mathbf{y}$, what might be called the fixed interval sequence estimator, is also the fixed interval smoothed estimator of any element of $\mathbf{x}$ from $\mathbf{y}$.

# 4 HW Set #4: due Thurs, Mar 11, 2010

1. Complete reading of Ch 7.
2. Work Problems 7.4, 7.7, 7.9, 7.19, and 7.21 in KSH.
3. Give an interpretation of the bottom branch of the equalizer derived in class.