Lecture 45
Bode Plots of Transfer Functions

A. Goals : Design Oriented Analysis
   1. **Design Oriented Analysis Overview**
   2. **Inspection of T(s) in normalized form**
   3. Five points in design oriented analysis

B. Review of Bode Plots
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   2. Log - Log Slopes vs f
   3. Normalized Form 1 + s/w
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         flyback / Buck - Boost
      c. Inverted Forms 1 + w/s
         1. Pole
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   5. Series Resonant R-L-C Circuit

Lecture 45
Bode Plots of Transfer Functions

A. Goal : Design Oriented Analysis

1. Design Oriented Analysis

How to approach a real (and hence, complicated) system

Problems:

- Complicated derivations
- Long equations
- Algebra mistakes

Design objectives:

- Obtain physical insight which leads engineer to synthesis of a good design
- Obtain simple equations that can be inverted, so that element values can be chosen to obtain desired behavior. Equations that cannot be inverted are useless for design!

Design-oriented analysis is a structured approach to analysis, which attempts to avoid the above problems

Our aim in design is to always keep the whole system view at hand so that we can alter some parts of the system and immediately see the results of the change in other parts of the system. The keystone will be the Bode plots of each part of the system that comprise the open loop response. From the open loop response Bode plots various design changes may be explored. We seek simple intuitive understanding of a transfer function via Bode Plots vs f

2. Inspection of T(s) in normalized form to:

a. Guess / estimate pole and zero location

b. Determine asymptotic behaviors

3. The five points in design oriented analysis we will emphasize are given below:

- Writing transfer functions in normalized form, to directly expose salient features
- Obtaining simple analytical expressions for asymptotes, corner frequencies, and other salient features, allows element values to be selected such that a given desired behavior is obtained
- Use of inverted poles and zeroes, to refer transfer function gains to the most important asymptote
- Analytical approximation of roots of high-order polynomials
- Graphical construction of Bode plots of transfer functions and polynomials, to
  - avoid algebra mistakes
  - approximate transfer functions
  - obtain insight into origins of salient features

Before we can do this we need to review how to construct and make Bode plots as they are the key to our design oriented approach. Asymptotic approximations to the full Bode plots are key to rapid design and analysis. Depending on whether or not we know the high frequency or low frequency behavior of the transfer function we may choose either normal pole/zero from or inverted pole/zero forms as we will discuss below. Each has it's advocates and we need to be familiar with both. What we don't want to do is to mix the forms in one T(s) expression, if at all possible.

For those students to whom Bode plots are hazy or even new we will give a brief review.

B. Bode Plot Review

Usually in a transfer function V_o/V_{in} has a value at each applied frequency. We use db for the transfer function magnitudes, as it will allow for easy asymptotic approximations to the curves.

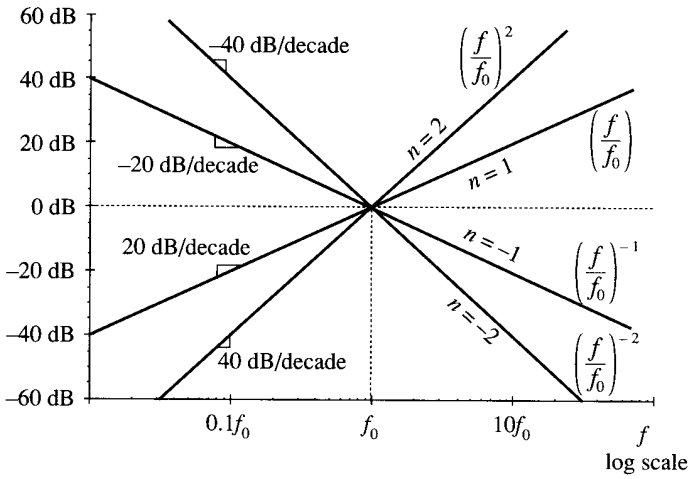
1. db values º 20 $\log_{10}$ G

To employ a db scale we always need a BASE value. For example 50kΩ on a base of 10 kΩ, is considered as 14 db. 20 db Ω is 20 db greater than a base of 10kΩ, or 100kΩ 60 db μA is a current 60 db greater than a base of 1mA or one Ampere. Do you get the new way of thinking in db?

Actual Magnitude	Magnitude in dB
½	-6 dB
1	0 dB
2	6 dB
5 = 10/2	20 dB - 6 dB = 14 dB
10	20dB
1000 = 10^3	3 * 20dB = 60 dB

What does -3db mean? What does +3 db mean? Can you tell the value of the T(s) ratio that gives 3db? Three db will be a useful rule of thumb to place the crude amplitude T(s) plots closer to the real ones as we will see later.

2. Log - log Plots and G vs f/f_o slopes in db units



f_o corresponds to a characteristic pole or zero of the transfer function

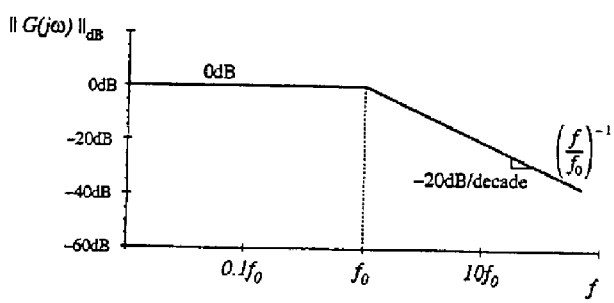
3. Normalized Forms for Transfer Functions

a. Single Isolated Pole $G(s) = 1/(1+s/w_p)$

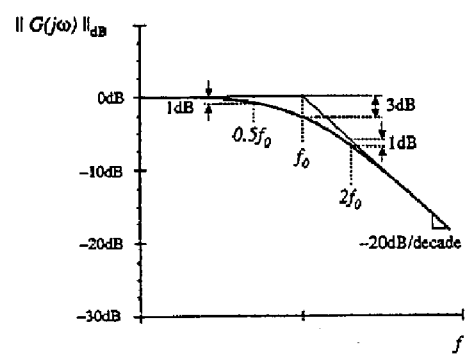
$w_p \equiv$ Pole break frequency in radians/sec.

Bode Plots for Single Pole

Crude Amplitude Plot



Accurate Amplitude



f_o is the characteristic frequency

When f is f_o /2 or $2f_o$ we go off the crude plot by only 1db to achieve the actual values. Likewise @f_o we are **off the crude plot by 3db** as shown above.

Evaluate exact magnitude:

at $f = f_0$:

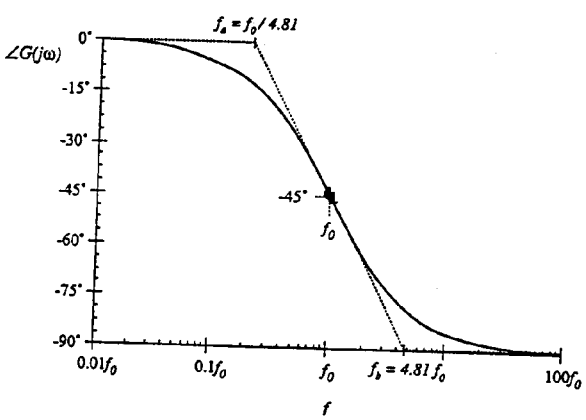
$$\| G(j\omega_0) \| = \frac{1}{\sqrt{1 + \left(\frac{\omega_0}{\omega_0}\right)^2}} = \frac{1}{\sqrt{2}}$$

$$\| G(j\omega_0) \|_{dB} = -20 \log_{10} \left(\sqrt{1 + \left(\frac{\omega_0}{\omega_0}\right)^2} \right) \approx -3 \text{ dB}$$

at $f = 0.5 f_0$ *and* $2 f_0$:

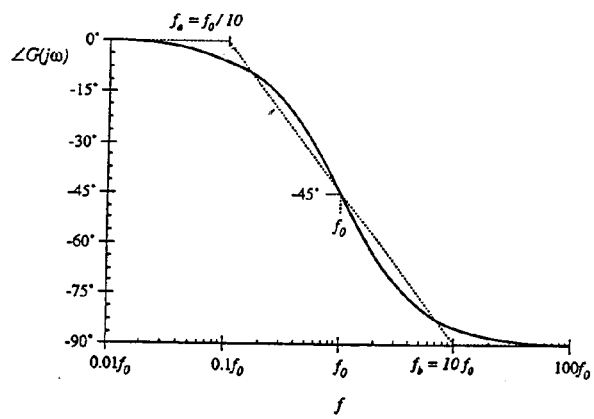
Similar arguments show that the exact curve lies 1dB below the asymptotes.

Crude Phase Plot



Real phase crosses asymptote only once

Accurate Phase Plot



Real phase crosses asymptote 3 times

Low frequency: 0°

High frequency: –90°

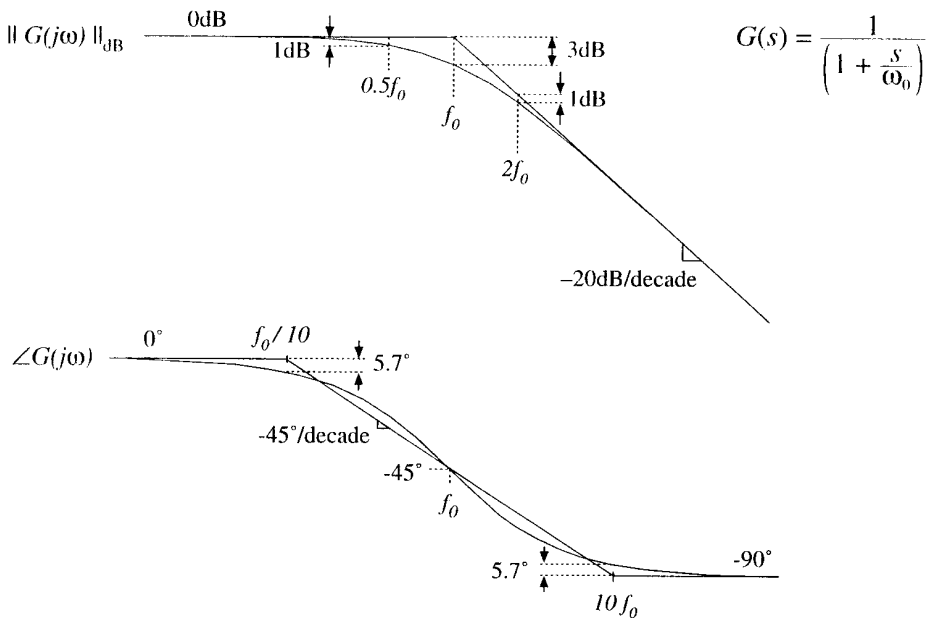
Low- and high-frequency asymptotes do not intersect

Hence, need a midfrequency asymptote

Try a midfrequency asymptote having slope identical to actual slope at the corner frequency f_0. One can show that the asymptotes then intersect at the break frequencies

$$f_a = f_0 \, e^{-\pi/2} \approx f_0 / 4.81$$
$$f_b = f_0 \, e^{\pi/2} \approx 4.81 \, f_0$$

These break frequencies above and below f_0 will be useful.



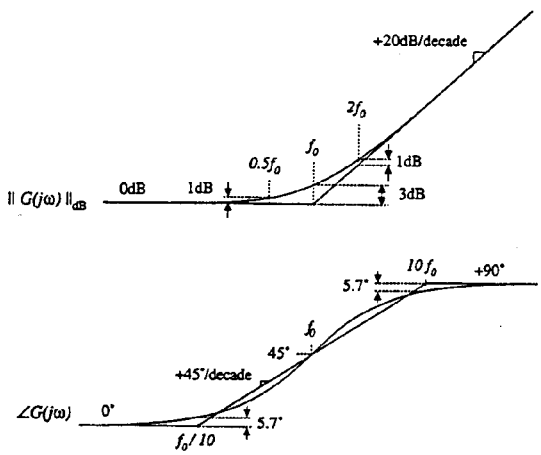
b. Isolated Right Half Plane Zero

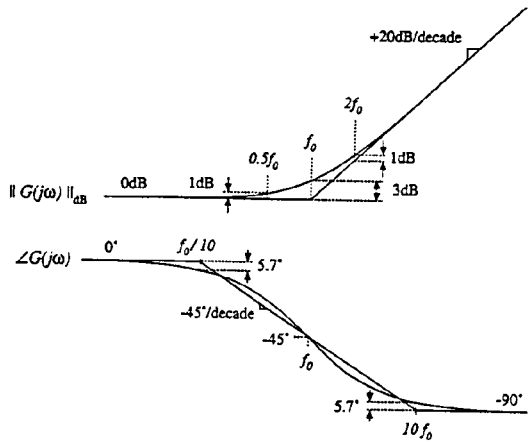
$G(s) = 1 - s/w_z$

Bode Plot Right Half Plane Zero vs. Left Half Plane Zero

$G(s) = 1 + s/w_z$	$G(s) = 1 - s/w_z$
Very usual in T(s) for many converters	Flyback / Buck-Boost T(s) have this unique right half zero feature

On page 8 we compare and contrast the right and left plane zeros behavior versus applied frequency. While the amplitude plots cannot tell the difference between the two, the phase plots certainly do. **What is the difference**?





Gain up vs. f
Phase up away from -180°

Gain up vs. f
Phase down toward -180°

These asymptotic plots of phase for left and right plane zeroes tell us the whole story.

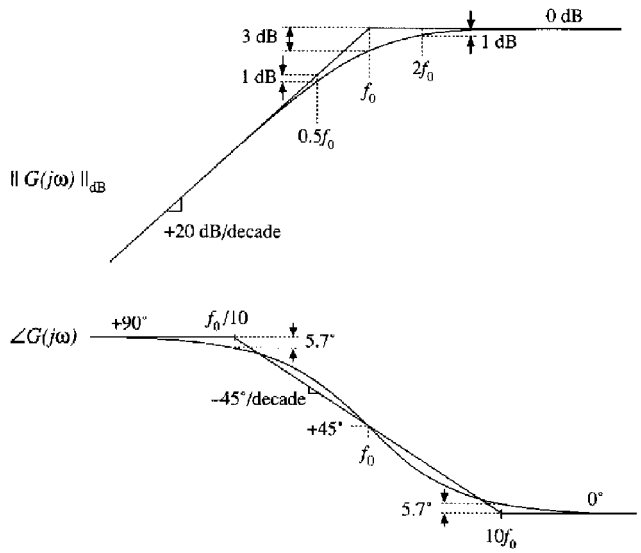
c. Inverted G(s) forms Have Unique Bode Plots

When we focus on high f response of T(s) or G(s) we sometimes utilize w/s forms for the poles or zeros.

1. **Inverted pole**

$$G(s) - \frac{1}{1 + w_p / s}$$

Bode plot of inverted pole has some unique properties:
Low f amplitude vs w decreases to -∞ at low f, $f < f_o$, unlike RHP zero where low f response saturates at some f, $f < f_o$.



For an inverted pole we the gain up and the phase down vs. f for f $\leq$ f_o

2. Inverted Zero

Normalized form, inverted zero:

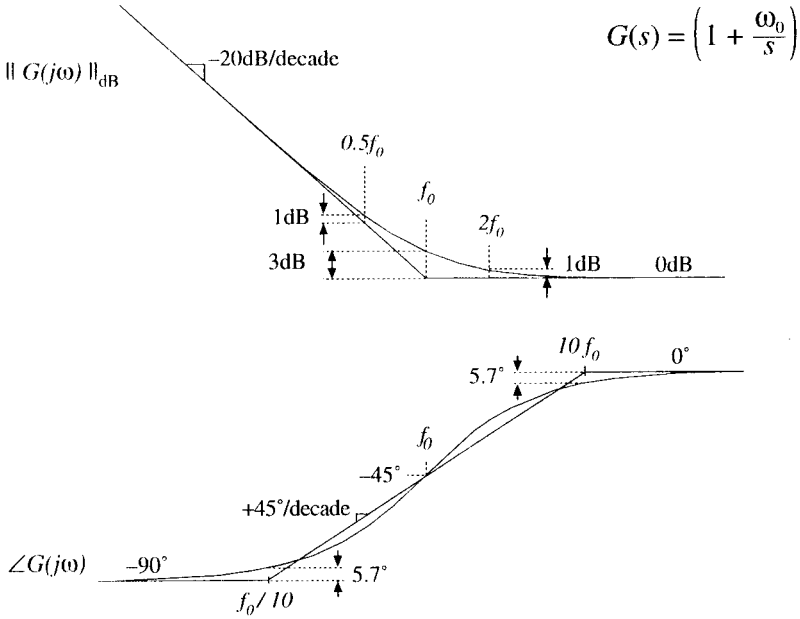
$$G(s) = \left(1 + \frac{\omega_0}{s}\right)$$

An algebraically equivalent form:

$$G(s) = \frac{\left(1 + \frac{s}{\omega_0}\right)}{\left(\frac{s}{\omega_0}\right)}$$

Again, the inverted-zero format emphasizes the high-frequency gain.

On page 10 we plot the frequency response of an inverted zero, with all the details of an asymptotic approach to both gain and phase plots.

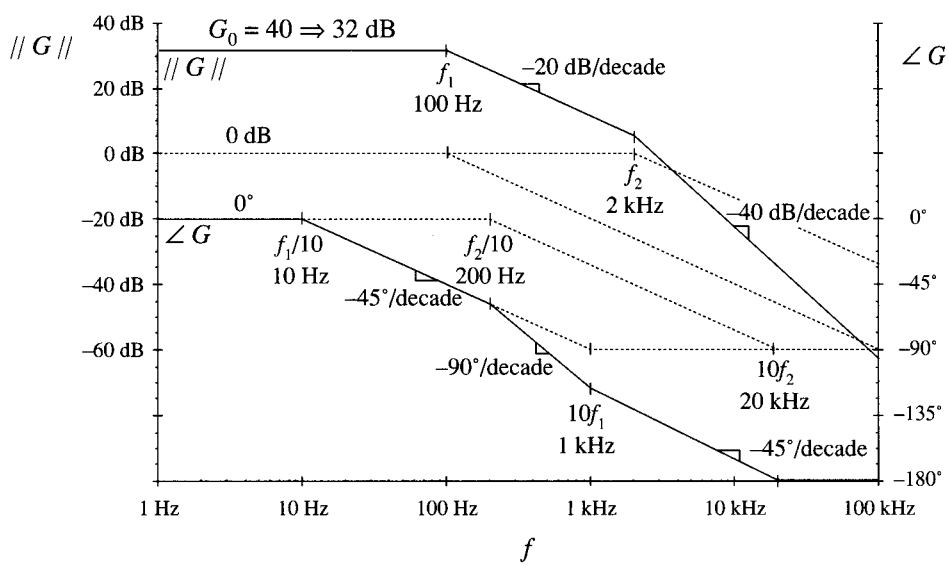


d. Various complex T(s) Plots versus Frequency

1. G(s) = 32 db or 40 on a linear scale

$$(1+\frac{s}{2p100\text{Hz}})(1+\frac{s}{2p2\text{kHz}})$$

We plot this below by asymptotes. For $1+s/w_o$, w_o is the break point.



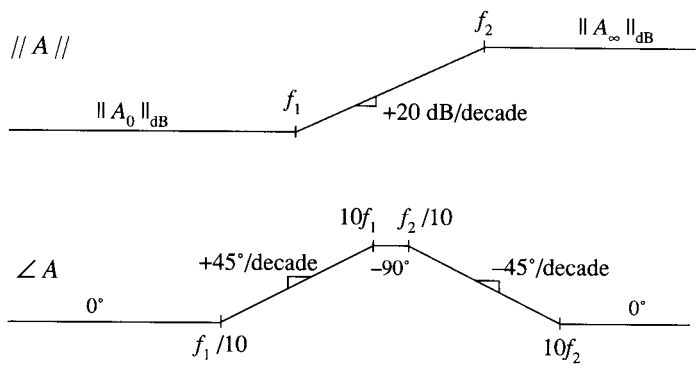
Note phase break frequencies are 10 times off from

amplitude break frequencies.

2. Well separated pole and zero means by a <u>factor > 100</u>

$$A(s) = A_o \frac{1 + s/w_z}{1 + s/w_p}$$ $w_p > 100\, w_z$ $f_z = 100 f_1$(below)

When $f \to 0$ we obtain $A(DC) = A_o$. Why is $A(f = \infty)$ higher in value? By exactly how much?



limit $s \to 0$ A_0 1/1 limit $s \to \infty$ $\frac{A_o s/w_z}{s/w_p} = A_o \frac{w_p}{w_z} = A_o \frac{f_z}{f_1}$

Phase starts early at $f_z/10$ with 45°/decade slope.
Phase ends late at f_p*10 with 45°/decade slope.

For $f > f_2$

$$\left\| A_0 \frac{\left(\cancel{1} + \frac{s}{\omega_1}\right)}{\left(\cancel{1} + \frac{s}{\omega_2}\right)} \right\|_{s=j\omega} = A_0 \frac{\left\| \frac{s}{\omega_1} \right\|_{s=j\omega}}{\left\| \frac{s}{\omega_2} \right\|_{s=j\omega}} = A_0 \frac{\omega_2}{\omega_1} = A_0 \frac{f_2}{f_1}$$

So the high-frequency asymptote is

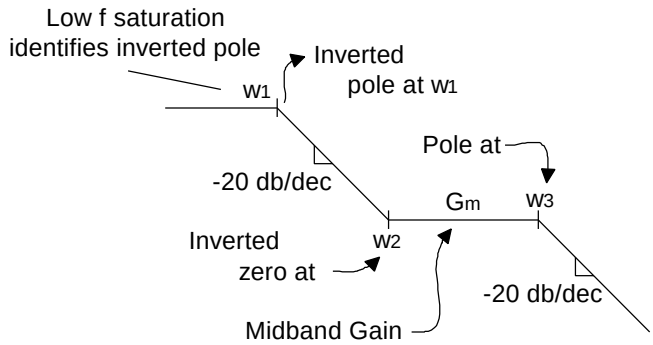
$$A_\infty = A_0 \frac{f_2}{f_1}$$

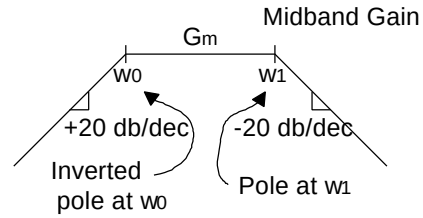
Another way to express $A(s)$: use inverted poles and zeroes, and express $A(s)$ directly in terms of A_∞

$$A(s) = A_\infty \frac{\left(1 + \frac{\omega_1}{s}\right)}{\left(1 + \frac{\omega_2}{s}\right)}$$

3. Given log - log plots Find T(s)

Two possible answers occur depending on the form chosen for T(s): normal pole and zero form or inverted poles and zeros. Consider the two T(f) amplitude plots below





Consider the inverted pole/zero form first and then normal form.

$$\Rightarrow G(s) = \frac{G_m(1+\frac{w_2}{s})}{(1+\frac{w_1}{s})(1+\frac{s}{w_3})} \qquad \Rightarrow G(s) = \frac{G_m}{(1+\frac{w_o}{s})(1+\frac{s}{w_1})}$$

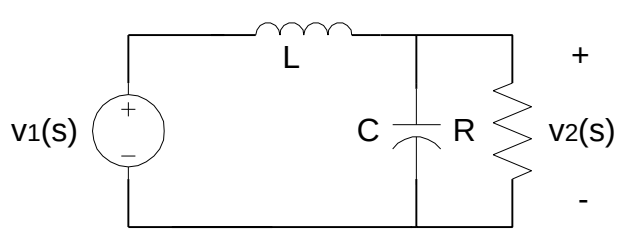
$$\text{OR: } \frac{w_3 G_m (s+w_2)}{(s+w_1)(s+w_3)} \qquad \text{OR: } \frac{w_1 G_m s}{(s+w_o)(s+w_1)}$$

4. Low-Pass Filter Resonant Circuit

These Bode plots are unique and different from all prior ones. The circuit diagram is at the bottom of the page:

$$\frac{v_2(s)}{v_1(s)} = \frac{1}{1 + \frac{s}{Qw_o} + (\frac{s}{w_o})^2}$$

Q ≡ Q factor of the resonant circuit. For Q ≤ 1/2 roots are real while for Q ≥ 1/2 roots are complex. Q is in linear units.



$w_o \equiv$ Corner radian Frequency = $\frac{1}{\sqrt{LC}}$ or f = $\frac{1}{2p\sqrt{LC}}$

$$G(s) = \frac{1}{1 + 2\zeta \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2} \qquad \text{or} \qquad G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

- When the coefficients of s are real and positive, then the parameters ζ, ω_0, and Q are also real and positive
- The parameters ζ, ω_0, and Q are found by equating the coefficients of s
- The parameter ω_0 is the angular corner frequency, and we can define $f_0 = \omega_0/2\pi$
- The parameter ζ is called the *damping factor.* ζ controls the shape of the exact curve in the vicinity of $f = f_0$. The roots are complex when $\zeta < 1$.
- In the alternative form, the parameter Q is called the *quality factor.* Q also controls the shape of the exact curve in the vicinity of $f = f_0$. The roots are complex when $Q > 0.5$.

In a second-order system, ζ and Q are related according to

$$Q = \frac{1}{2\zeta}$$

Q is a measure of the dissipation in the system. A more general definition of Q, for sinusoidal excitation of a passive element or system is

$$Q = 2\pi \frac{\text{(peak stored energy)}}{\text{(energy dissipated per cycle)}}$$

For a second-order passive system, the two equations above are equivalent. We will see that Q has a simple interpretation in the Bode diagrams of second-order transfer functions.

We will find below that Q(low pass filter) $\equiv R\sqrt{\frac{C}{L}}$.

This may seem wrong as higher R values mean higher Q for the low-pass filter. Q for the series R-L-C circuit, as distinct from the low-pass filter, differs in it's R dependence as shown below in section 5. In the low-pass filter case $L \to 0$, $Q \to \infty$ reallymeans we have only a RC filter. The

transfer function of the low-pass filter is easily found:

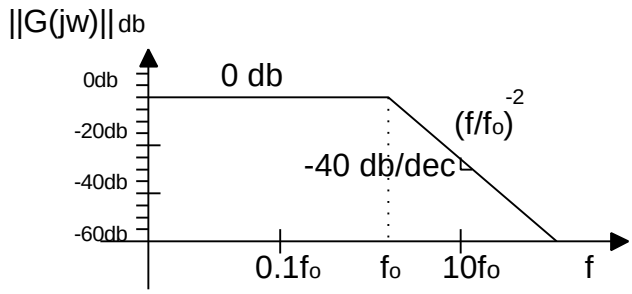
$$\left|\frac{V_2(s)}{V_1(s)}\right| = \frac{1}{\sqrt{[1-(\frac{w}{w_o})^2]^2 + \frac{(w/w_o)^2}{Q^2}}}$$

For w << w_o: $\frac{V_2}{V_1}=1$

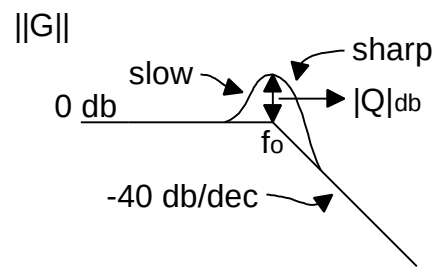
For w >> w_o: $\frac{V_2}{V_1} \approx \left(\frac{w}{w_o}\right)^{-2}$

We plot this response including the resonant bump below.
The resonant bump near $f=f_0$ is asymmetric in shape.

Crude Plot



Accurate Plot with "Q Bump"



$$\frac{V_2(s)}{V_1(s)} = \frac{1}{1+sR_LC+s^2LC} = \frac{1}{1+\frac{s}{Qw_o}+\frac{s^2}{w_o^2}}$$

$$= \frac{1}{(1+s/w_1)(1+s/w_2)}$$

For the resonant bump to occur w_1, w_0 are complex, not real!
We do not have a simple Bode plot with only straight line asymptotes, as this is a resonant circuit with $w_o = \frac{1}{\sqrt{LC}}$!

Rather we have to learn the proper way to treat resonant circuits which involves the linear asymptotes at frequencies far from f_0 and a resonant bump near $f=f_0$.

$$\frac{1}{Qw_o} = R_l C \quad Q = \frac{1}{R_l C w_o} = \frac{\sqrt{\frac{L}{C}}}{R}$$ In series R-L-C R↓ Q↑

$$\downarrow$$

$$\frac{1}{\sqrt{LC}}$$

$$\angle \frac{V_2(s)}{V_1(s)} = \tan^{-1} \left| \frac{\frac{1}{Q}\frac{w}{w_o}}{1-(\frac{w}{w_o})^2} \right| = \phi(f)$$

In summary for the low-pass filter:

Two-pole low-pass filter example: we found that

$$G(s) = \frac{v_2(s)}{v_1(s)} = \frac{1}{1 + s\frac{L}{R} + s^2 LC}$$

Equate coefficients of like powers of s with the standard form

$$G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

Result:

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

$$Q = R\sqrt{\frac{C}{L}}$$

We now need to see some trends of T(s) phase plots for the low-pass filter near $f=f_0$ as well as far from resonance. The behavior is very different from the Bode plots we covered before that did not have a resonance.

Trend of phase shift of low-pass T(s), f, vs. f with the varying Q factor as chosen by circuit components

Total change of 180° occurs over two decades of frequency around f_o for low Q and even more rapidly for high Q. The location of the TWO phase angle break frequencies is the key.

Rule of thumb: $f_{lower} = 10^{-1/2Q} f_o$ when phase shift begins

$f_{upper} = 10^{+1/2Q} f_o$ when phase shift ends

This has a special pair of break frequencies for the case

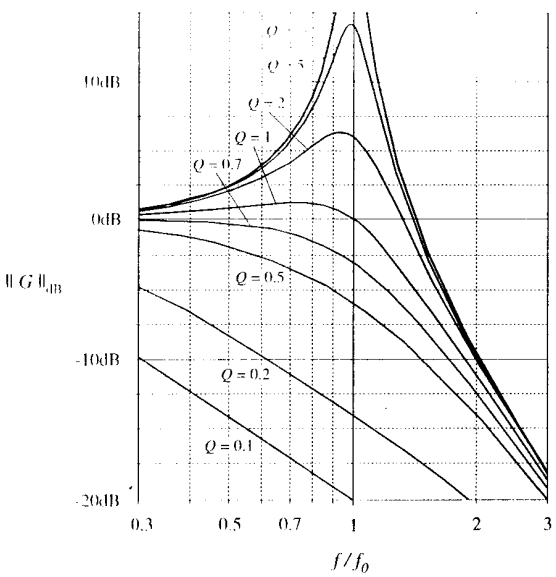
For Q = 1/2 (minimum value) or beginning of low Q case

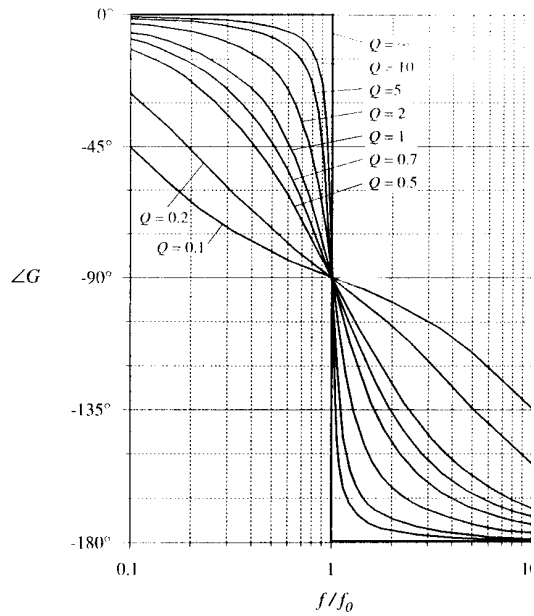
$f_L = f_o 10$ and $f_u = 10 f_u$

Phase slope is 90°/ decade 180° over 2 decades.

For high Q conditions in the circuit values the break frequencies f_{lower} and f_{higher} are closer together and for Q = ∞, $f_L = f_H = f_o$.

Below we give the full-blown amplitude (with Q peaking) and phase plots for T(s) for the low-pass filter. On page 17 we give the phase plots only from Erickson which emphasize the way to spot high Q via the rate of change of phase.





The "Q peaking" in the amplitude plots can be estimated by a set of rules we will develop later, so that we need not have to plot out the curve in detail to see the major features of "Q peaking". This will be done in lectures 46-47.

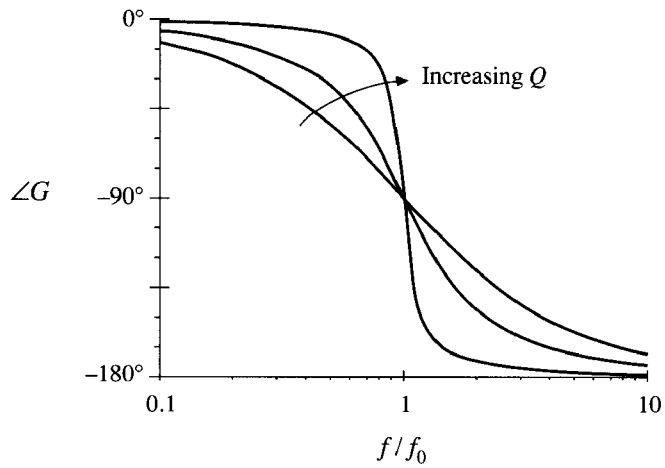


Fig. 8.19 Phase plot, second-order poles. Increasing Q causes a sharper phase change.

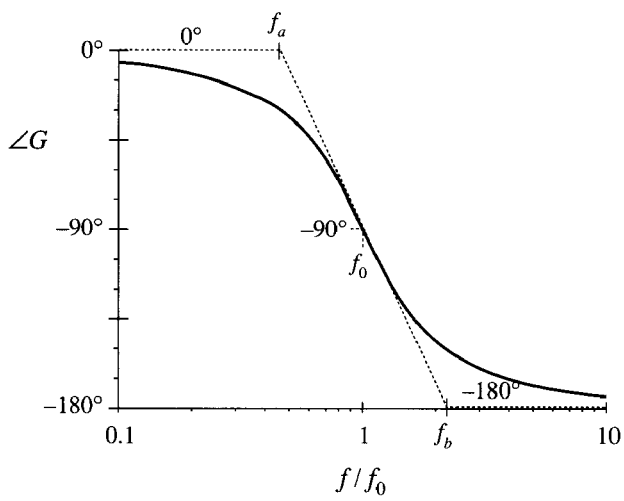
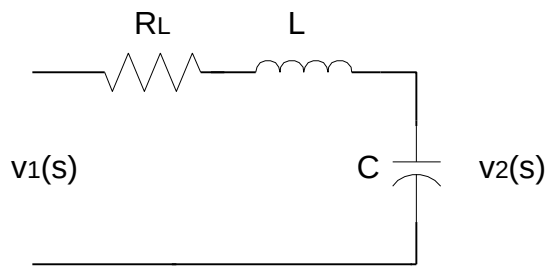


Fig. 8.20 One choice for the midfrequency phase asymptote of the two-pole response, which correctly predict the actual slope at $f = f_0$.

For a second look at resonant circuits and their T(s) behavior via Bode plots we will leave the low-pass filter and look at the series resonance circuit. It will reveal a different Q behavior, different phase behavior and unique dynamical behavior. We do this because in resonant converters this series R-L-C circuit is as crucial to operation as the low-pass filter is to switch mode buck, boost and buck-boost. It also shows us that Q is not a fixed concept the same for all resonant circuits.

5. Series R-L-C Resonant circuit



$V_1(s) = I(s)[R_L + sL + 1/sC]$

The characteristic equation is:

$s^2 + Rs/L + 1/LC = 0$

Solutions for the roots are:

$$s = \frac{\frac{-R}{L} \pm \sqrt{\frac{R^2}{L} - \frac{4}{LC}}}{2}$$

$\frac{R}{2L}$ is termed the damping parameter ζ as given on page 13 of this lecture. $w_o = \frac{1}{\sqrt{LC}}$ is the radian resonance frequency.

$s = -z \pm \sqrt{z^2 - w_o^2}$. We have several ways to solve **for the natural time response to a delta function inpu**t:

$z^2 - w_o^2 > 0$ We have exponential decaying solutions .

$z^2 - w_o^2 < 0$ We have decaying sinusoidal solutions .

$z = 0$ The step response solutions are pure sinusoids.

In the series resonance circuit the characteristic impedance of an L-C pair is $Z_o = \sqrt{\frac{L}{C}}$. also define the quality factor Q as:

$$Q = \frac{Z_o}{R}$$

With this set of definitions we find the characteristic equation becomes: $s^2 + \frac{w_o}{Q}s + w_o^2 = 0$ and the damping factor

Is $z = \frac{w_o}{2Q}$.

Hence solutions are: $s = z\left(-1 \pm \sqrt{1-4Q^2}\right)$

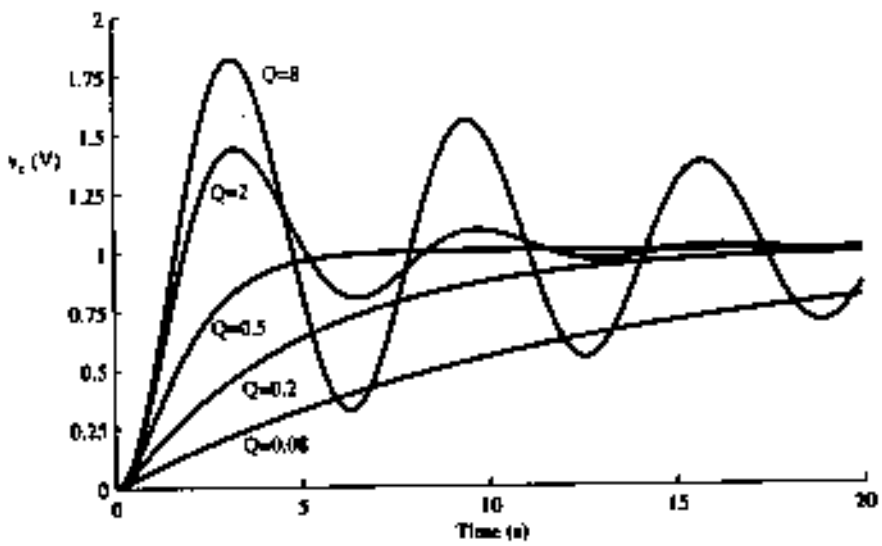
We restate the division between different dynamical solutions depending on the chosen circuit "Q". The case Q = ½ is pivotal:

Q < ½ two real roots damped exponential response
Q = ½ two repeated real roots critically damped
Q > ½ two complex roots decaying oscillatory response

We find for a step voltage input the capacitor voltage is:

$$V_c = V_{in} - V_{in}e^{-\zeta t}\left(\frac{z}{w}\text{Sin } wt + \text{Cos } wt\right)$$

This is plotted below

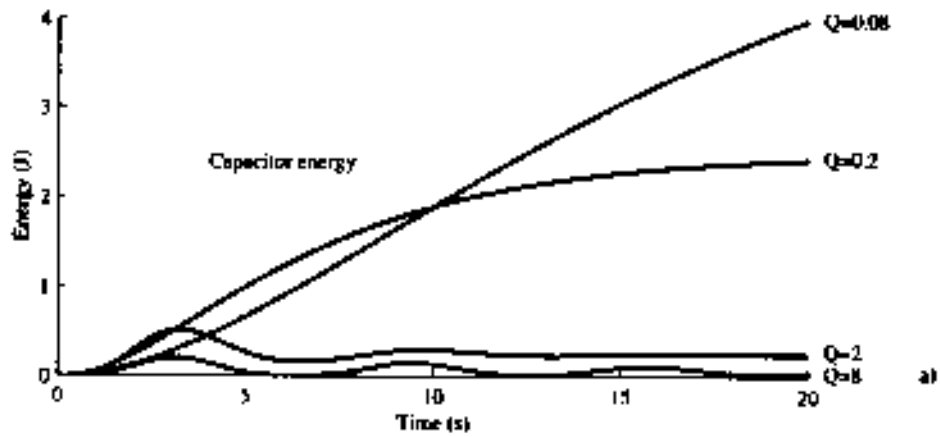


For long times the energy stored in the capacitor goes to ½ CV_{IN}^2. The inductor current has a different time response.

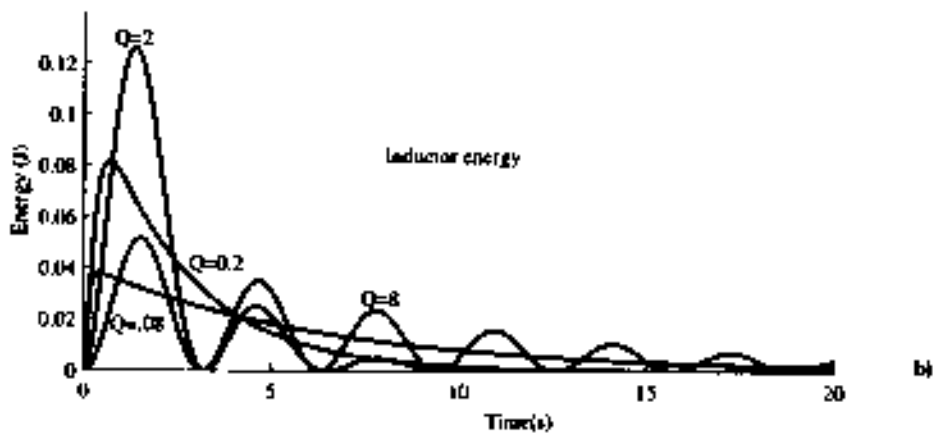
$$I_L = \zeta CV_{in}e^{-\zeta t}\left(\frac{z}{w}\text{Sin } wt + \text{Cos } wt\right) - wCV_{in}e^{-\zeta t}\left(\frac{z}{w}\text{Sin } wt - \text{Cos } wt\right)$$

As time goes on the inductor current goes to zero and the energy stored in the inductor goes to zero. Inbetween we have a roller coaster of energy exchange.

The changing transient energy storage between the capacitor and the inductor versus time in the series R-L-C circuit for a step voltage input is shown below.



Note how inductor energy decays to zero with time.

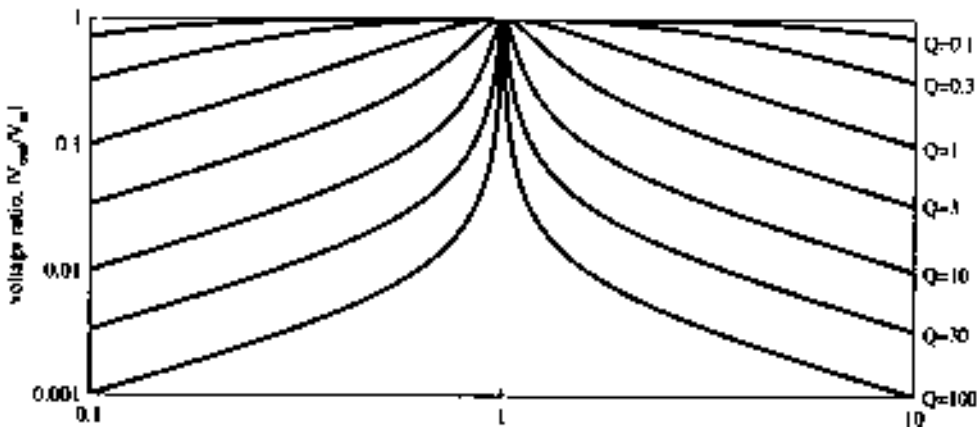


Ultimately, $\mathcal{E} = ½ CV_{in}^2$ but during the transient response the energy in the inductor has wide swings depending on the Q chosen for the circuit before settling down to zero.

If we analyze the response of a series R-L-C to a periodic wave (square, triangle, etc.) and plot V_R/V_{in} at each harmonic we find the general T(s) relationship for the voltage across the resistor, $V_R(f)$, as shown on the top of page 21:

$$\frac{V_R}{V_{in}} = \frac{R/Z_o}{jn\frac{w}{w_o} + \frac{j}{\frac{nw}{w_o}} + R/Z_c}$$

Considering only the fundamental at $f_r = \frac{1}{\sqrt{LC}}$ **this T(s) for the resistor voltage** varies as shown below in what many consider the more classic resonance shape:



For Q > 100 the response is very selective. High Q means high selectivity about f_r. This seems too good to be true as there first appears no price is paid for high Q. This is an illusion we will remove next as there is a price to be paid. WHAT IS IT?

Consider the capacitor voltage T(s) at each harmonic frequency, $f_H = n\, f_{IN}$, even the case n=1

$$\frac{V_C}{V_{in}} = \frac{1}{1 - n^2\frac{w^2}{{w_o}^2} + j\frac{nw}{w_o Q}}$$

When the input frequency component $nw = w_o$

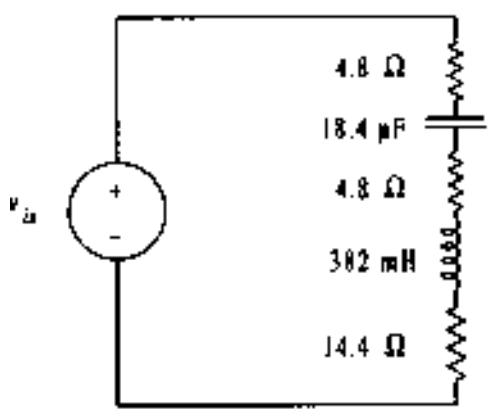
$$\frac{V_C}{V_{in}}(nw = w_o) = QV_{in}$$

For V_{in} of 120V rms V_c = 12,000V rms for a Q of 100.

Similarly for V_L but 180° out of phase with V_c we find another 12,000 V level. **Can L and C withstand voltages 100*V_{in}?**

Consider a resonant filter at 60 Hz for a 120V, 1KW load. This implies we have a particular value load resistor. $R_L = (120)^2/P = 14.4\Omega$ load.

To achieve a given Q for the circuit the individual components must have Q's well in excess of the target circuit Q. For a filter with a desired Q = 10 at f = 60 Hz = $\frac{1}{2p\sqrt{LC}}$ we need L and C with 3 times higher Q levels. For the chosen series R-L-C with Q=10 we can then find the required characteristic impedance from the relation Q = Z_o/R_L or 144 = Z_o = $\sqrt{\frac{L}{C}}$. We then find L = 382mH and C = 18.4 μF as one possible solution set. Assuming Q(L) = 30 implies that associated with the inductor is an equivalent series resistance ,R_L = 4.8Ω. For Q(C) = 30 means ESR R_c = 4.8Ω. The total R-L-C circuit with all associated equivalent series resistance's and the load resistance of 14.4 Ohms is shown below:



Note that the circuit Q is = Z_o/R(total) = 144/24 = 6 way below expected values from L and C components each with Q = 30. To do better higher Q components are needed.