Lecture 43
State Variable Approach to AC Converter Models

A. State Space Averaging

1. General Concepts
   \( \bar{x} \): states, \( v \) or \( u \): independent inputs
   a. Energy variables: \( \bar{x} \)
   b. Independent inputs: \( \bar{v} \) or \( u \)
   c. State equations
      \[
      K\bar{x} = A\bar{x} + B\bar{v} \\
      \bar{Y} = C\bar{x} + E\bar{v}
      \]

2. Illustrative Circuit Example

3. Methodology in state space for \(<\>_{Ts}\) averaging
   a. \( \bar{X}_1 \) for first interval \(dT_s\)
      \( \bar{X}_2 \) for second interval \(d'T_s\)
   b. Time weighting over \(T_s\) of matrices
   c. Small Signal Excursions
      1. \( \bar{x} \) Independent state equations
      2. \( \bar{y} \) Dependent variable equations
   d. Transform time domain \( \rightarrow \) S domain
      \[
      \hat{x}(t) \rightarrow \hat{x}(s) = f(\hat{x}, \hat{d}, \hat{v}) \\
      \hat{y}(t) \rightarrow \hat{y}(s) = f(\hat{x}, \hat{d}, \hat{v})
      \]

4. Examples
   a. DC Lossy Buck-Boost
   b. Erickson Problem 7.8
Lecture 43
State Variable Approach to AC Converter Models

A. State Space Averaging

- A formal method for deriving the small-signal ac equations of a switching converter
- Equivalent to the modeling method of the previous sections
- Uses the state-space matrix description of linear circuits
- Often cited in the literature
- A general approach: if the state equations of the converter can be written for each subinterval, then the small-signal averaged model can always be derived
- Computer programs exist which utilize the state-space averaging method

The state equations of a system are employed and placed in matrix form:

- A canonical form for writing the differential equations of a system
- If the system is linear, then the derivatives of the state variables are expressed as linear combinations of the system independent inputs and state variables themselves
- The physical state variables of a system are usually associated with the storage of energy
- For a typical converter circuit, the physical state variables are the inductor currents and capacitor voltages
- Other typical physical state variables: position and velocity of a motor shaft
- At a given point in time, the values of the state variables depend on the previous history of the system, rather than the present values of the system inputs
- To solve the differential equations of a system, the initial values of the state variables must be specified

1. System variables are of several types: state variables, \( x \), independent system inputs \( u \) or \( v \) and \( y \) will be dependent variables. Energy ↔ state variables, use “\( x \)”
a. Follow the energy / momentum variables or state variables, because both are conserved

1. For electrical systems the energy resides in:
   a. Inductor currents
   b. Capacitor voltage
   c. Resistor I - V

2. For mechanical systems energy and momentum involve:
   a. Velocities angular speed
   b. Positions

b. **System independent inputs - the “u” or “v” variables**

1. For electrical systems
   a. Drive voltage / current
   b. Duty cycles of switches

2. For mechanical systems - the “v” variables
   a. Forces / Torques

Independent inputs to electromechanical systems are given by “u” or “v”

c. United electro-mechanical models are possible by using (a) and (b) together using an agreed upon standard mathematical formalism.

\[ \ddot{x}(t) \] is a vector with all state variables (energy storage)
\[ \ddot{v}(t) \text{ or } u(t) \] is a vector with all independent variables (driving forces)

\[ K\ddot{x} = A\ddot{x} + B\ddot{u} \leftarrow \text{input vector of external sources,} \]
\[ \downarrow \quad \downarrow \]
Matrix \quad Matrices with
of L’s \quad constants of
and C’s \quad proportionality
of the circuit
State Space analysis is very popular in modern control theory where the rule is “the more variables you can sense the better off you are.” In fact one tries to feedback ALL STATE VARIABLES so you can tailor the system transfer function to better achieve the dynamical response you seek. Later, in chapter 11, we will employ two feedback loops one for current and one for voltage as an example: We repeat the two loop current/voltage control schematic below

1. Std $V_o$ compared to $V_{\text{ref}}$ as an outer control loop
2. Inner current feedback loop on $i_L$ compared to $i(\text{control})$

Another example of feedback with dual loops would be the boost converter below with proportional / integral control. In simple proportional control the error signal can never be zero
as we need a small error signal to drive $d(t)$ creation. The error can be made small with high gain but this creates other problems. For example small $V_o$ variations could cause big $d(t)$ variations. This doesn’t create instability in the sense of a growing disturbance but big variations of $d(t)$ cause bigger swings on the way to recovery.

One solution is to employ integration to the error signal, $e$, so that the output will change until the error is exactly zero and $\int e\,dt$ is also zero. This effect occurs even at very low loop gain as the integrator with low gain still gets $e$ zero in steady state. We can then use both proportional control and integral control together, giving a control parameter $K_{pi}$.

$$K_{pi} = K_i \int (V_{ref} - V_o)\,dt + K_p(V_{ref} - V_{out})$$

Generally $K_i$ is low to avoid integrator over hunting (integrator wind-up) and $K_p$ is large for fast large signal disturbances.

Below the dual loop boost converter employs the output voltage error as a virtual current reference. $I_L$ is usually triangular / sawtooth in shape and acts also as a stabilizing ramp as we will show in Ch. 11.
In the state variable approach we begin with the state equations of a linear system, which summarizes known relations in a specific system or circuit topology for independent inputs $u$ or $v$ and state variables $x$. All other dependent variables in the circuit are given by the vector $\tilde{y}$ or matrix $[Y]$

\[
[Y] = [C] [x] + [E] [u]
\]

Output Matrices with proper constants of vector proportionality

Note the mathematical completeness assumption that all y’s are a linear combination of only x’s and u’s. There is some ambiguity at first as to what actually constitutes independent inputs and what constitutes dependent variables, e.g. $i_g$ input to converter is usually chosen as a dependent variable while $v_g$ is chosen as independent.

Summarize methodology

\[
K\ddot{x} = A\ddot{x} + B\ddot{u}
\]

$x$: states associated with Energy storage
$u$: inputs you specify and are the driving forces
$\tilde{y} = C\ddot{x} + E\ddot{u}$

$y$: dependent variables that the driving forces, $u$, and the $x$ states fully specify.
2. **Specific Circuit Model in State Space Matrix Formalism**
   Let’s get specific and see what all this talk means.

![Circuit Diagram]

Independent state variables  
K without mutual coupling  Independent Inputs  
\[
\begin{bmatrix}
\dot{V}_1 \\
\dot{V}_2 \\
\dot{i}
\end{bmatrix} = 
\begin{bmatrix}
\dot{V}_o \\
\dot{i}_R
\end{bmatrix}
\]

\[
\begin{bmatrix}
C_1 & 0 & 0 \\
0 & C_2 & 0 \\
0 & 0 & L
\end{bmatrix}
\]

\[
\begin{bmatrix}
i_{in}
\end{bmatrix}
\]

Then without much thinking we can combine these four into matrix form using state space matrix formalism.

\[
k \dot{\mathbf{x}} = A \mathbf{x} + B \mathbf{u}
\]

Below we further justify the specific choice of A & B matrices based on the well known circuit node and loop equations.
In the way of a check we find the following:

\[ i_{c1} = C_1 \frac{dV_1}{dt} = \text{int} - \frac{V_1}{R} - i(t) \Rightarrow \text{Top row of A and K} \]

\[ i_{c2} = C_2 \frac{dV_2}{dt} = i(t) - \frac{V_2}{R_2 + R_3} \Rightarrow \text{Second rows of A and K} \]

\[ V_L = L \frac{di(t)}{dt} = V_1 - V_2 \Rightarrow \text{Third rows of A and K} \]

In summary, the same equations:

\[ i_{c1}(t) = C_1 \frac{dV_1(t)}{dt} = i_{\text{int}}(t) - \frac{v_1(t)}{R} - i(t) \]

\[ i_{c2}(t) = C_2 \frac{dV_2(t)}{dt} = i(t) - \frac{v_2(t)}{R_2 + R_3} \]

\[ v_L(t) = L \frac{di(t)}{dt} = v_1(t) - v_2(t) \]

Express in matrix form:

\[
\begin{bmatrix}
C_1 & 0 & 0 \\
0 & C_2 & 0 \\
0 & 0 & L
\end{bmatrix}
\begin{bmatrix}
\frac{dV_1(t)}{dt} \\
\frac{dV_2(t)}{dt} \\
\frac{di(t)}{dt}
\end{bmatrix} =
\begin{bmatrix}
-\frac{1}{R_1} & 0 & -1 \\
0 & -\frac{1}{R_2 + R_3} & 1 \\
1 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
v_1(t) \\
v_2(t) \\
i(t)
\end{bmatrix} +
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
i_{\text{int}}(t)
\end{bmatrix}
\]

\[ K \frac{dx(t)}{dt} = A x(t) + B u(t) \]
Likewise we can combine \( \bar{x} \) and \( \bar{u} \) to form the dependent variables.

\[
V_o = V_2 \frac{R_3}{R_2 + R_3}, \quad i_{R_1}(t) = \frac{V_1(t)}{R_1}
\]

That is in the form: \( Y = CX + EU \)

\[
\begin{bmatrix}
\begin{bmatrix} v_{out} \\ i_{R_1}(t) \end{bmatrix} \\
\end{bmatrix} = \begin{bmatrix}
\begin{bmatrix}
0 \\
\frac{R_3}{R_2 + R_3} \\
\frac{1}{R_1} \\
0 \\
0
\end{bmatrix}
\end{bmatrix} \begin{bmatrix}
\begin{bmatrix} v_1(t) \\ v_2(t) \\ i(t) \end{bmatrix} \\
\end{bmatrix} + \begin{bmatrix}
0 \\
0
\end{bmatrix} [i_{in}(t)]
\]

\[
\begin{bmatrix}
\begin{bmatrix} y(t) \\
x(t) \\
u(t) \end{bmatrix} \\
\end{bmatrix} = \begin{bmatrix}
0 \\
C \\
0
\end{bmatrix} \begin{bmatrix}
\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \\
\end{bmatrix} + \begin{bmatrix}
0 \\
0
\end{bmatrix} [i_{in}(t)]
\]

Simply stated we are doing nothing new. We are simply agreeing to write all circuit loop, node and dependent relations in an easy to visualize matrix form, that we all agree on. Once this is done then matrix math will easily be done via standard methods of perturbation theory to obtain small signal averaged models. We do this in three major steps.

* Averaging between the switch states during \( T_s \)
* Calculate quiescent equations
* Calculate small signal equations

This is similar in spirit to circuit averaging but the mathematical means are different.

3. **State Space Methodology for \( <>T_s \) Averaging of switch states**

a. Separate State space matrices for

\[
\begin{bmatrix}
D_1 T_s & D'_2 T_s & D_3 T_s \\
\text{interval} & \text{interval} & \text{If it exists in DCM},
\end{bmatrix}
\]
Forget for now the DCM of operation and consider only \( \overline{x}_1(dT_s) \) and \( \overline{x}_2(d'T_s) \) and consider only a \( V_d \) (input). We then get two separate state equations for the two switch intervals.

\[
\overline{x}_1 = A_1 \overline{x} + B_1 V_d \\
\overline{x}_2 = A_2 \overline{x} + B_2 V_d
\]

Simplify \( V_o \) terms of \( x \) only

\[
V_o = C_1 x \\
V_o = C_2 x
\]

during \( dT_s \) \hspace{1cm} during \( d'T_s \)

During subinterval 1, we have

\[
K \frac{dx(t)}{dt} = A_1 x(t) + B_1 u(t) \\
y(t) = C_1 x(t) + E_1 u(t)
\]

So the elements of \( x(t) \) change with the slope

\[
\frac{dx(t)}{dt} = K^{-1} \left[ A_1 x(t) + B_1 u(t) \right]
\]

Small ripple assumption: the elements of \( x(t) \) and \( u(t) \) do not change significantly during the subinterval. Hence the slopes are essentially constant and are equal to

\[
\frac{dx(t)}{dt} = K^{-1} \left[ A_1 \langle x(t) \rangle_{T_1} + B_1 \langle u(t) \rangle_{T_1} \right]
\]

The change during the first interval is:

\[
\frac{dx(t)}{dt} = K^{-1} \left[ A_1 \langle x(t) \rangle_{T_1} + B_1 \langle u(t) \rangle_{T_1} \right]
\]

Net change in state vector over first subinterval:

\[
x(dT_s) = x(0) + (dT_s) K^{-1} \left[ A_1 \langle x(t) \rangle_{T_1} + B_1 \langle u(t) \rangle_{T_1} \right]
\]

We can do a similar step for the second interval as shown on page 11.
That is:
Use similar arguments.
State vector now changes with the essentially constant slope
\[
\frac{dx(t)}{dt} = K^{-1} \left( A_2 \langle x(t) \rangle_{T_s} + B_2 \langle u(t) \rangle_{T_s} \right)
\]

The value of the state vector at the end of the second subinterval is therefore
\[
x(T_s) = x(dT_s) + (dT_s) K^{-1} \left[ A_2 \langle x(t) \rangle_{T_s} + B_2 \langle u(t) \rangle_{T_s} \right]
\]

We can now get the net change over the switching period:
We have:
\[
x(dT_s) = x(0) + (dT_s) K^{-1} \left[ A_1 \langle x(t) \rangle_{T_s} + B_1 \langle u(t) \rangle_{T_s} \right]
\]
\[
x(T_s) = x(dT_s) + (dT_s) K^{-1} \left[ A_2 \langle x(t) \rangle_{T_s} + B_2 \langle u(t) \rangle_{T_s} \right]
\]

Eliminate \(x(dT_s)\), to express \(x(T_s)\) directly in terms of \(x(0)\):
\[
x(T_s) = x(0) + dT_s K^{-1} \left[ A_1 \langle x(t) \rangle_{T_s} + B_1 \langle u(t) \rangle_{T_s} \right] + d'T_s K^{-1} \left[ A_2 \langle x(t) \rangle_{T_s} + B_2 \langle u(t) \rangle_{T_s} \right]
\]
Collect terms:
\[
x(T_s) = x(0) + T_s K^{-1} \left[ (d(t)A_1 + d'(t)A_2) \langle x(t) \rangle_{T_s} + B_1 \langle u(t) \rangle_{T_s} \right] + T_s K^{-1} \left[ (d(t)B_1 + d'(t)B_2) \langle u(t) \rangle_{T_s} \right]
\]

b. Average State Variable Over \(T_s\) by Time Weighting
We next calculate an average by weighting each state variable matrix by the appropriate duty cycle.
\[\bar{x} = [A_1 d + A_2 d''] \bar{x} + [B_1 d + B_2 d'] V_d\]
\[V_o = [C_1 d + C_2 d'] \bar{x}\]
Consider first a converter with two state variables: \(i_L\) and \(v_C\). Also only consider CCM operation with only two time intervals \(D_1\) and \((1-D_1)\). We find for external sources, \(u\) or \(v\).
\[ \dot{x} = A_1 x + B_1 u \quad \text{for switch period } D_1 T_s \]
\[ \dot{x} = A_2 x + B_2 u \quad \text{for switch period } D_2 T_s \]
\[ (1-D_1) = D_2 \]

If we are operating where the ripple is triangular ($f_{sw}$ very high) then all time derivatives are simply constants so that from $t = 0$ to $t = D_1 T_s$
\[ x(\Delta t) = x(0) + \dot{x} \Delta t, \Delta t = D_1 T_s \]
\[ x(D_1 T_s) = x(0) + (A_1 x + B_1 u) D_1 T_s \]

This $x$ value is then the initial condition for the second switch period $(1-D)$. So that at $T_s$
\[ x(T_s) = x(D_1 T_s) + (A_2 x + B_2 u) D_2 T_s \]
\[ = x(0) + (A_1 x + B_1 u) D_1 T_s + (A_2 x + B_2 u) D_2 T_s \]

Combining like terms
\[ x(T_s) = x(0) + [D_1 A_1 + (1-D_1) A_2] x T_s + [D_1 B_1 + (1-D_1) B_2] u T_s \]

Notice the average matrices have been formed.
\[ \overline{A} \text{ (T_s average)} = D_1 A_1 + (1-D_1) A_2 = \overline{A} \]
\[ \overline{B} \text{ (T_s average)} = D_1 B_1 + (1-D_1) B_2 = \overline{B} \]

\[ \overline{A} \text{ and } \overline{B} \text{ are the duty cycle weighted averages of the state space average.} \]
\[ x(T_s) = x(0) + [\overline{A} x + \overline{B} u] T_s \]

That is, the averaged system equations over the switch period are
\[ \frac{dx}{dt} = x(T_s) - x(0) \]
\[ \dot{x} = \overline{A} \text{ (T_s average)} x + \overline{B} \text{ (T_s average)} u \]

This is the continuous approximation to the original switching system.

To recap,
1. Circuit equations are written for each switch state.
2. A weighted average of A and B matrices are made via duty cycle weighting.
Consider the buck converter below:

We write the state equations for $L$ and $C$.

\[ v_L(\text{inductor}) = \frac{L}{d} \frac{di_L}{dt} \]

For $D_1T_s$ transistor on: \( v_L = D_1V_{\text{in}} - V_c \)

For $D_2T_s$ diode on: \( v_L = -V_c \)

\[ i_c(\text{capacitor}) = i_L - i(\text{load}) = \frac{C}{d} v_c/dt \]

Dividing $i_c$ by $L$ and $v_L$ by $C$ we obtain

\[
\begin{bmatrix}
\frac{di_L}{dt} \\
\frac{dv_c}{dt}
\end{bmatrix}
= \begin{bmatrix}
0 & -1 \\
\frac{1}{L} & \frac{-1}{C} \frac{1}{RC}
\end{bmatrix}
\begin{bmatrix}
i_L \\
v_c
\end{bmatrix}
+ \begin{bmatrix}
D_1V_{\text{in}} \\
0
\end{bmatrix}
\]

\[
A_1= A_2 = \overline{A} \\
\text{for Buck}
\]

\[
B_{\text{changes}} \text{ with switching}
\]

\[
B_{1u} = \begin{bmatrix}
\frac{V_{\text{in}}}{L} \\
0
\end{bmatrix}, \quad B_{2u} = \begin{bmatrix}
0 \\
0
\end{bmatrix}, \quad \overline{B} = \begin{bmatrix}
\frac{D_1V_{\text{in}}}{L} \\
0
\end{bmatrix}
\]

Next consider the boost converter below:
We write state equations for $L$ and $C$

\[ v_L(\text{inductor}) = L \frac{di_L}{dt} \]

For $D_1T_s$ transistor on: \[ v_L = V_{in} \]
For $D_2T_s$ diode on: \[ v_L = D_1V_{in} - V_c \]

\[ i_c(\text{capacitor}) = D_2i_L - \frac{V_c}{R} = \frac{Cdv_c}{dt} \]

\[
A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -1 \\ \frac{L}{C} & -1 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 0 & -D_2 \\ \frac{L}{C} & -1 \end{bmatrix}
\]

Now $B_1 = B_2 = \bar{B} = \begin{bmatrix} \frac{V_{in}}{L} \\ 0 \end{bmatrix}$.

c. **General Quiescent Operation point and Small Signal Excursions valid only at that point**

\[ x \rightarrow X + \hat{x}, \quad v_o \rightarrow V_o + \hat{v}_o, \quad d \rightarrow D + \hat{d} \]

Look familiar except for the matrix bookkeeping?

Then expand neglect all higher order terms, keeping only DC and AC terms.

For simplicity small signal variations $\hat{v}_d$ on $v_d$ are assumed zero for now and $v_d \equiv V_d$. Using $\bar{x}$ equation:

1. \[ \bar{x} = Ax + BV_d + \hat{x}[A] + \hat{d}[(A_1 - A_2)X + (B_1 - B_2)V_d] \]

   \[
   \downarrow \quad \downarrow \quad \downarrow \quad \downarrow
   \quad \text{dc terms} \quad \text{ac terms} \quad \text{dc terms}
   \]

   A(average) $\equiv A_1D + A_2D'$ note: time average on $A_1, A_2$

   B(average) $\equiv B_1D + B_2D'$ and the dc values $x, V_d$
Steady State
\[ \ddot{x} = 0 = Ax + B V_d \]
\[ x = A^{-1} B V_d \]

AC Perturbation
\[ \dot{x} = \dot{x} A + d \text{ [average]} \]

2. Next we use \( V_o = [C_1 d + C_2 d'] \) \( \ddot{x} \) and perturb/linearize
\[ V_o + \dot{V}_o = CX \quad + \quad [(C_1 - C_2) X] \dot{d} \]
\[ \quad \downarrow \quad \downarrow \]
\[ \text{dc term} \quad \text{ac terms coefficient [ ]} \]
\[ \text{is time averaged} \]
\[ C \equiv C_1 D + C_2 D' \]

Steady State
\[ V_o = CX = - C A' B V_d \]

AC Perturbation
\[ \dot{V}_o = C \dot{x} + [(C_1 - C_2) X] \dot{d} \]

d. **Transform (time equations) into transfer Laplace transforms and transfer functions**
\[ \dot{x} \rightarrow S \dot{x} = Ax(s) + [(A_1 - A_2) X + (B_1 - B_2) V_d] * \dot{d}(s) \]
\[ \dot{x}(s) = [sI - A]^{-1}[(A_1 - A_2) X + (B_1 - B_2) V_d] \dot{d}(s) \]
\[ \downarrow \]
\[ \text{unity matrix} \]
\[ \dot{V}_o(s) = C[sI - A]^{-1}[(A_1 - A_2) X + (B_1 - B_2) V_d] \dot{d}(s) + [(C_1 - C_2) X] \]
\[ \frac{\dot{V}_o(s)}{\dot{d}(s)} = C[sI - A]^{-1}[(A_1 - A_2) X + (B_1 - B_2) V_d] \dot{d}(s) + (C_1 - C_2) X \]

In summary, the time domain forms:
\[ K \frac{d\dot{x}(t)}{dt} = A \dot{x}(t) + B \dot{u}(t) + \{(A_1 - A_2) X + (B_1 - B_2) U\} \dot{d}(t) \]
\[ \hat{y}(t) = C \hat{x}(t) - E \hat{u}(t) + \{(C_1 - C_2) X + (E_1 - E_2) U\} \hat{d}(t) \]
can easily be changed to Laplace form and \[ \frac{\hat{x}(s)}{\hat{d}(s)} \]
calculated or any other transfer function of interest.

1. Lets try the forward converter with equivalent series resistance's for both C and L - r_c and r_L. \( x_1 \to i_L \) and \( x_2 \to V_c \) as shown below

During DT_s (switch on) we obtain state loop equations
- \[ V_d + L \dot{x}_1 + r_L x_1 + R(x_1 - C x_2) = 0 \quad (KVL \#1) \]
- \[ -x_2 - C r_c \dot{x}_2 + R(x_1 - C x_2) = 0 \quad (KVL \#2) \]
The simplified secondary circuit is shown below to aid understanding.
Now use the standard state space form.

\[
\begin{aligned}
\dot{x} &= [A_1] \rightarrow [x_1] + B_1 V_d \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -\frac{R r_c + R r_L + r_c r_L}{L(R + r_c)} & -\frac{R}{L(R + r_c)} \\ \frac{R}{C(R + r_c)} & -\frac{1}{C(R + r_c)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} V_d
\end{aligned}
\]

During D'T_s (switch off series diode on)

\[
A_2 = A_1 \quad B_2 = 0
\]

\[
V_o \text{ for both DT}_s \text{ and D'T}_s \text{ intervals is } \quad R(x_1 - C \dot{x}_2)
\]

\[
V_o = R(x_1 - C \dot{x}_2) = \frac{R r_c}{R + r_c} x_1 + \frac{R}{R + r_c} x_2
\]

\[
V_o = \begin{bmatrix} R r_c \\ R \\ R \\ R + r_c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

During both DT_s and D'T_s

\[
V_o = C_1 \bar{x} = C_2 \bar{x}
\]

Time Averaged Matrices over the switch period T_s are obtained as follows.

A = A_2 = A_1 \quad \text{from AD} + (1-D)A = A

B = DB_1

C = C_1 = C_2

Using the simplification R >> r_c + r_L we know from circuit values
Steady State or DC Conditions

\[ 0 = Ax + BV_d \rightarrow x = A^{-1} BV_d \]
\[ V_o = Cx = -CA^{-1} BV_d \]

\[ \frac{V_o}{V_d} = -CA^{-1} B. \text{ Where } C = [r_c1], B = \begin{bmatrix} D \\ \frac{L}{L} \\ 0 \end{bmatrix} \]

and \( A^{-1} = \frac{LC}{1 + \frac{(r_c + r_L)}{R}} \begin{bmatrix} -\frac{1}{CR} & \frac{1}{L} \\ \frac{1}{C} & \frac{(r_c + r_L)}{L} \end{bmatrix} \).

Multiplying out we find for the DC case:

\[ \frac{V_o}{V_d} = D \frac{R + r_c}{R + r_c + r_L} \approx D \]

**AC Small Signal Model**

\[ s\hat{x}(s) = A\hat{x}(s) + \hat{d}(s)[(A_1 - A_2)X + (B_1 - B_2)V_d] \]

\[ \hat{x}(s) = [sI - A]^{-1}[(A_1 - A_2)X + (B_1 - B_2)V_d]\hat{d}(s) \]
\[ \dot{V}_o(s) = C \dot{x}(s) + [(C_1 - C_2)X] \dot{d}(s) \]

\[ \uparrow \quad [sI - A]^{-1} \begin{bmatrix} (A_1 - A_2)X \\ (B_1 - B_2) V_d \end{bmatrix} + \dot{d}(s) \]

\[ \frac{\dot{V}_o(s)}{\dot{d}(s)} = [sI - A]^{-1} \begin{bmatrix} (A_1 - A_2)X \\ (B_1 - B_2) V_d \end{bmatrix} + \dot{d}(s) + [(C_1 - C_2)X] \]

\[ = V_d \frac{1 + s r_c C}{LC\left[s^2 + s\left(\frac{1}{RC + \frac{(r_c + r_L)}{L}}\right) \frac{1}{LC}\right]} \]

Please note that the forward converter transfer function has the following properties:

1. single zero \( w_z = \frac{1}{r_c C} \)

2. double pole \( w_o = \frac{1}{\sqrt{LC}} \)

3. Since \( r_c C \) is usually very small \( \Rightarrow w_z > w_o \)

The general shape of \( V_o/d \) versus frequency is then plotted below on page 20.
Looking ahead to feedback conditions, for avoiding oscillation we want 76° phase margin @ unity gain in the open loop part of the gain. That is the actual phase angle of the \( \frac{V_o}{d} \) minus 180° should be \( \geq 76° \).  

\[ 76° \geq \phi - 180° \]
2. Looking even further ahead to the flyback converter operating in CCM we will find:

\[
\frac{\dot{V}_o(s)}{d(s)} = V_d \cdot f(D) \cdot \frac{(1 + s \cdot w_{z1})(1 - s / w_{z2})}{as^2 + bs + c}
\]

That is the flyback converter has a RHP zero \( w = f(R, L \cdot f(0)) \). This makes for a very unstable situation, as shown in the open loop plots below. In stark contrast the flyback operating DCM does not have right half plane zero! From \( \frac{V_o}{d} \sim (1 - s / w_z) \) can you make the case that the output has an undesirably high phase lag at high f?

Flyback converter Open Loop Bode Plots have some oddities in that the low f gain is non-linear. This implies that the precise frequency range with -40 db/decade drop depends on the low f gain.
Notice in the phase plot we do not have enough $\phi$ margin and the open loop system response tells we come close to instability at unity gain $\phi \rightarrow 180^\circ$. This is undesirable.

3. **Summary of Phase Margin in Open Loop Plots**

Below we show four different open-loop Bode plots. For HW#3 please tell which are stable and which are unstable.

What occurs in (c) for small component variations, temperature changes, component aging? Most designers prefer a phase margin of 60° to address these problems. We will visit this in detail in upcoming lectures.
4. Further Examples
Lossy Switch

a. Buck - Boost Erickson’s text pages 212 - 217

\[ v(t) \]
\[ v_g(t) \]
\[ i(t) \]
\[ i_g(t) \]
\[ C \]
\[ R \]
\[ L \]
\[ Q_1 \]
\[ D_1 \]

Trans on → \( R_{on} \) } DC losses
Diode on → \( V_D \) } \([V] = [V_g, V_D]\)

DTs
\[
[x] = \begin{bmatrix}
i \\
v
\end{bmatrix}
\]

D'Ts
\[
[y] = [i_g]
\]

\[
\frac{L}{C} \frac{d}{dt} \begin{bmatrix} i(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} -R_s & 0 \\ 0 & -\frac{1}{R} \end{bmatrix} \begin{bmatrix} i(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_g(t) \\ V_D \end{bmatrix}
\]

\[
\frac{L}{C} \frac{d}{dt} \begin{bmatrix} i(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{1}{R} \end{bmatrix} \begin{bmatrix} i(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_g(t) \\ V_D \end{bmatrix}
\]

\[
K \frac{dx(t)}{dt} = A_1 x(t) + B_1 u(t)
\]

\[
K \frac{dx(t)}{dt} = A_2 x(t) + B_2 u(t)
\]
\[ \begin{bmatrix} i_g \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} i(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} V_g \\ V_D \end{bmatrix} \]

\[ \begin{bmatrix} i_g \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} i(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} V_g \\ V_D \end{bmatrix} \]

\[
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\begin{bmatrix} y(t) \\ C_1 \quad x(t) \quad E_1 \quad u(t) \quad y(t) \quad C_2 \quad x(t) \quad E_2 \quad u(t) \end{bmatrix}
\]

Next, we get the \(<>_{Ts}\) average matrices \(A, B, C, E\), by time weighting by \(D\) and \(D'\).

\[
A = DA_1 + D'A_2 = D \begin{bmatrix} -R_{on} & 0 \\ 0 & -\frac{1}{R} \end{bmatrix} + D' \begin{bmatrix} 0 & 1 \\ -1 & -\frac{1}{R} \end{bmatrix} = \begin{bmatrix} -DR_{on} & D' \\ -D' & -\frac{1}{R} \end{bmatrix}
\]

\[
B = DB_1 + D'B_2 = \begin{bmatrix} D & -D' \\ 0 & 0 \end{bmatrix}
\]

\[
C = DC_1 + D'C_2 = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}
\]

\[
E = DE_1 + D'E_2 = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}
\]

\[
K\ddot{x} = A_{av}\ddot{x} + B_{av}\ddot{v} \}
\quad x + X_0 + \dot{x} \}
\quad y = Y_0 + \dot{y} \}
\quad \ddot{y} = C_{av}\ddot{x} + E_{av}\ddot{v} \}
\quad d = D_0 + \dot{d} \}
\]

1. Steady State Solutions: \(\ddot{x} = 0\)

\[
\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -DR_{on} & D' \\ -D' & -\frac{1}{R} \end{bmatrix} \begin{bmatrix} I \\ V \end{bmatrix} + \begin{bmatrix} D & -D' \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_g \\ V_D \end{bmatrix}
\]
\[
\begin{align*}
\Rightarrow \begin{bmatrix} I \\ V \end{bmatrix} &= [A_{av}]^{-1} \begin{bmatrix} B \\ V_g \\ V_D \end{bmatrix} \\
[I_g] &= [D \ 0] \begin{bmatrix} I \\ V \end{bmatrix} + [0 \ 0] \begin{bmatrix} V_g \\ V_D \end{bmatrix} \Rightarrow [I_g] = [C_{av}] \begin{bmatrix} I \\ V \end{bmatrix} \\
[I_g] &= C_{av} A_{av}^{-1} B \begin{bmatrix} V_g \\ V_D \end{bmatrix}
\end{align*}
\]

dc solutions:
Equation for I,V output and Input I<sub>g</sub>

\[
\begin{align*}
\begin{bmatrix} I \\ V \end{bmatrix} &= [A_{av}]^{-1} \begin{bmatrix} B \\ V_g \\ V_D \end{bmatrix} = \begin{bmatrix} 1 \\ D \ 1 \\ D' \end{bmatrix} + \frac{1}{D'^2 R} \begin{bmatrix} D \\ 1 \\ -D' \end{bmatrix} \begin{bmatrix} V_g \\ V_D \end{bmatrix} \\
[I_g] &= C_{av} A_{av}^{-1} B \begin{bmatrix} V_g \\ V_D \end{bmatrix} = \begin{bmatrix} 1 \\ D'^2 R \end{bmatrix} + \frac{1}{D^2 R} \begin{bmatrix} D^2 D \\ D'^2 RD'R \end{bmatrix} \begin{bmatrix} V_g \\ V_D \end{bmatrix}
\end{align*}
\]

DC Model Circuit

![DC Model Circuit Diagram]
2. **AC Perturbed / Linear Solutions**

\[
\begin{align*}
K\dot{x} &= A\dot{x} + Bu + (A - A)x + (B - B)u \quad \hat{d} \\
\dot{y} &= C\dot{x} + Eu + (C - C)x + (E - E)u \\
\end{align*}
\]

for [ ] in front of \( \hat{d} \) terms

\[
(A - A)x + (B - B)u = -V + V - IR + V \\
in \dot{x}
\]

\[
(C - C)x + (E - E)u = [I] \quad \text{in } \dot{y}
\]

In standard state variable form the matrix math is:

\[
\begin{bmatrix}
L & 0 \\
0 & C
\end{bmatrix}
\begin{bmatrix}
\dot{i}(t) \\
\dot{\nu}(t)
\end{bmatrix}
= \begin{bmatrix}
-D\text{on} & D' \\
D' & -1/R
\end{bmatrix}
\begin{bmatrix}
\dot{i}(t) \\
\dot{\nu}(t)
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\nu}_D(t) \\
\dot{\nu}_D(t)
\end{bmatrix}
+ \begin{bmatrix}
V_g - V - IR + V_D
\end{bmatrix}\hat{d}
\]

\[
[i_g(t)] = [D 0]
\begin{bmatrix}
\dot{i}(t) \\
\dot{\nu}(t)
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\nu}(t) \\
\dot{V}_D(t)
\end{bmatrix}
+ \begin{bmatrix}
0 \\
I
\end{bmatrix}\hat{d}(t)
\]

Hard to get a “physical hold” on matrix equations.
In fact, we cannot easily get a complete circuit model of the whole converter for ac analysis. We can only separate out each set of three equations and then get a partial model.

Equation:

\[
L \frac{d\hat{i}(t)}{dt} = D'\dot{\nu}(t) - D\text{on}\dot{i}(t) + D\dot{\nu}_g(t) + (V_g - V - IR + V_D)\hat{d}(t)
\]
\[ sL\hat{i}(s) = D'\hat{v}(s) - DR_{on}\hat{i}(s) + D\hat{V}_g(s) + (V_g - V - IR_{on} + V_D)\hat{d}(s) \]

The circuit representing these equations:

Equation:
\[ C \frac{d\hat{v}(t)}{dt} = -D'\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t) \]

\[ Cs\hat{v}(s) = -D'\hat{i}(s) - \frac{\hat{V}(s)}{R} + I\hat{d}(s) \]

Circuit:

Equation:
\[ \hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t) \]

\[ \hat{i}_g(s) = D\hat{i}(s) + I\hat{d}(s) \]
Circuit:

Later we will get to employ $\frac{\hat{V}(s)}{d(s)}$ in a complex feedback loop so it’s open loop form is good to know well at this point.

**b. Erickson Problem 7.8**

Buck converter with $V_g$ possessing a source impedance $R_g$

$\begin{align*}
[x] & \rightarrow i_L \text{ energy terms } \} V \text{ is } V_c \text{ a state vector } V_C \\
[U] & \rightarrow V_g \text{ independent inputs} \\
[y] & \rightarrow i_g \text{ dependent output}
\end{align*}$

$DT_s (Q_{on})$:
\[
\begin{bmatrix}
K & \ddot{x} & A_1 & X & B_1 & U
\end{bmatrix}
\]

\[
\begin{bmatrix}
L & 0 & \frac{d}{dt} & i
\end{bmatrix}
= \begin{bmatrix}
-R_g & -1
\end{bmatrix}
\begin{bmatrix}
i
\end{bmatrix}
+ \begin{bmatrix}
1
\end{bmatrix}
\begin{bmatrix}
V_g
\end{bmatrix}
\]

\[
[i_g] = [1 \ 0]
\begin{bmatrix}
i
\end{bmatrix}
+ [0]
\begin{bmatrix}
V_g
\end{bmatrix}
\]

\[
[i_g] = [0 \ 0]
\begin{bmatrix}
i
\end{bmatrix}
+ [0][V_g]
\]

D'T_s (Q off/ D on) :

\[
\begin{bmatrix}
L & 0 & \frac{d}{dt} & i
\end{bmatrix}
= \begin{bmatrix}
0 & -1
\end{bmatrix}
\begin{bmatrix}
i
\end{bmatrix}
+ \begin{bmatrix}
0
\end{bmatrix}
\begin{bmatrix}
V_g
\end{bmatrix}
\]

\[
[i_g] = [0 \ 0]
\begin{bmatrix}
i
\end{bmatrix}
+ [0][V_g]
\]

\[
[i_g] = [0 \ 0]
\begin{bmatrix}
i
\end{bmatrix}
+ [0][V_g]
\]

\[
[y] = [C_1 \ E_1 \ U]
\]

\[
[y] = [C_2 \ E_2]
\]
Get time-averaged A, B, C, E

\[
A = D \begin{bmatrix} -R_g & 1 \\ 1 & 1/\frac{1}{R} \end{bmatrix} + D' \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -DR_g & 0 \\ 1 & -1/\frac{1}{R} \end{bmatrix}
\]

\[
B = D \begin{bmatrix} 1 \\ 0 \end{bmatrix} + D' \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} D \\ 0 \end{bmatrix}
\]

\[
C = D[1 \ 0] + D'[0 \ 0] = [0 \ 0]
\]

\[
E = D[0 \ 0] + D'[0 \ 0] = [0]
\]

**DC Relations:**

\[
0 = Ax + Bu \quad \{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{DR_g}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} I \\ V \end{bmatrix} + \begin{bmatrix} D \\ 0 \end{bmatrix} + [V_g] \}
\]

\[
y = Cx + Eu \quad \{ \begin{bmatrix} I_g \end{bmatrix} = [D \ 0] \begin{bmatrix} I \\ V \end{bmatrix} + [0] + [V_g] \}
\]

**OR:** \( I_g = DI \), \( 0 = -DR_g I - V + DV_g \) \& \( 0 = I - V/R \)

**Equivalent DC Circuit Model:**

\[
\Rightarrow AC \text{ Relations: }
\]

\[
(A_1 - A_2)x + (B_1 - B_2)u = \begin{bmatrix} V_g -IR_g \\ 0 \end{bmatrix}, (C_1 - C_2)x + (E_1 - E_2)u = [I]
\]

30
\[ \frac{d}{dt} \begin{bmatrix} \hat{i} \\ \hat{\nu} \end{bmatrix} = \begin{bmatrix} -\frac{DR_g}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{\nu} \end{bmatrix} + \begin{bmatrix} D \\ 0 \end{bmatrix} \begin{bmatrix} \hat{\nu} \end{bmatrix} + \begin{bmatrix} \frac{V_g - IR_g}{L} \\ 0 \end{bmatrix} \begin{bmatrix} \hat{d} \end{bmatrix} \]

\[ \frac{d}{dt} \hat{x} = A \hat{x} + B \hat{u} + [(A_1 - A_2)x + (B_1 - B_2)u] \hat{d} \]

\[ \begin{bmatrix} \hat{i}_g \end{bmatrix} = \begin{bmatrix} D & 0 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{\nu} \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} \hat{\nu} \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} \hat{d} \end{bmatrix} \]

\[ \hat{y} = C \hat{x} + E \hat{u} + [(C_1 - C_2)x + (E_1 - E_2)u] \hat{d} \]

OR:

\[ L \frac{d\hat{i}}{dt} = -DR_g \hat{i} - \hat{\nu} + (V_g - IR_g) \hat{d} \]

\[ C \frac{d\hat{\nu}}{dt} = -\hat{i}(t) - \frac{\hat{\nu}}{R} \quad \& \quad \hat{i}_g = D \hat{i} + I \hat{d} \]

Equivalent AC circuit:
Ac/dc transformer model:

\[
\begin{align*}
\frac{\hat{V}}{\hat{d}} & \approx (V_g - IR_g) \left( \frac{R}{R + DR_g} \right) \\
\frac{1}{1 + s \left( \frac{L}{R + DR_g} + CR || DR_g \right)} + s^2 \left( \frac{LCR}{R + DR_g} \right)
\end{align*}
\]

\[
Ls\hat{i}(s) = -DR_g \hat{i}(s) - \hat{v}(s) + [V_g(s) - IR_g]^* \hat{d}(s)
\]

\[
Cs\hat{v}(s) = \hat{i}(s) - \frac{\hat{V}(s)}{R}
\]

\[
\hat{i}_g(s) = DI(s) + I\hat{d}(s)
\]

Solve for \( \frac{\hat{V}(s)}{\hat{d}(s)} \)