Lecture 43
State Variable Approach to AC Converter Models

A. State Space Averaging

1. General Concepts

   $\vec{x}$: states, v or u: independent inputs

   a. Energy variables : $\vec{x}$

   b. Independent inputs: $\vec{v}$ or u

   c. State equations

   $$K\dot{\vec{x}} = A\vec{x} + B\vec{v}$$

   $$\vec{Y} = C\vec{x} + E\vec{v}$$

2. Illustrative Circuit Example

3. Methodology in state space for $\langle\rangle_{Ts}$ averaging

   a. $\vec{X}_1$ for first interval dTs

      $\vec{X}_2$ for second interval d'Ts

   b. Time weighting over T_s of matrices

   c. Small Signal Excursions

      1. $\vec{x}$ Independent state equations

      2. $\vec{y}$ Dependent variable equations

   d. Transform time domain$\rightarrow$ S domain

   $$\hat{x}(t) \rightarrow \hat{x}(s) = f(\hat{x}, \hat{d}, \hat{v})$$

   $$\hat{y}(t) \rightarrow \hat{y}(s) = f(\hat{x}, \hat{d}, \hat{v})$$

4. Examples

   a. DC Lossy Buck-Boost

   b. Erickson Problem 7.8

Lecture 43
State Variable Approach to AC Converter Models

A. State Space Averaging

- A formal method for deriving the small-signal ac equations of a switching converter
- Equivalent to the modeling method of the previous sections
- Uses the state-space matrix description of linear circuits
- Often cited in the literature
- A general approach: if the state equations of the converter can be written for each subinterval, then the small-signal averaged model can always be derived
- Computer programs exist which utilize the state-space averaging method

The state equations of a system are employed and placed in matrix form:

- A canonical form for writing the differential equations of a system
- If the system is linear, then the derivatives of the *state variables* are expressed as linear combinations of the system independent inputs and state variables themselves
- The physical state variables of a system are usually associated with the storage of energy
- For a typical converter circuit, the physical state variables are the inductor currents and capacitor voltages
- Other typical physical state variables: position and velocity of a motor shaft
- At a given point in time, the values of the state variables depend on the previous history of the system, rather than the present values of the system inputs
- To solve the differential equations of a system, the initial values of the state variables must be specified

1. **System variables are of several types**: state variables, x, independent system inputs u or v and y will be dependent variables. Energy ↔ state variables, use “x”

a. Follow the energy / momentum variables or state variables, because both are conserved

1. For electrical systems the energy resides in:
    a. Inductor currents
    b. Capacitor voltage
    c. Resistor I - V
2. For mechanical systems energy and momentum involve:
    a. Velocities angular speed
    b. Positions

b. **System independent inputs - the "u" or "v" variables**

1. For electrical systems
    a. Drive voltage / current
    b. Duty cycles of switches
2. For mechanical systems - the "v" variables
    a. Forces / Torques

Independent inputs to electromechanical systems are given by "u" or "v"

c. United electro-mechanical models are possible by using (a) and (b) together using an agreed upon standard mathematical formalism.

$\vec{x}(t)$ is a vector with <u>all</u> state variables (energy storage)

$\vec{v}(t)$ or u(t) is a vector with <u>all</u> independent variables (driving forces)

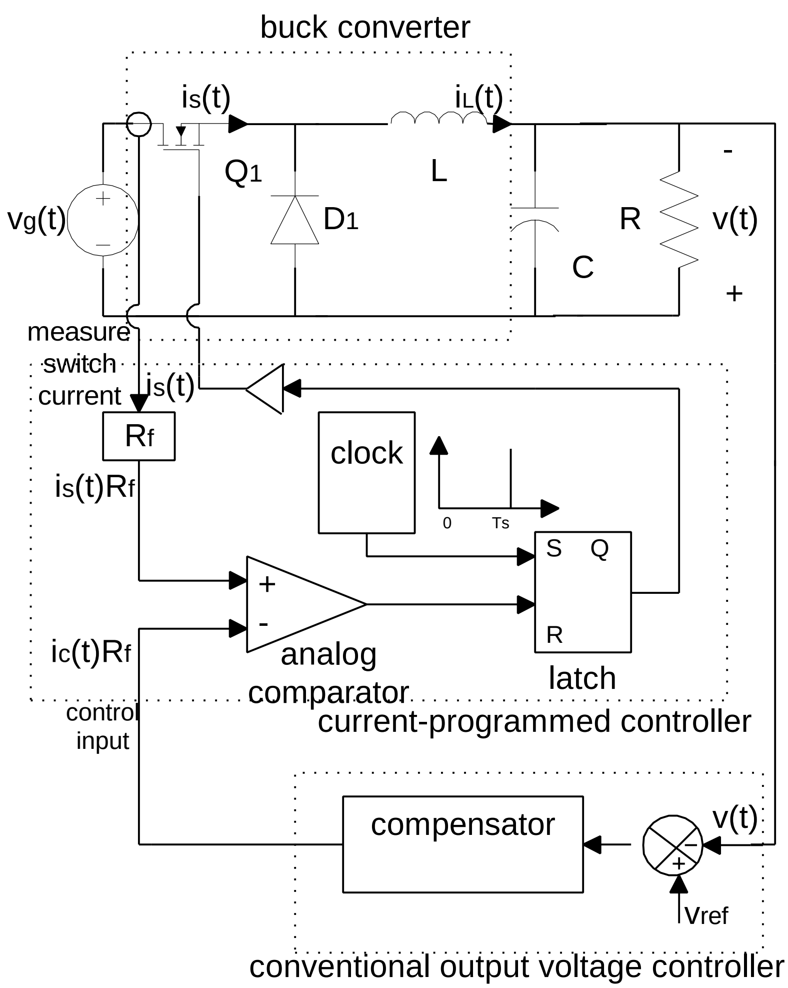
$$K\vec{x} = A\vec{x} + B\vec{u} \quad \leftarrow$$ input vector of external sources,

$\downarrow$ Matrix of L's and C's

$\downarrow$ Matrices with constants of proportionality of the circuit

State Space analysis is very popular in modern control theory where the rule is "the more variables you can sense the better off you are." In fact one tries to feedback ALL STATE VARIABLES so you can tailor the system transfer function to better achieve the dynamical response you seek. Later, in chapter 11, we will employ two feedback loops one for current and one for voltage as an example: We repeat the two loop current/voltage control schematic below

(1) Std V_o compared to V_{ref} as an outer control loop
(2) Inner current feedback loop on i_L compared to i(control)



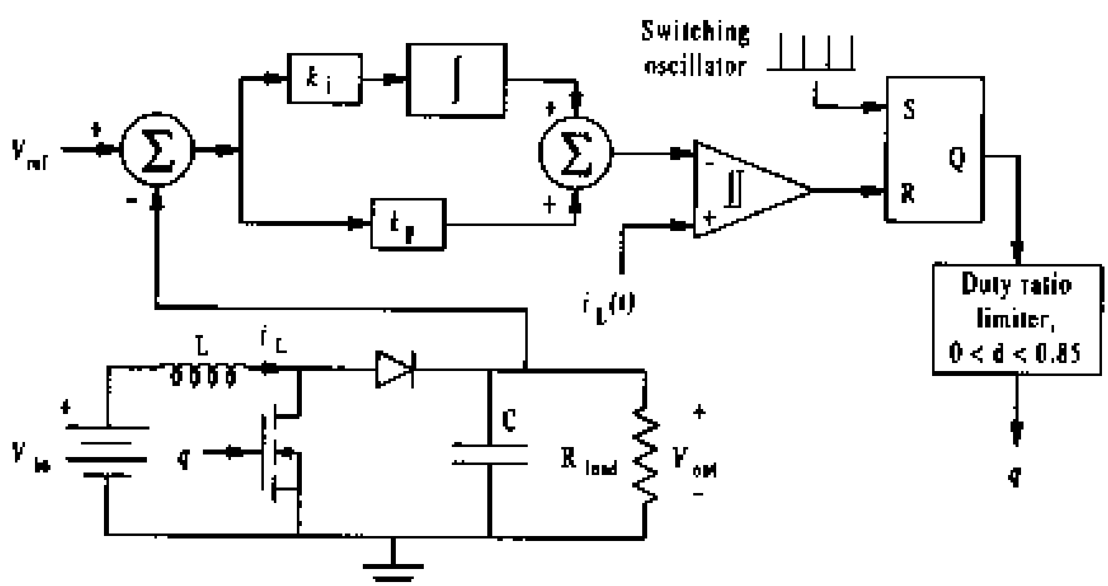
Another example of feedback with dual loops would be the boost converter below with proportional / integral control. In simple proportional control the error signal can never be zero as we need a small error signal to drive d(t) creation. The error can be made small with high gain but this creates other problems. For example small V_o variations could cause big d(t) variations. This doesn't create instability in the sense of a growing disturbance but big variations of d(t) cause bigger swings on the way to recovery.

One solution is to employ integration to the error signal, e, so that the output will change until the error is exactly zero and $\int e*dt$ is also zero. This effect occurs even at very low loop gain as the integrator with low gain still gets e zero in steady state. We can then use both proportional control and integral control together, giving a control parameter K_{pi}.

$K_{pi} = K_i \int (V_{ref} - V_o)dt + K_p(V_{ref} - V_{out})$

Generally K_i is low to avoid integrator over hunting (integrator wind-up) and K_p is large for fast large signal disturbances.

Below the dual loop boost converter employs the output voltage error as a virtual current reference. I_L is usually triangular / sawtooth in shape and acts also as a stabilizing ramp as we will show in Ch. 11.



$$[K][x] = [A][x] + [B][u]$$

In the state variable approach we begin with the state equations of a linear system, which summarizes known relations in a specific system or circuit topology for independent inputs u or v and state variables x.
All other <u>dependent variables</u> in the circuit are given by the vector $\vec{y}$ or matrix [Y]

$$\underset{\substack{\text{Output}\\ \text{vector}}}{\underset{\downarrow}{[Y]}} \quad = \quad \underset{\downarrow}{[C]}\,[x] \; + \; \underset{\downarrow}{[E]}\,[u]$$

Matrices with proper constants of proportionality

Note the mathematical completeness assumption that all y's are a linear combination of <u>only</u> x's and u's. There is some ambiguity at first as to what actually constitutes independent inputs and what constitutes dependent variables, e.g. i_g input to converter is usually chosen as a dependent variable while v_g is chosen as independent.
Summarize methodology

$$K\vec{x} = A\vec{x} + B\vec{u}$$

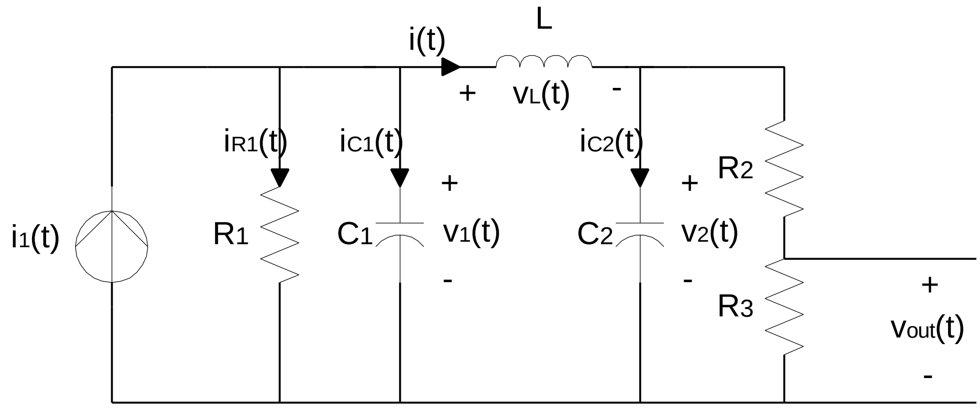
x: states associated with Energy storage
u: inputs you specify and are the driving forces

$$\vec{y} = C\vec{x} + E\vec{u}$$

y: dependent variables that the driving forces, u, and the x states fully specify.

2. **Specific Circuit Model in State Space Matrix Formalism**

Let's get specific and see what all this talk means.



Independent state variables

$$\vec{x} = \begin{bmatrix} V_1 \\ V_2 \\ i \end{bmatrix}$$

K without mutual coupling

$$k = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & L \end{bmatrix}$$

Independent Inputs

$$\vec{u} = [i_{in}]$$

Dependent variables

$$\vec{y} = \begin{bmatrix} V_o \\ i_{R_1} \end{bmatrix}$$

Then without much thinking we can combine these four into matrix form using state space matrix formalism. $k\dot{\vec{x}} = Ax+Bu$. Below we further justify the specific choice of A & B matrices based on the well known circuit node and loop equations.

$$\begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & L \end{bmatrix} \begin{bmatrix} \frac{d\,v_1(t)}{dt} \\ \\ \frac{d\,v_2(t)}{dt} \\ \\ \frac{di(t)}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1} & 0 & -1 \\ \\ 0 & -\frac{1}{R_2+R_3} & 1 \\ \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ i(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} i_{in}(t) \end{bmatrix}$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$K \qquad \frac{dx(t)}{dt} \quad = \qquad A \qquad x(t) \; + \; B \; u(t)$$

In the way of a check we find the following:

$$i_{C_1} = C_1 \frac{dV_1}{dt} = i_{int} - \frac{V_1}{R} - i(t) \Rightarrow \text{Top row of A and K}$$

$$i_{C_2} = C_2 \frac{d\,V_2}{dt} = i(t) - \frac{V_2}{R_2 + R_3} \Rightarrow \text{Second rows of A and K}$$

$$V_L = L \frac{di(t)}{dt} = V_1 - V_2 \Rightarrow \text{Third rows of A and K}$$

In summary,

The same equations:

$$i_{C1}(t) = C_1 \frac{dv_1(t)}{dt} = i_{in}(t) - \frac{v_1(t)}{R} - i(t)$$

$$i_{C2}(t) = C_2 \frac{dv_2(t)}{dt} = i(t) - \frac{v_2(t)}{R_2 + R_3}$$

$$v_L(t) = L \frac{di(t)}{dt} = v_1(t) - v_2(t)$$

Express in matrix form:

$$\underbrace{\begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & L \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} \frac{dv_1(t)}{dt} \\ \frac{dv_2(t)}{dt} \\ \frac{di(t)}{dt} \end{bmatrix}}_{\frac{d\mathbf{x}(t)}{dt}} = \underbrace{\begin{bmatrix} -\frac{1}{R_1} & 0 & -1 \\ 0 & -\frac{1}{R_2+R_3} & 1 \\ 1 & -1 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} v_1(t) \\ v_2(t) \\ i(t) \end{bmatrix}}_{\mathbf{x}(t)} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} i_{in}(t) \end{bmatrix}}_{\mathbf{u}(t)}$$

Likewise we can combine $\vec{x}$ and $\vec{u}$ to form the dependent variables.

$$V_o = V_2 \frac{R_3}{R_2 + R_3}, \quad i_{R_1}(t) = \frac{V_1(t)}{R_1}$$

That is in the form: y = Cx + Eu

$$\underbrace{\begin{bmatrix} v_{out} \\ i_{R1}(t) \end{bmatrix}}_{y(t)} = \underbrace{\begin{bmatrix} 0 & \frac{R_3}{R_2 + R_3} & 0 \\ \frac{1}{R_1} & 0 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} v_1(t) \\ v_2(t) \\ i(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{E} \underbrace{[i_{in}(t)]}_{u(t)}$$

Simply stated we are doing nothing new. We are simply agreeing to write all circuit loop, node and dependent relations in an easy to visualize matrix form, that we all agree on. Once this is done then matrix math will easily be done via standard methods of perturbation theory to obtain small signal averaged models. We do this in three major steps.

* Averaging between the switch states during T_s
* Calculate quiescent equations
* Calculate small signal equations

This is similar in spirit to circuit averaging but the mathematical means are different.

3. State Space Methodology for <>Ts Averaging of switch states

a. Separate State space matrices for

D_1T_s	D'_2T_s	D_3T_s
interval	interval	If it exists in DCM,

Forget for now the DCM of operation and consider only $\vec{x}_1(dT_s)$ and $\vec{x}_2(d'T_s)$ and consider only a V_d (input). We then get two separate state equations for the two switch intervals.

$\vec{x}_1 = A_1\vec{x} + B_1 V_d$ $\qquad$ $x_2 = A_2\vec{x} + B_2 V_d$

Simplify V_o terms of x only

$V_o = C_1 x$ $\qquad$ $V_o = C_2 x$

during dTs $\qquad$ during d'Ts

During subinterval 1, we have

$$\mathbf{K}\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}_1\,\mathbf{x}(t) + \mathbf{B}_1\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}_1\,\mathbf{x}(t) + \mathbf{E}_1\,\mathbf{u}(t)$$

So the elements of $\mathbf{x}(t)$ change with the slope

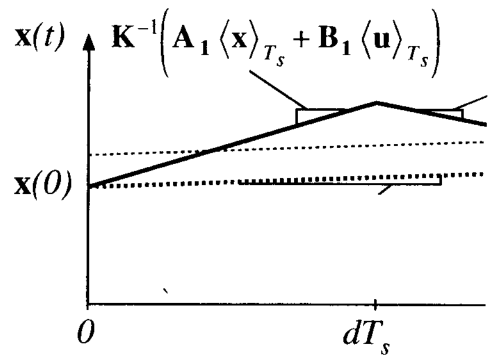
$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{K}^{-1}\Big(\mathbf{A}_1\,\mathbf{x}(t) + \mathbf{B}_1\,\mathbf{u}(t)\Big)$$

Small ripple assumption: the elements of $\mathbf{x}(t)$ and $\mathbf{u}(t)$ do not change significantly during the subinterval. Hence the slopes are essentially constant and are equal to

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{K}^{-1}\Big(\mathbf{A}_1\left\langle\mathbf{x}(t)\right\rangle_{T_s} + \mathbf{B}_1\left\langle\mathbf{u}(t)\right\rangle_{T_s}\Big)$$

The change during the first interval is:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{K}^{-1}\Big(\mathbf{A}_1\left\langle\mathbf{x}(t)\right\rangle_{T_s} + \mathbf{B}_1\left\langle\mathbf{u}(t)\right\rangle_{T_s}\Big)$$



Net change in state vector over first subinterval:

$$\underbrace{\mathbf{x}(dT_s)}_{\substack{\textit{final}\\ \textit{value}}} = \underbrace{\mathbf{x}(0)}_{\substack{\textit{initial}\\ \textit{value}}} + \underbrace{\left(dT_s\right)}_{\substack{\textit{interval}\\ \textit{length}}}\ \underbrace{\mathbf{K}^{-1}\Big(\mathbf{A}_1\left\langle\mathbf{x}(t)\right\rangle_{T_s} + \mathbf{B}_1\left\langle\mathbf{u}(t)\right\rangle_{T_s}\Big)}_{\textit{slope}}$$

We can do a similar step for the second interval as shown on page 11.

That is:

Use similar arguments.

State vector now changes with the essentially constant slope

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{K}^{-1}\left(\mathbf{A}_2 \left\langle \mathbf{x}(t) \right\rangle_{T_s} + \mathbf{B}_2 \left\langle \mathbf{u}(t) \right\rangle_{T_s}\right)$$

The value of the state vector at the end of the second subinterval is therefore

$$\underbrace{\mathbf{x}(T_s)}_{\substack{final\\value}} = \underbrace{\mathbf{x}(dT_s)}_{\substack{initial\\value}} + \underbrace{\left(d'T_s\right)}_{\substack{interval\\length}} \underbrace{\mathbf{K}^{-1}\left(\mathbf{A}_2 \left\langle \mathbf{x}(t) \right\rangle_{T_s} + \mathbf{B}_2 \left\langle \mathbf{u}(t) \right\rangle_{T_s}\right)}_{slope}$$

We can now get the net change over the switching period:

We have:

$$\mathbf{x}(dT_s) = \mathbf{x}(0) + \left(dT_s\right) \mathbf{K}^{-1}\left(\mathbf{A}_1 \left\langle \mathbf{x}(t) \right\rangle_{T_s} + \mathbf{B}_1 \left\langle \mathbf{u}(t) \right\rangle_{T_s}\right)$$

$$\mathbf{x}(T_s) = \mathbf{x}(dT_s) + \left(d'T_s\right) \mathbf{K}^{-1}\left(\mathbf{A}_2 \left\langle \mathbf{x}(t) \right\rangle_{T_s} + \mathbf{B}_2 \left\langle \mathbf{u}(t) \right\rangle_{T_s}\right)$$

Eliminate $\mathbf{x}(dT_s)$, to express $\mathbf{x}(T_s)$ directly in terms of $\mathbf{x}(0)$:

$$\mathbf{x}(T_s) = \mathbf{x}(0) + dT_s\mathbf{K}^{-1}\left(\mathbf{A}_1 \left\langle \mathbf{x}(t) \right\rangle_{T_s} + \mathbf{B}_1 \left\langle \mathbf{u}(t) \right\rangle_{T_s}\right) + d'T_s\mathbf{K}^{-1}\left(\mathbf{A}_2 \left\langle \mathbf{x}(t) \right\rangle_{T_s} + \mathbf{B}_2 \left\langle \mathbf{u}(t) \right\rangle_{T_s}\right)$$

Collect terms:

$$\mathbf{x}(T_s) = \mathbf{x}(0) + T_s\mathbf{K}^{-1}\left(d(t)\mathbf{A}_1 + d'(t)\mathbf{A}_2\right) \left\langle \mathbf{x}(t) \right\rangle_{T_s} + T_s\mathbf{K}^{-1}\left(d(t)\mathbf{B}_1 + d'(t)\mathbf{B}_2\right) \left\langle \mathbf{u}(t) \right\rangle_{T_s}$$

b. Average State Variable Over T_s by Time Weighting

We next calculate an average by weighting each state variable matrix by the appropriate duty cycle.

$$\vec{x} = [A_1 d + A_2 d'] \, \vec{x} + [B_1 d + B_2 d'] \, V_d$$

$$V_o = [C_1 d + C_2 d'] \, \vec{x}$$

Consider first a converter with two state variables: i_L and v_c. Also only consider CCM operation with only two time intervals D_1 and $(1\text{-}D_1)$. We find for external sources, u or v.

$\dot{x} = A_1 x + B_1 u$ for switch period D_1T_s
$\dot{x} = A_2 x + B_2 u$ for switch period D_2T_s

$(1-D_1) = D_2$

If we are operating where the ripple is triangular (f_{sw} very high) then all time derivatives are simply constants so that from t = 0 to $t = D_1T_s$

$x(\Delta t) = x(0) + \dot{x}\Delta t$, $\Delta t = D_1T_s$

$x(D_1T_s) = x(0) + (A_1 x + B_1 u)D_1T_s$

This x value is then the initial condition for the second switch period (1-D). So that at T_s

$x(T_s) = x(D_1T_s) + (A_2 x + B_2 u)D_2T_s$

$= x(0) + (A_1 x + B_1 u)D_1T_s + (A_2 x + B_2 u)D_2T_s$

Combining like terms

$x(T_s) = x(0) + [D_1A_1 + (1-D_1)A_2]xT_s + [D_1B_1 + (1-D_1)B_2]uT_s$

Notice the average matrices have been formed.

$\overline{A}$ (T_s average) $= D_1A_1 + (1-D_1)A_2 = \overline{A}$

$\overline{B}$ (T_s average) $= D_1B_1 + (1-D_1)B_2 = \overline{B}$

$\overline{A}$ and $\overline{B}$ are the duty cycle weighted averages of the state space average.

$x(T_s) = x(0) + [\overline{A}x + \overline{B}u]T_s$

That is, the averaged system equations over the switch period are $dx/dt = x(T_s) - x(0)$

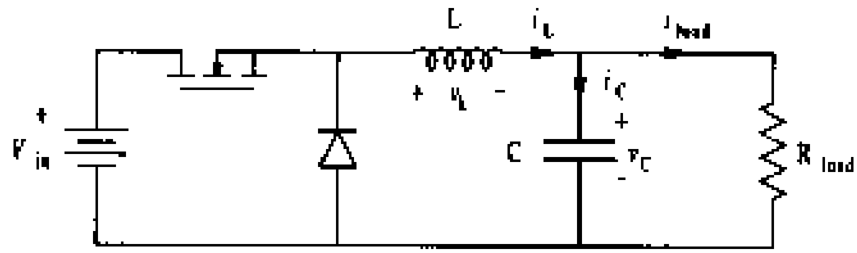
$\dot{x} = \overline{A}$ (T_s average)x + $\overline{B}$ (T_s average)u

This is the continuous approximation to the original switching system.

To recap,

1. Circuit equations are written for each switch state.
2. A weighted average of A and B matrices are made via duty cycle weighting.

Consider the buck converter below:



We write the state equations for L and C.

v_L(inductor) = $L di_L/dt$

For D_1T_s transistor on: $v_L = D_1V_{in} - V_c$

For D_2T_s diode on: $v_L = -V_c$

i_c(capacitor) = i_L - i(load) = Cdv_c/dt

Dividing i_c by L and v_L by C we obtain

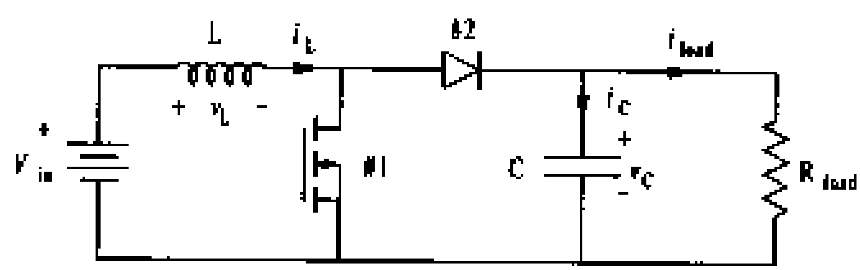
$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv_c}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{RC} \end{bmatrix} \begin{bmatrix} i_L \\ v_c \end{bmatrix} + \begin{bmatrix} \frac{D_1V_{in}}{L} \\ 0 \end{bmatrix}$$

$A_1=A_2=\overline{A}$ for Buck

B changes with switching

$$B_1u = \begin{bmatrix} \frac{V_{in}}{L} \\ 0 \end{bmatrix}, \quad B_2u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} \frac{D_1V_{in}}{L} \\ 0 \end{bmatrix}$$

Next consider the boost converter below:



We write state equations for L and C

v_L(inductor) = $L di_L/dt$

For D_1T_s transistor on: $v_L = V_{in}$

For D_2T_s diode on: $v_L = D_1V_{in} - V_c$

i_c(capacitor) = $D_2 i_L - V_c/R = C dv_c/dt$

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{-1}{RC} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{RC} \end{bmatrix}, \quad \overline{A} = \begin{bmatrix} 0 & \frac{-D_2}{L} \\ \frac{D_2}{C} & \frac{-1}{RC} \end{bmatrix}$$

Now $B_1 = B_2 = \overline{B} = \begin{bmatrix} \frac{V_{in}}{L} \\ 0 \end{bmatrix}$.

c. **General Quiescent Operation point and Small Signal Excursions valid only at that point**

$x \to X + \hat{x}, \; v_o \to V_o + \hat{v}_o, \; d \to D + \hat{d}$

Look familiar except for the matrix bookkeeping?
Then expand neglect all higher order terms, keeping only DC and AC terms.
For simplicity small signal variations $\hat{v}_d$ on v_d are assumed zero for now and $v_d \equiv V_d$. Using $\vec{x}$ equation:

1. $$\vec{x} = \underbrace{Ax + BV_d}_{\text{dc terms}} + \underbrace{\hat{x}[A] + \hat{d}[(A_1 - A_2)X}_{\text{ac terms}} + \underbrace{(B_1 - B_2)V_d]}_{\text{dc terms}}$$

A(average) $\equiv A_1D + A_2D'$ note: time average on A_1, A_2

B(average) $\equiv B_1D + B_2D'$ and the dc values x, V_d

<u>Steady State</u> $\vec{x} = 0 = Ax + BV_d$

$x = A^{-1} BV_d$

<u>AC Perturbation</u> $\hat{x} = \hat{x}A + d$ [average]

2. Next we use $V_o = [C_1 d + C_2 d'] \vec{x}$ and perturb/linearize

$$V_o + \hat{v}_o = \underbrace{CX}_{\text{dc term}} + \underbrace{[(C_1 - C_2)X]\hat{d}}_{\text{ac terms coefficient [ ] is time averaged}}$$

$C \equiv C_1 D + C_2 D'$

<u>Steady State</u> $V_o = CX = - C A' B V_d$

<u>AC Perturbation</u> $\hat{v}_o = C\hat{x} + [(C_1 - C_2)X]\hat{d}$

d. **Transform (time equations) into transfer Laplace transforms and transfer functions**

$$\hat{x} \rightarrow S\hat{x} = Ax(s) + [(A_1 - A_2)X + (B_1 - B_2)V_d] * \hat{d}(s)$$

$$\hat{x}(s) = \underbrace{[sI - A]^{-1}}_{\text{unity matrix}}[(A_1 - A_2)x + (B_1 - B_2)v_d]\hat{d}(s)$$

$$\hat{v}_o(s) = C[sI - A]^{-1}[(A_1 - A_2)x + (B_1 - B_2)v_d]\hat{d}(s) + [(C_1 - C_2)X]$$

$$\frac{\hat{v}_o(s)}{\hat{d}(s)} = C[sI - A]^{-1}[(A_1 - A_2)X + (B_1 - B_2)V_d] + (C_1 - C_2)X$$

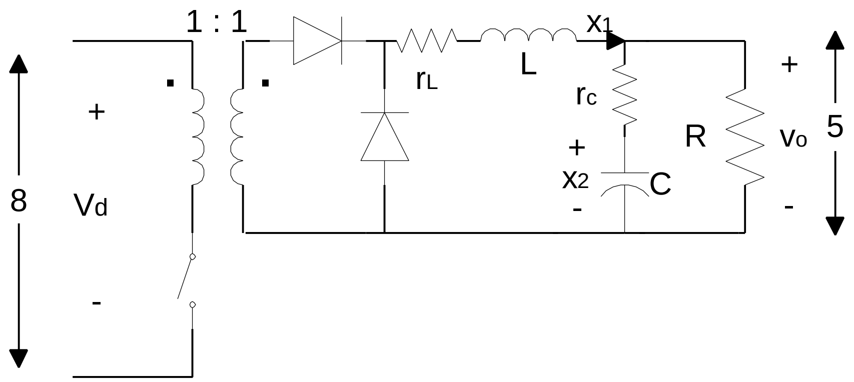
In summary, the time domain forms:

$$K\frac{d\hat{x}(t)}{dt} = A\,\hat{x}(t) + B\,\hat{u}(t) + \{(A_1 - A_2)X + (B_1 - B_2)U\}\hat{d}(t)$$

$$\hat{y}(t) = C\,\hat{x}(t) - E\,\hat{u}(t) + \{(C_1 - C_2)X + (E_1 - E_2)U\}\hat{d}(t)$$

can easily be changed to Laplace form and $\dfrac{\hat{x}(s)}{\hat{d}(s)}$ calculated or any other transfer function of interest.

1. Lets try the forward converter with equivalent series resistance's for both C and L - r_c and r_L. x1 $\rightarrow$ i_L and x_2 $\rightarrow$ V_c as shown below



V_d=8V V_o=5V
r_L=20m ohms
L=5uH
r_c=10m ohms
C=2,000uF
R=200m ohms
f_s=200kHz

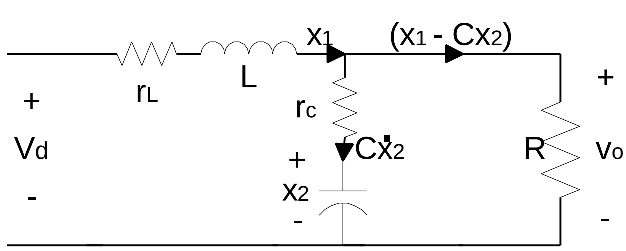
During DT_s (switch on) we obtain state loop equations

$$-V_d + L\dot{x}_1 + r_L x_1 + R(x_1 - C\dot{x}_2) = 0 \qquad \text{(KVL \#1)}$$

\---

$$-x_2 - C r_c \dot{x}_2 + R(X_1 - C\dot{x}_2) = 0 \qquad \text{(KVL \#2)}$$

The simplified secondary circuit is shown below to aid understanding.



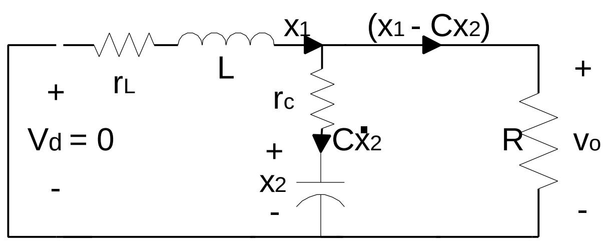
Now use the standard state space form.

$$\begin{array}{ccccc} \dot{x} = & [A_1] & [x_1] + & B_1 & V_d \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{Rr_c + Rr_L + r_c r_L}{L(R+r_c)} & -\dfrac{R}{L(R+r_c)} \\ \dfrac{R}{C(R+r_c)} & -\dfrac{1}{C(R+r_c)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix} V_d$$

During D'T$_s$ (switch off series diode on)

$A_2 = A_1 \quad B_2 = 0$



V$_o$ <u>for both</u> DT$_s$ and D'T$_s$ intervals is $R(x_1 - C\dot{x}_2)$

$$V_o = R(x_1 - C\dot{x}_2) = \frac{R\,r_c}{R+r_c}x_1 + \frac{R}{R+r_c}x_2$$

$$V_o = \begin{bmatrix} \dfrac{R\,r_c}{R+r_c} & \dfrac{R}{R+r_c} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

During both DT$_s$ and D'T$_s$

$$V_o = C_1\vec{x} = C_2\vec{x}$$

Time Averaged Matrices over the switch period T$_s$ are obtained as follows.

$A = A_2 = A_1$ from $AD + (1-D)A = A$

$B = DB_1$

$C = C_1 = C_2$

Using the simplification $R \gg r_c + r_L$ we know from circuit values

$$A \rightarrow \begin{bmatrix} \frac{-r_c + r_2}{L} & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{CR} \end{bmatrix}$$, $C \rightarrow [\, r_c \; 1]$, B is the same - no resistors

Steady State or DC Conditions

$0 = Ax + BV_d \rightarrow x = A^{-1}\,BV_d$

$V_o = Cx = -CA^{-1}\,BV_d$

$$\frac{V_o}{V_d} = -C\,A^{-1}B.$$ Where $C = [r_c 1]$, $B = \begin{bmatrix} \frac{D}{L} \\ 0 \end{bmatrix}$

and $$A^{-1} = \frac{LC}{1 + \frac{(r_c + r_L)}{R}} \begin{bmatrix} -\frac{1}{CR} & \frac{1}{L} \\ -\frac{1}{C} & \frac{(r_c + r_L)}{L} \end{bmatrix}.$$ Multiplying out we find for the DC case:

$$\frac{V_o}{V_d} = D\,\frac{R + r_c}{R + r_c + r_L} \approx D$$

AC Small Signal Model

$$s\hat{x}(s) = A\hat{x}(s) + \hat{d}(s)\left[\,(A_1 - A_2)\,X + (B_1 - B_2)\,V_d\right]$$

$$\hat{x}(s) = [sI - A\,]^{-1}\left[(A_1 - A_2)\,X + (B_1 - B_2)\,V_d\right]\hat{d}(s)$$

$$\hat{V}_o(s) = C\hat{x}(s) + [(C_1 - C_2)X]\hat{d}(s)$$

$$\uparrow \quad [sI - A]^{-1} \begin{bmatrix} (A_1 - A_2)X \\ + \\ (B_1 - B_2)V_d \end{bmatrix} \hat{d}(s)$$

$$\frac{\hat{V}_o(s)}{\hat{d}(s)} = [sI - A]^{-1} \begin{bmatrix} (A_1 - A_2)X \\ + \\ (B_1 - B_2)V_d \end{bmatrix} \hat{d}(s) + [(C_1 - C_2)X]$$

$$= V_d \frac{1 + sr_cC}{LC[s^2 + s(\frac{1}{RC + \frac{(r_c + r_L)}{L}}) \frac{1}{LC}]}$$

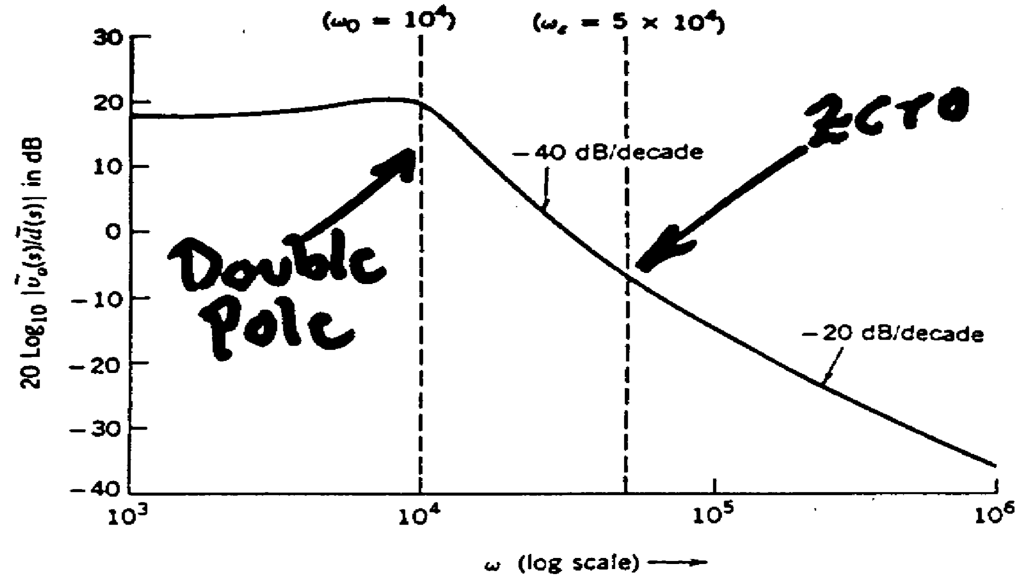
Please note that the forward converter transfer function has the following properties:

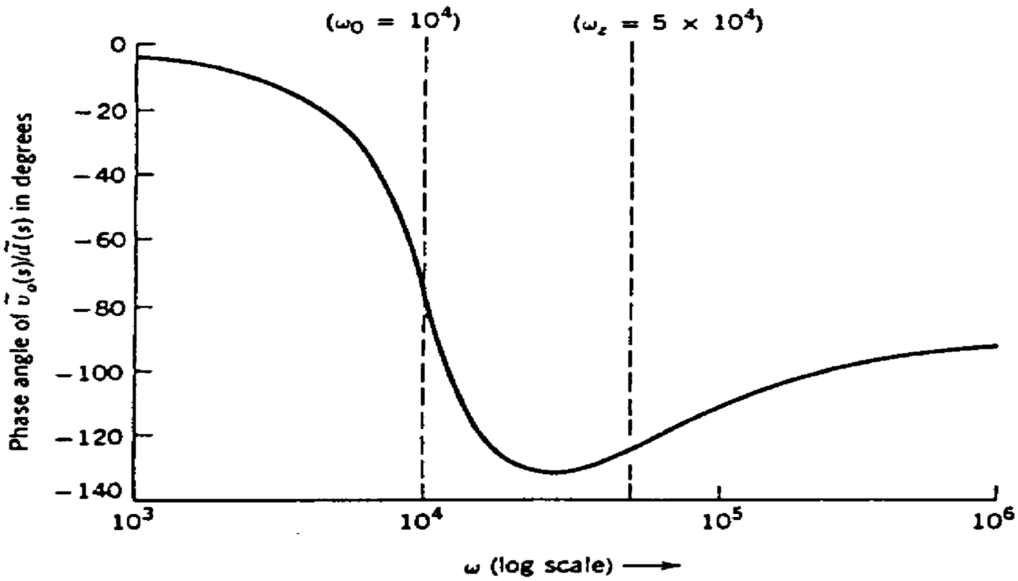
1. single zero @ $w_z = \frac{1}{r_cC}$

2. double pole @ $w_o = \frac{1}{\sqrt{LC}}$

3. Since r_cC is usually very small $\Rightarrow w_z > w_o$

The general shape of V_o/d versus frequency is then plotted below on page 20.

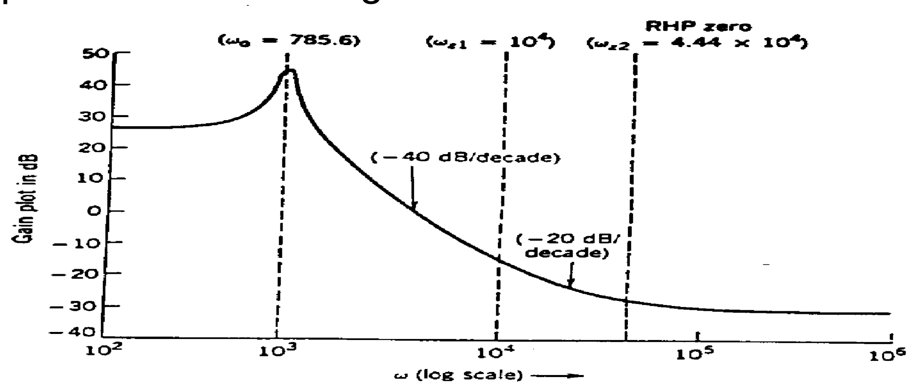


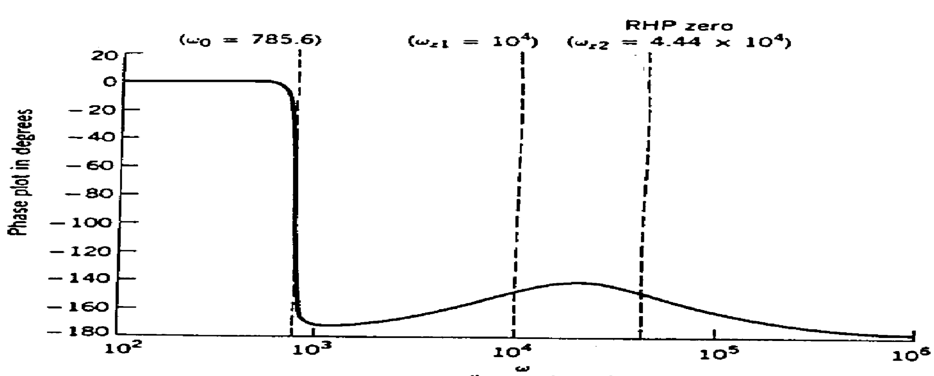


Looking ahead to feedback conditions, for avoiding oscillation we want 76° phase margin @ unity gain in the open loop part of the gain. That is the actual phase angle of the $\frac{v_o}{\hat{d}}$ minus 180° should be $\geq$ 76°. $76° \geq \phi - 180°$

2. Looking even further ahead to the flyback converter operating in CCM we will find: $\frac{\hat{V}_o(s)}{\hat{d}(s)} = V_d\ f(D)\ \frac{(1+s\,w_{z1})(1-s/\,w_{z2})}{as^2+bs+c}$. That is the flyback converter has a RHP zero w = f (R, L* f(0)). **This makes for a very unstable situation,** as shown in the open loop $\frac{v_o}{\hat{d}}$ plots below. In stark contrast the flyback operating DCM does not have right half plane zero! From $\frac{V_o}{\hat{d}} \sim (1-s/\,w_z)$ can you make the case that the output has an undesirably high phase lag at high f?

Flyback converter Open Loop Bode Plots have some oddities in that the low f gain is non - linear. This implies that the precise frequency range with -40 db/decade drop depends on the low f gain.

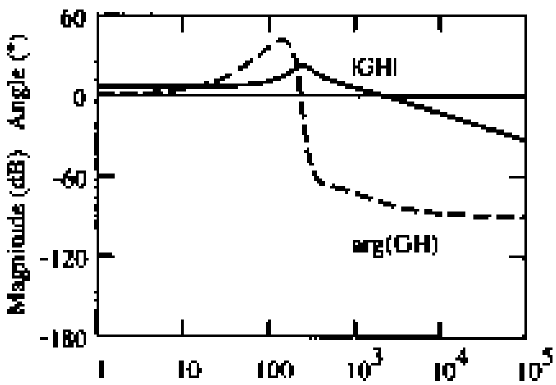




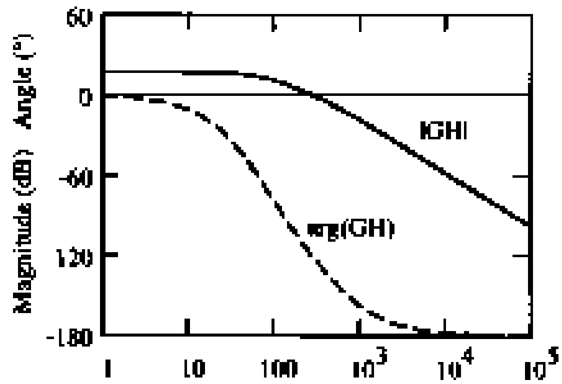
Notice in the phase plot we do not have enough ϕ margin and the open loop system response tells we come close to instability at unity gain $\phi \rightarrow 180°$. This is undesirable.

3. **Summary of Phase Margin in Open Loop Plots**

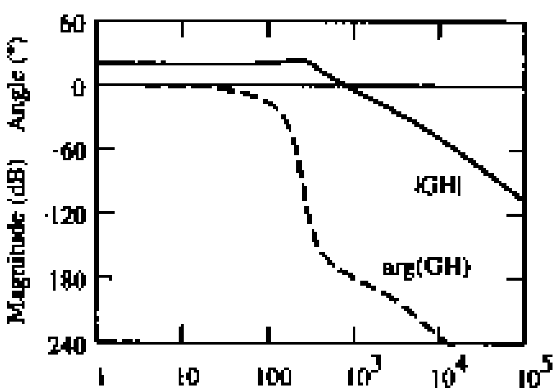
Below we show four different open-loop Bode plots. **For HW#3 please tell which are stable and which are unstable.**



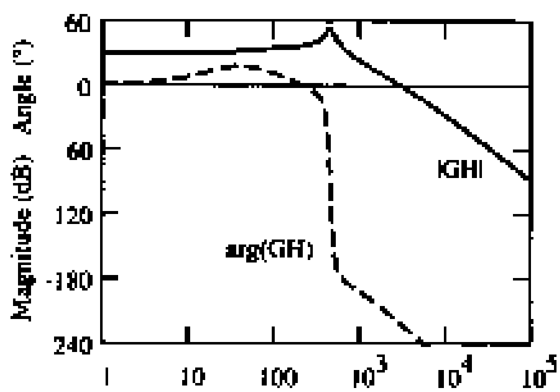
a) 100 degree phase margin



b) 60 degree phase margin



c) 5 degree phase margin

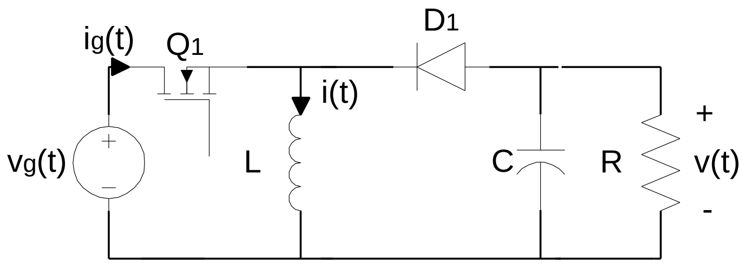


d) Open-loop unstable

What occurs in (c) for small component variations, temperature changes, component aging? Most designers prefer a phase margin of 60° to address these problems. We will visit this in detail in upcoming lectures.

4. Further Examples
Lossy Switch

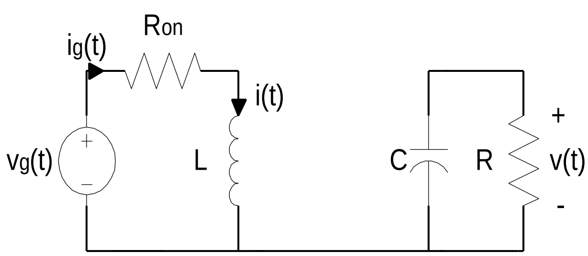
a. Buck - Boost Erickson's text pages 212 - 217

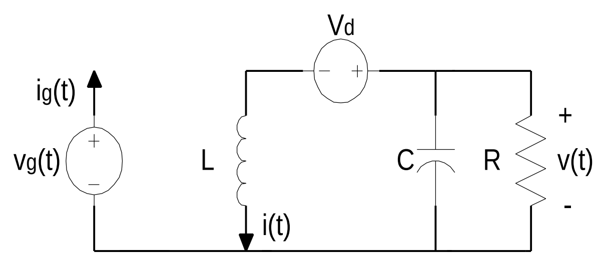


Trans on $\rightarrow$ R_{on} } DC losses
Diode on $\rightarrow$ V_D } $[V] = [V_g, V_D]$

DTs $\qquad [x] = \begin{bmatrix} i \\ v \end{bmatrix}$ $\qquad$ D'Ts

$[y] = [i_g]$





$$\underbrace{\begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix}}_{K} \underbrace{\frac{d}{dt}\begin{bmatrix} i(t) \\ v(t) \end{bmatrix}}_{\frac{dx(t)}{dt}} = \underbrace{\begin{bmatrix} -R_{on} & 0 \\ 0 & -\frac{1}{R} \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} i(t) \\ v(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{B_1} \underbrace{\begin{bmatrix} v_g(t) \\ V_D \end{bmatrix}}_{u(t)} \qquad \underbrace{\begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix}}_{K} \underbrace{\frac{d}{dt}\begin{bmatrix} i(t) \\ v(t) \end{bmatrix}}_{\frac{dx(t)}{dt}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -\frac{1}{R} \end{bmatrix}}_{A_2} \underbrace{\begin{bmatrix} i(t) \\ v(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}}_{B_2} \underbrace{\begin{bmatrix} v_g(t) \\ V_D \end{bmatrix}}_{u(t)}$$

$$[i_g] = [1\ \ 0]\begin{bmatrix} i(t) \\ v(t) \end{bmatrix} + [0\ \ 0]\begin{bmatrix} V_g \\ V_D \end{bmatrix}$$

$$[i_g] = [0\ \ 0]\begin{bmatrix} i(t) \\ v(t) \end{bmatrix} + [0\ \ 0]\begin{bmatrix} v_g \\ V_D \end{bmatrix}$$

↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓

y(t) C_1 x(t) E_1 u(t) y(t) C_2 x(t) E_2 u(t)

Next, we get the $\langle\rangle_{Ts}$ average matrices A, B, C, E, by time weighting by D and D'.

$$A = DA_1 + D'A_2 = D\begin{bmatrix} -R_{on} & 0 \\ 0 & -\frac{1}{R} \end{bmatrix} + D'\begin{bmatrix} 0 & 1 \\ -1 & -\frac{1}{R} \end{bmatrix} = \begin{bmatrix} -DR_{on} & D' \\ -D' & -\frac{1}{R} \end{bmatrix}$$

$$B = DB_1 + D'B_2 = \begin{bmatrix} D & -D' \\ 0 & 0 \end{bmatrix}$$

$$C = DC_1 + D'C_2 = [D\ \ \ 0]$$

$$E = DE_1 + D'E_2 = [0\ \ \ 0]$$

$$\begin{aligned} K\dot{\vec{x}} &= A_{av}\vec{x} + B_{av}\vec{v} \quad \} \qquad x + X_o + \hat{x} \\ & \qquad\qquad\qquad\quad\ \ \} \qquad y = Y_o + \hat{y} \\ \vec{y} &= C_{av}\vec{x} + E_{av}\vec{v} \quad \} \qquad d = D_o + \hat{d} \end{aligned}$$

1. Steady State Solutions: $\dot{\vec{x}} = 0$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -DR_{on} & D' \\ -D' & -\frac{1}{R} \end{bmatrix}\begin{bmatrix} I \\ V \end{bmatrix} + \begin{bmatrix} D & -D' \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_g \\ V_D \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} I \\ V \end{bmatrix} = [A_{av}]^{-1} [B] \begin{bmatrix} V_g \\ V_D \end{bmatrix}$$

$$[I_g] = [D\ \ 0] \begin{bmatrix} I \\ V \end{bmatrix} + [0\ \ 0] \begin{bmatrix} V_g \\ V_D \end{bmatrix} \quad \Rightarrow \quad [I_g] = [C_{av}] \begin{bmatrix} I \\ V \end{bmatrix}$$

$$[I_g] = C_{av} A_{av}^{-1} B \begin{bmatrix} \frac{V_g}{V_D} \end{bmatrix}$$

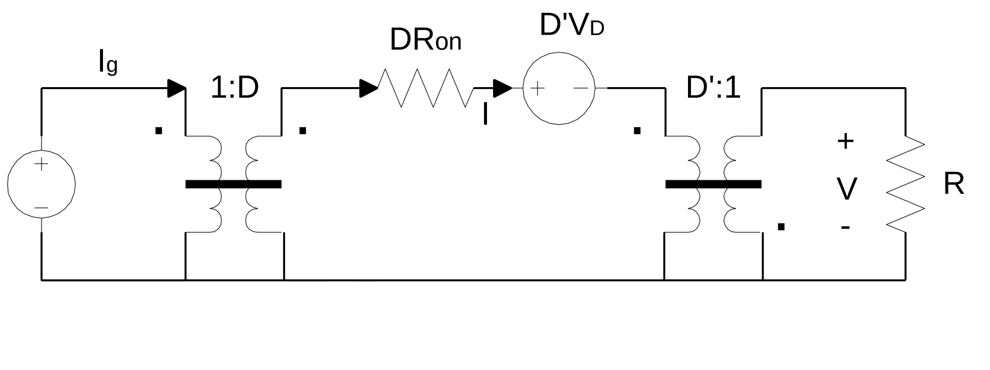
<u>dc solutions:</u>

Equation for I,V output and Input I_g

$$\begin{bmatrix} I \\ V \end{bmatrix} = [A_{av}]^{-1} [B] \begin{bmatrix} V_g \\ V_D \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1+\dfrac{DR_{on}}{D'^2 R}} \end{bmatrix} \begin{bmatrix} \dfrac{D}{D'^2 R} & \dfrac{1}{D'R} \\ -\dfrac{D}{D'} & 1 \end{bmatrix} \begin{bmatrix} V_g \\ V_D \end{bmatrix}$$

$$[I_g] = C_{av} A_{av}^{-1} B \begin{bmatrix} \frac{V_g}{V_D} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1+\dfrac{DR_{on}}{D'^2 R}} \end{bmatrix} \begin{bmatrix} \dfrac{D^2 D}{D'^2 R D'R} \end{bmatrix} \begin{bmatrix} V_g \\ V_D \end{bmatrix}$$

DC Model Circuit



2. **AC Perturbed / Linear Solutions**

$$K\hat{x} = A\hat{x} + B\hat{u} + \begin{bmatrix}(A_1 - A_2)X + \\ (B_1 - B_2)u\end{bmatrix}\hat{d}$$

$$\hat{y} = C\hat{x} + E\hat{u} + \begin{bmatrix}(C_1 - C_2)X + \\ (E_1 - E_2)u\end{bmatrix}\hat{d}$$

for [ ] in front of $\hat{d}$ terms

$$(A_1 - A_2)X + (B_1 - B_2)U = \begin{bmatrix}-V \\ I\end{bmatrix} + \begin{bmatrix}V_g - IR_{on} + V_D \\ 0\end{bmatrix} = \begin{bmatrix}V_g - V - IR_{on} + V_D \\ I\end{bmatrix}$$

in $\hat{x}$

$(C_1 - C_2)X + (E_1 - E_2)U = [I]$ in $\hat{y}$

In standard state variable form the matrix math is:

$$\begin{bmatrix}L & 0 \\ 0 & C\end{bmatrix}\frac{d}{dt}\begin{bmatrix}\hat{i}(t) \\ \hat{v}(t)\end{bmatrix} = \begin{bmatrix}-DR_{on} & D' \\ -D' & -\frac{1}{R}\end{bmatrix}\begin{bmatrix}\hat{i}(t) \\ \hat{v}(t)\end{bmatrix} + \begin{bmatrix}D & -D' \\ 0 & 0\end{bmatrix}\begin{bmatrix}\hat{v}_g(t) \\ \hat{v}_D(t)\end{bmatrix} + \begin{bmatrix}V_g - V - IR_{on} + V_D \\ I\end{bmatrix}\hat{d}$$

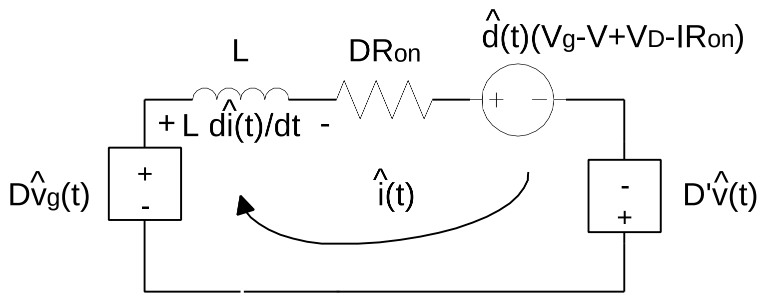
$$[\hat{i}_g(t)] = [D\ 0]\begin{bmatrix}\hat{i}(t) \\ \hat{v}(t)\end{bmatrix} + \begin{bmatrix}0 & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}\hat{v}(t) \\ v_D(t)\end{bmatrix} + \begin{bmatrix}0 \\ I\end{bmatrix}\hat{d}(t)$$

Hard to get a “physical hold” on matrix equations.
In fact, we cannot easily get a complete circuit model of the whole converter for ac analysis. We can only separate out each set of three equations and then get a partial model.
Equation:

$$L\frac{d\hat{i}(t)}{dt} = D'\hat{v}(t) - DR_{on}\hat{i}(t) + D\hat{v}_g(t) + (V_g - V - IR_{on} + V_D)\hat{d}(t)$$

$$sL\hat{i}(s) = D'\hat{v}(s) - DR_{on}\hat{i}(s)$$
$$+ D\hat{V}_g(s) + (V_g - V - IR_{on} + V_D)\hat{d}(s)$$

The circuit representing these equations:

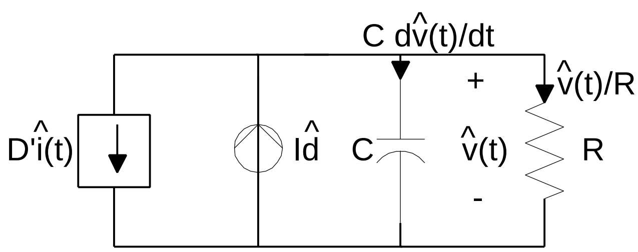


Equation:

$$C\frac{d\hat{v}(t)}{dt} = -D'\,\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\,\hat{d}(t)$$

$$Cs\hat{v}(s) = -D'\hat{i}(s) - \frac{\hat{V}(s)}{R} + I\hat{d}(s)$$

Circuit:

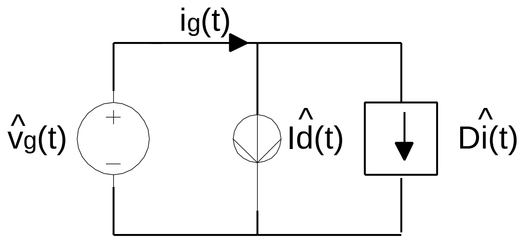


Equation:

$$\hat{i}_g(t) = D\,\hat{i}(t) + I\,\hat{d}(t)$$

$$\hat{i}_g(s) = D\,\hat{i}(s) + I\,\hat{d}(s)$$

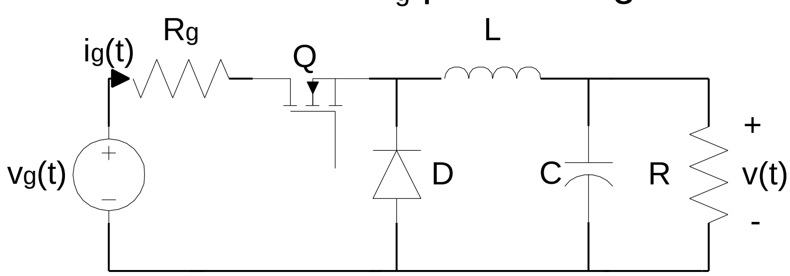
Circuit:



Later we will get to employ $\frac{\hat{V}(s)}{\hat{d}(s)}$ in a complex feedback loop so it's open loop form is good to know well at this point.

b. Erickson Problem 7.8

Buck converter with V_g possessing a source impedance R_g

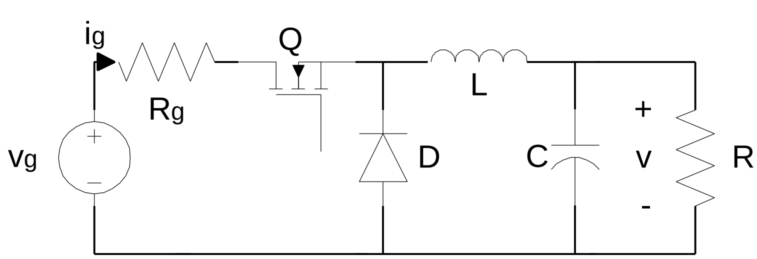


[x] $\rightarrow$ i_L energy terms } V is V_c a state vector
Vc

[U] $\rightarrow$ V_g independent inputs

[y] $\rightarrow$ i_g dependent output

DT_s (Q_{on}) :



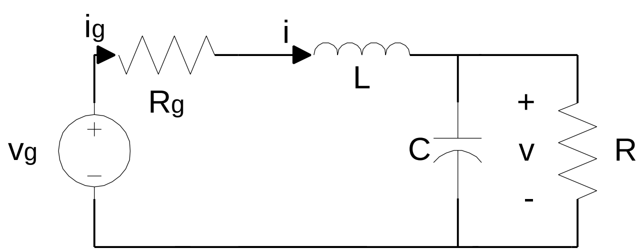
$$\begin{array}{cccccc} K & \vec{x} & A_1 & X & B_1 & U \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

$$\begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i \\ v \end{bmatrix} = \begin{bmatrix} -R_g & -1 \\ -1 & \frac{1}{R} \end{bmatrix} \begin{bmatrix} i \\ v \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} V_g \end{bmatrix}$$

$$[i_g] = [1 \;\; 0] \begin{bmatrix} i \\ v \end{bmatrix} + [0]\,[V_g]$$

$$\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ y & C_1 & E_1 & U \end{array}$$

$D'T_s$ (Q off/ D on) :



$$\begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i \\ v \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -\frac{1}{R} \end{bmatrix} \begin{bmatrix} i \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} [V_g]$$

$$\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ k & \vec{x} & A_2 & B_2 \end{array}$$

$$[i_g] = [0 \;\; 0] \begin{bmatrix} i \\ v \end{bmatrix} + [0][V_g]$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ y & C_2 & E_2 \end{array}$$

Get time - averaged A, B, C, E

$$A = D\begin{bmatrix} -R_g & -1 \\ 1 & -1/R \end{bmatrix} + D'\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} -DR_g & -1 \\ 1 & -1/R \end{bmatrix}$$

$$B = D\begin{bmatrix} 1 \\ 0 \end{bmatrix} + D'\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} D \\ 0 \end{bmatrix}$$

C = D[1 0] + D'[0 0] = [0 0]

E = D[0 0] + D'[0 0] = [0]

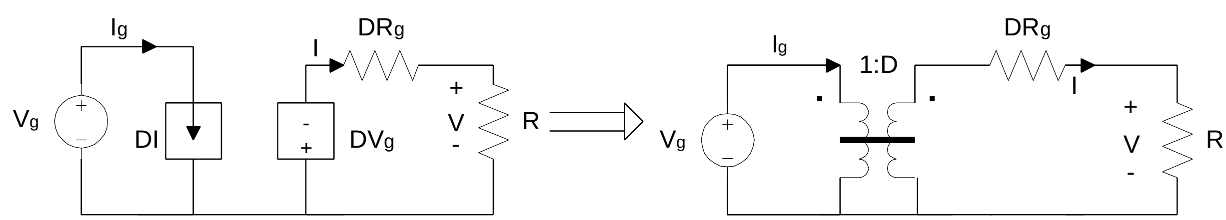
<u>DC Relations:</u>

$$0=A\underline{x}+B\underline{u} \quad \{\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{DR_g}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}\begin{bmatrix} I \\ V \end{bmatrix} + \begin{bmatrix} \frac{D}{L} \\ 0 \end{bmatrix} + \begin{bmatrix} V_g \end{bmatrix}$$

$$\underline{y}=C\underline{x}+E\underline{u} \quad \{\begin{bmatrix} I_g \end{bmatrix} = \begin{bmatrix} D & 0 \end{bmatrix}\begin{bmatrix} I \\ V \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} + \begin{bmatrix} V_g \end{bmatrix}$$

<u>OR:</u> $I_g = DI$, $\quad 0 = -DR_gI - V + DV_g \quad \& \quad 0 = I - V/R$

<u>Equivalent DC Circuit Model:</u>



$\Rightarrow$ <u>AC Relations:</u>

$$(A_1 - A_2)\underline{x} + (B_1 - B_2)\underline{u} = \begin{bmatrix} \frac{V_g - IR_g}{L} \\ 0 \end{bmatrix}, (C_1 - C_2)\underline{x} + (E_1 - E_2)\underline{u} = [I]$$

$$\frac{d}{dt}\begin{bmatrix}\hat{i}\\ \hat{v}\end{bmatrix} = \begin{bmatrix} -\frac{DR_g}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC}\end{bmatrix}\begin{bmatrix}\hat{i}\\ \hat{v}\end{bmatrix} + \begin{bmatrix}\frac{D}{L}\\ 0\end{bmatrix}\begin{bmatrix}\hat{v}_g\end{bmatrix} + \begin{bmatrix}\frac{V_g - IR_g}{L}\\ 0\end{bmatrix}\begin{bmatrix}\hat{d}\end{bmatrix}$$

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

$\frac{d}{dt}\hat{x} =$ A $\underline{\hat{x}}$ + B $\underline{\hat{u}}$ + $[(A_1 - A_2)x + (B_1 - B_2)u]\hat{d}$

$$\begin{bmatrix}\hat{i}_g\end{bmatrix} = \begin{bmatrix}D & 0\end{bmatrix}\begin{bmatrix}\hat{i}\\ \hat{v}\end{bmatrix} + \begin{bmatrix}0\end{bmatrix}\begin{bmatrix}\hat{v}_g\end{bmatrix} + \begin{bmatrix}I\end{bmatrix}\begin{bmatrix}\hat{d}\end{bmatrix}$$

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

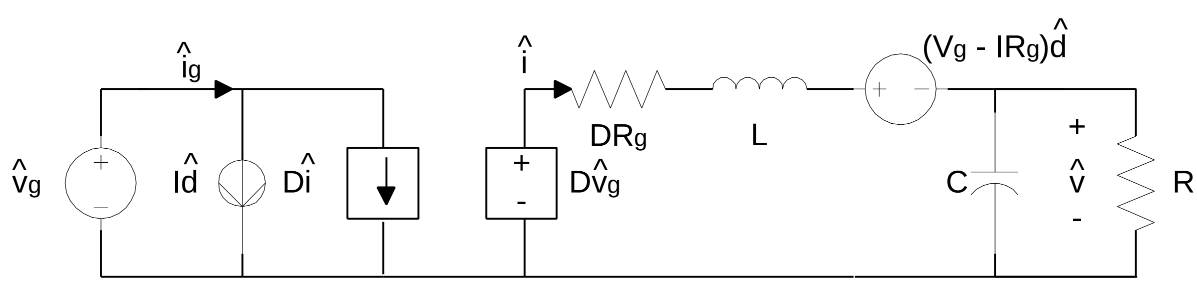
$\underline{\hat{y}} =$ C $\underline{\hat{x}}$ + E $\underline{\hat{u}}$ + $[(C_1 - C_2)x + (E_1 - E_2)u]\hat{d}$

<u>OR:</u>

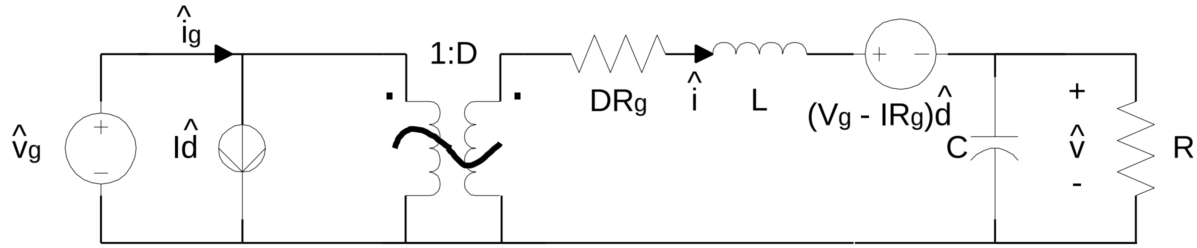
$$L\,\frac{d\hat{i}}{dt} = -DR_g\,\hat{i} - \hat{v} + (V_g - IR_g)\,\hat{d}$$

$$C\,\frac{d\hat{v}}{dt} = -\hat{i}(t) - \frac{\hat{v}}{R} \qquad \& \qquad \hat{i}_g = D\,\hat{i} + I\,\hat{d}$$

<u>Equivalent AC circuit:</u>



Ac/dc transformer model:



$$\frac{\hat{v}}{\hat{d}} \approx \left(V_g - IR_g\right)\left(\frac{R}{R+DR_g}\right)\frac{1}{1+s\left(\frac{L}{R+DR_g}+CR\|DR_g\right)+s^2\left(\frac{LCR}{R+DR_g}\right)}$$

$$Ls\hat{i}(s) = - DR_g\hat{i}(s) - \hat{v}(s) + [V_g(s) - IR_g]*\hat{d}(s)$$

$$Cs\hat{v}(s) = \hat{i}(s) - \frac{\hat{V}(s)}{R}$$

$$\hat{i}_g(s) = D\hat{i}(s) + I\hat{d}(s)$$

Solve for $\frac{\hat{V}(s)}{\hat{d}(s)}$