Digital Image Processing
Lectures 21 & 22

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Spring 2015
Transform Domain Operations

Idea is to map an image, using a unitary transform, to the *feature domain* where the desired features are more localized. The filtering is then carried out in the transform domain using a “zonal filter”. The inverse unitary mapping then converts the result back to the spatial domain (see Fig. 1). The transform domain operation provides better insight about the features that need to be removed or enhanced in the image.

The filtering is performed on the transformed image using algebraic operations i.e.

\[
Y(k, l) = H(k, l)X(k, l)
\]

where \(X(k, l) = \mathcal{A}\{x(m, n)\}\) and \(Y(k, l) = \mathcal{A}\{y(m, n)\}\) are the 2-D unitary transforms of the original and the filtered images, \(x(m, n)\) and \(y(m, n)\), respectively, \(H(k, l) = \mathcal{A}\{h(m, n)\}\) is the response of the 2-D zonal filter, i.e. the unitary transform of the impulse response, \(h(m, n)\), and \(\mathcal{A}\{.\}\) is the unitary mapping.
Then, inverse unitary transformation gives

\[ y(m, n) = A^{-1}\{Y(k, l)\} \]

The simplest filter that can be constructed is an ideal 2–D filter.

\[ H_I(\Omega_1, \Omega_2) = \begin{cases} 
1, & D(\Omega_1, \Omega_2) \leq D_0 \\
0, & otherwise 
\end{cases} \]

where \( D_0 \) is the cut-off frequency. For \textit{circularly symmetric} ideal filters, \( D(\Omega_1, \Omega_2) = \sqrt{(\Omega_1^2 + \Omega_2^2)} \). The choice of \( D_0 \) presents a trade-off between noise removal ability and the blurring artifacts. Fig. 2 (a) shows the sampled frequency response of an ideal 2–D LPF, i.e. \( H_I(k, l) = H_I(\Omega_1, \Omega_2)|_{\Omega_1 = \frac{2\pi k}{N_1}, \Omega_2 = \frac{2\pi l}{N_2}} \). Taking the IDFT of \( H_I(k, l)'s \) of size \( N_1 \times N_2 \) yields the impulse response (Fig. 2(b)) or the filter coefficients, \( h_I(m, n) = IDFT\{H_I(k, l)\}'s \), for the corresponding 2–D FIR filter of order \( N_1 \times N_2 \) (frequency sampling approach for the design of 2–D FIR filters).
Bandwidth $D_0$ can be chosen by specifying $\%$ energy, $\beta$, retained in the filtered image wrt total energy, i.e.

$$\beta = \left[\frac{\sum \sum_{k,l \in S_0} E(k,l)}{\sum \sum_{k=0}^{N-1} E(k,l)}\right] \times 100\%$$

$E(k,l) = |X(k,l)|^2$ and $S_0$ is the set of $(k,l)$ values within the passband region ($N_1 = N_2 = N$).

Figs. 3 (a)-(f) show the filtered Peppers images and the corresponding DFTs for increasing values of $D_0$ namely 20%, 50% and 70% of the total radius (i.e. $N$).
For $D_0 = 0.2$ more than 99% of the energy is retained while the filtered image still exhibits some blurring (loss of high frequency components). As $D_0$ increases blurring reduces but the *ringing* artifacts due to sharp cutoff are still there. As $D_0$ increases the number of rings in a given region increases, while producing more finely spaced rings. Note that 100% energy is obtained when $D_0 = \frac{\sqrt{2}}{2} N$. 

$D_0 = 20\% \quad \beta = 99.12\%$
$D_0 = 50\%$  \hspace{1cm} $\beta = 99.73\%$

$D_0 = 70\%$  \hspace{1cm} $\beta = 99.84\%$

Figure 3: Filtered Images and Magnitude Spectra.
Ringing effects can be circumvented by designing filters that exhibit smooth transition region using the *windowing* method for FIR filter design. Figs. 4(a) and (b) show the 2-D frequency response of a filter and the contour plot of its impulse response designed by windowing the impulse response of the ideal filter for $D_0 = 0.2$ using a Hamming window. The filter possesses a smooth cut-off characteristic as desired. Figs. 4(c) and (d) show the resultant filtered Peppers image and its DFT, respectively. Although the blurring effect is still noticeable, no ringing is evident in the processed image.

Figure 4: Frequency Response & Impulse Response Contour of the Designed Filter.
Remarks:

1. To avoid circular convolution results implement the zero-padding procedure (see Lecture 10) prior to taking DFT and transform domain filtering.

2. Other non-ideal filters that can be used to reduce the ringing artifacts are 2D Butterworth and trapezoidal filters.
For Low-pass Butterworth filter transfer function is:

\[ H(\Omega_1, \Omega_2) = \frac{1}{1 + (\sqrt{2} - 1)[D(\Omega_1, \Omega_2)/D_0]^{2n}} \]

where \( n \) is the order of the filter that determines the sharpness of the response. At \( D(\Omega_1, \Omega_2) = D_0 \) we have \( H(\Omega_1, \Omega_2) = 1/\sqrt{2} \) i.e. the cut-off point.

For low-pass trapezoidal filter, we have

\[
H(\Omega_1, \Omega_2) = \begin{cases} 
1 & D(\Omega_1, \Omega_2) \leq D_0 \\
\frac{1}{D_0-D_1}[D(\Omega_1, \Omega_2) - D_1], & D_0 \leq D \leq D_1 \\
0 & \text{otherwise}
\end{cases}
\]

**Example 1:**
Consider the problem of approximating an ideal LPF with frequency response

\[ H_I(\Omega_1, \Omega_2) = \begin{cases} 
1, & |\Omega_1| < a < \pi, |\Omega_2| < b < \pi \\
0, & \text{otherwise}
\end{cases} \]

by a filter whose impulse response is
\[ h_D(m,n) = \begin{cases} 
A, & m = n = 0 \\
B, & m = \pm 1, n = 0 \\
C, & m = 0, n = \pm 1 \\
0, & \text{otherwise} 
\end{cases} \]

What values of unknown \( A, B, \) and \( C \) will minimize the design objective function

\[ \varepsilon = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |H_I(\Omega_1, \Omega_2) - H_D(\Omega_1, \Omega_2)|^2 d\Omega_1 d\Omega_2 \]

**Solution:**

From the definition of \( h_D(m,n) \)

\[ h_D(m,n) = A\delta(m,n) + B(\delta(m-1,n) + \delta(m+1,n)) + C(\delta(m,n-1) + \delta(m,n+1)) \]

which gives

\[ H_D(\Omega_1, \Omega_2) = A + 2B \cos \Omega_1 + 2C \cos \Omega_2 \]

Now, taking partial derivatives wrt \( A, B \) and \( C \) and using the fact that area under \( \cos \) is zero within \([-\pi, \pi]\) gives
\[ \frac{\partial \varepsilon}{\partial A} = 0 \implies A = \frac{\int_{-a}^{a} \int_{-b}^{b} d\Omega_1 d\Omega_2}{\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} d\Omega_1 d\Omega_2} = \frac{ab}{\pi^2} \]

\[ \frac{\partial \varepsilon}{\partial B} = 0 \implies B = \frac{\int_{-a}^{a} \int_{-b}^{b} \cos \Omega_1 d\Omega_1 d\Omega_2}{2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cos^2 \Omega_1 d\Omega_1 d\Omega_2} = \frac{b \sin a}{\pi^2} \]

\[ \frac{\partial \varepsilon}{\partial C} = 0 \implies C = \frac{\int_{-a}^{a} \int_{-b}^{b} \cos \Omega_2 d\Omega_1 d\Omega_2}{2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cos^2 \Omega_2 d\Omega_1 d\Omega_2} = \frac{a \sin b}{\pi^2} \]

**Remark:**

This method is referred to as the method of *Least Squares* for optimal FIR filter design.

**2-D Matched Filtering in Transform Domain**

Matched filtering can be carried in the transform domain using 2-D DFT, i.e.

\[ C_{x,t}(k, l) = X(k, l)T^*(k, l) \]

where \( C_{x,t}(k, l) \) (cross-power spectrum) and \( T(k, l) \) are 2-D DFTs of \( c_{x,t}(p, q) \) and \( t(p, q) \), respectively and \( * \) stands for complex conjugate operation.
Idea behind root filtering and Cepstral (juxtaposition of first four letters of the word Spectral) is to apply a nonlinear mapping to the magnitude spectrum in order to redistribute the energy of image in the transform domain while the phase is unchanged. Several types of nonlinear mapping functions can be employed for different tasks. In root filtering the magnitude component is raised to a power $\alpha < 1$ while the phase component is retained (see Fig. 6) i.e.

$$Y(k, l) = |X(k, l)|^\alpha \exp(j\theta(k, l)), \quad 0 \leq \alpha \leq 1$$

This operation tends to deemphasize those frequency components with large magnitude while boosting the magnitude of those that have smaller contribution.
Thus, high frequency components corresponding to edges and textural features that generally have small magnitudes will be emphasized in comparison with those of the lower frequencies. This will yield an image with enhanced edges and hence better visual appearance.

Another useful nonlinear transformation consists of taking the log of the magnitude function while keeping the phase the same, i.e.

$$S(k, l) = \log |X(k, l)| \exp(j\theta(k, l))$$

Now, the inverse transform of $S(k, l)$, denoted by $c(m, n)$ is called the generalized Cepstrum or generalized Homomorphic transform of $x(m, n)$ (see Fig. 7 for inverse mapping). This operation tends to reduce the dynamic range of components in the Cepstral domain and hence increases the dynamic range of pixels in the image domain upon reconstruction.
Consequently, this operation provides certain amount of high frequency enhancement since natural images tend to have large magnitudes at low spatial frequencies. Several types of operations (e.g., point operation) can be performed in the Cepstral domain (see next section). If the entire Cepstral transform is denoted by $\mathcal{F}$, the block diagram in Fig. 8 shows a typical Cepstral domain operation.

![Cepstral Domain Operation Diagram]

To prevent $\log$ operation to become negative infinity at spatial frequencies where $X(k,l) = 0$, we use

$$S(k,l) = \left[\log(a + b|X(k,l)|)\right] \exp(j\theta(k,l))$$

where $a$ and $b$ control the shape of the $\log$ function.

**Contrast Enhancement with Homomorphic Filters**

Unlike histogram manipulation methods, contrast enhancement using this method can be done without loss of any grey level resolution. Fig. 9(a) shows the result of applying contrast enhancement (linear mapping) in the Cepstral domain to the Dollar image.
The histogram of the enhanced image is shown in Fig. 9(b). As can be observed, contrast enhancement is achieved without losing grey level resolution at all.

Figure 9: Enhanced dollar image & its histogram.
Area 3: Image Restoration

Any image acquired by different types of imaging sensors is subject to noise and other degradations caused by stochastic and/or deterministic sources as:

1. Sensor/channel noise (additive)
2. Multiplicative noise (e.g., speckle in coherent imaging systems)
3. Clutter (correlated noise), e.g., in radar and sonar,
4. Blurring phenomenon (deterministic) caused by:
   - (a) Motion between camera and object,
   - (b) De-focusing,
   - (c) Atmospheric turbulence as a result of changes in the refractive index of media in satellite imaging,
   - (d) Finite aperture size in radar or sonar.

Image restoration can be viewed as an inverse process where the effects of degradations are removed as much as possible using certain a priori knowledge of the degradation processes.
Goal:
Design a restoration filter to recover or estimate the original image from the degraded image taking into account the properties of the degradation phenomena.

![Image Restoration Operation](image.png)

A typical image formation system consisting of additive noise and blur is shown in Fig. 11. The noisy observed image is given by

$$y(u, v) = x(u, v) * h(u, v) + \eta(u, v)$$

where \(\eta(u, v)\) represents additive noise image and \(h(u, v)\) is the PSF of the blur, which is assumed to be linear space invariant (LSI).

![Simple Image Formation System](image2.png)
Some examples of LSI blur and corresponding PSF and frequency response are given in the following table.

<table>
<thead>
<tr>
<th>Type</th>
<th>( h(u,v) )</th>
<th>( H(\omega_1, \omega_2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal Motion Blur</td>
<td>( \frac{1}{\alpha_0} \text{rect}\left[ \frac{u}{\alpha_0} - \frac{1}{2} \right] \delta(v) )</td>
<td>( \alpha_0 e^{-j\frac{\omega_1 \alpha_0}{2}} \text{sinc}\left(\frac{\omega_1 \alpha}{2\pi}\right) )</td>
</tr>
<tr>
<td>Atmospheric Turbulence</td>
<td>( e^{-\pi \alpha^2 (u^2 + v^2)} )</td>
<td>( \frac{1}{\alpha^2} e^{-\left(\frac{\omega_1^2 + \omega_2^2}{4\pi \alpha^2}\right)} )</td>
</tr>
<tr>
<td>Rectangular Scanning Aperture</td>
<td>( \text{rect}\left[ \frac{u}{\alpha}, \frac{v}{\beta} \right] )</td>
<td>( \alpha \beta \text{sinc}\left(\frac{\alpha \omega_1}{2\pi}\right) \text{sinc}\left(\frac{\beta \omega_2}{2\pi}\right) )</td>
</tr>
<tr>
<td>CCD local interactions</td>
<td>( \sum_{k,l=-1}^{1} \alpha_{k,l} \delta(u-k \Delta, v-l \Delta) )</td>
<td>( \sum_{k,l=-1}^{1} \alpha_{k,l} e^{-j(\omega_1 k + \omega_2 l) \Delta} )</td>
</tr>
<tr>
<td>Diffraction-limited (coherent)</td>
<td>( \text{absinc}(au) \text{sinc}(bv) )</td>
<td>( \text{rect}\left[ \frac{\omega_1}{2\pi a}, \frac{\omega_2}{2\pi b} \right] )</td>
</tr>
<tr>
<td>Diffraction-limited (incoherent)</td>
<td>( \text{sinc}^2(au) \text{sinc}^2(bv) )</td>
<td>( \text{tri}\left[ \frac{\omega_1}{2\pi a}, \frac{\omega_2}{2\pi b} \right] )</td>
</tr>
</tbody>
</table>
Clearly, these are obtained based upon certain physical characteristics about the blurring phenomenon. The following is an example that illustrates how physical properties can be used to yield the PSF.

**Example:** An object \( x(u, v) \) being imaged moves uniformly along \( u \)-direction at velocity \( \nu \). If the exposure time is \( T \), find \( h(u, v) \).

**Solution:** The image intensity at time \( t \) is

\[
x_t(u, v) = x(u - \nu t, v)
\]

Observed image over \( T \) seconds is

\[
y(u, v) = \frac{1}{T} \int_0^T x(u - \nu t, v) dt
\]

Let \( \nu t = \alpha \) and let \( \nu T = \alpha_0 \) then \( \nu \) \( dt \) = \( \frac{d\alpha}{\nu} \) and thus

\[
y(u, v) = \frac{1}{\alpha_0} \int_0^{\alpha_0} x(u - \alpha, v) d\alpha
\]
or

\[ y(u, v) = \frac{1}{\alpha_0} \int \int_{-\infty}^{\infty} \text{rect} \left[ \frac{\alpha}{\alpha_0} - \frac{1}{2} \right] \delta(\beta) x(u - \alpha, v - \beta) d\alpha d\beta \]

Now comparing with

\[ y(u, v) = \int \int h(\alpha, \beta) x(u - \alpha, v - \beta) d\alpha d\beta \]

gives

\[ h(u, v) = \frac{1}{\alpha_0} \text{rect} \left( \frac{u}{\alpha_0} - \frac{1}{2} \right) \delta(v) \]

i.e. the result in the table.