Problem 1

a) \( y(m,n) = T[x(m,n)] = 3x(m,n) + q \)

\[
\begin{align*}
T[a x(m,n) + b x_2(m,n)] &= 3ax(m,n) + 3b x_2(m,n) + q \\
a T[x_1(m,n)] + b T[x_2(m,n)] &= 3ax_1(m,n) + 9 + 3b x_2(m,n) + q
\end{align*}
\]

\( \text{Non-Linear} \)

\[
\begin{align*}
T[x(m-k,n-l)] &= 3x(m-k,n-l) + q \\
y(m-k,n-l) &= 3x(m-k,n-l) + q
\end{align*}
\]

\( \text{Space-Invariant} \)

\( h(m,n) = T[S(m,n)] = 3S(m,n) + q \)

\[
|y(m,n)| < M < \infty \quad \text{then} \quad |y(m,n)| < 3M + q < \infty \quad \Rightarrow \quad \text{Stable}
\]

b) \( y(m,n) = T[x(m,n)] = m^2 a x(m,n) \)

\[
\begin{align*}
T[a x_1(m,n) + b x_2(m,n)] &= am^2 a x_1(m,n) + bm^2 a x_2(m,n) \\
a T[x_1(m,n)] + b T[x_2(m,n)] &= am^2 a x_1(m,n) + bm^2 a x_2(m,n)
\end{align*}
\]

\( \text{Linear} \)

\[
\begin{align*}
T[x(m-k,n-l)] &= m^2 a x(m-k,n-l) \\
y(m-k,n-l) &= (m-k)^2 (n-l)^2 x(m-k,n-l)
\end{align*}
\]

\( \text{Space-Varying} \)

\( h(m,n; k, l) = T[S(m-k,n-l)] = m^2 a S(m-k,n-l) = \begin{cases} k^2 l^2 & m=k, n=l \\ 0 & \text{otherwise} \end{cases} \)

\[|y(m,n)| = m^2 a |x(m,n)| \rightarrow \infty \quad \text{as} \quad m \text{ or } n \rightarrow \infty \quad \Rightarrow \quad \text{Unstable} \]
c) \( y(m, n) = T[x(m, n)] = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} x(k, l) \)

\[
T[a x(m, n) + b x_a(m, n)] = a \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} x(k, l) + b \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} x_a(k, l)
\]

\[
a T[x(m, n)] + b T[x_a(m, n)] = a \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} x(k, l) + b \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} x_a(k, l)
\]

Linear

\[
T[x(m-m', n-n')] = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} x(k-m', l-n')
\]

let \( k-m' = k' \) and \( l-n' = l' \) then

\[
T[x(m-m', n-n')] = \sum_{k'=0}^{\infty} \sum_{l'=0}^{\infty} x(k', l')
\]

\[
y(m-m', n-n') = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} x(k, l)
\]

Space-Invariant

\[
h(m, n) = T[\delta(m, n)] = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \delta(k, l) = \begin{cases} 1 & m=0, n=0 \\ 0 & \text{otherwise} \end{cases} \Rightarrow \text{IIR}
\]

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |h(m, n)| = \infty \Rightarrow \text{Unstable}
\]

d) \( y(m, n) = T[x(m, n)] = e^{-|x(m, n)|} \)

\[
T[a x(m, n) + b x_a(m, n)] = e^{-|a x(m, n) + b x_a(m, n)|}
\]

\[
a T[x(m, n)] + b T[x_a(m, n)] = a e^{-|x(m, n)|} + b e^{-|x_a(m, n)|}
\]

Non-Linear

\[
T[x(m-k, n-l)] = e^{-|x(m-k, n-l)|}
\]

\[
y(m-k, n-l) = e^{-|x(m-k, n-l)|} \quad \text{Space-Invariant}
\]

\[
h(m, n) = T[\delta(m, n)] = e^{-|\delta(m, n)|} = \begin{cases} e^{-1} & m=n=0 \\ 1 & \text{otherwise} \end{cases} \Rightarrow \text{IIR}
\]

\[ |y(m, n)| = 1 \text{ for any } x(m, n) \Rightarrow \text{Stable} \]
Problem 2

a) 2D linear convolution yields the following results

\[
\begin{bmatrix}
0 & -1 & 3 & 1 \\
-1 & -2 & 11 & 2 & 2 \\
-3 & 1 & 11 & 6 & -3 \\
0 & -2 & -5 & -3 & 0
\end{bmatrix}
\]

b) Check:

\[
\sum \sum y(m, n) = 16
\]

\[
\sum \sum h(m, n) = 1 \quad \checkmark
\]

\[
\sum \sum x(m, n) = 16
\]

Problem 3

\[
x(m, n) = a^m b^n u(m, n)
\]

\[
z(m, n) = x(m, n) * y(m, n) = \sum_k \sum_l x(k, l) y(m-k, n-l)
\]

\[
= \sum_k a^k b^l u(k, l) \sum_{r=0}^{\infty} s(m-rN, n-rN)
\]

\[
= \sum_{r=0}^{\infty} a^{m-rN} b^{n-rN} u(m-rN, n-rN)
\]
Problem 4
a) An image with quarter plane (1st quadrant) ROS is shown in Figure 1.

In performing convolution of two such images, we get the scenario shown in Figure 2.

Thus, as can be seen, the convolution will be zero whenever \( m < 0 \) and \( n < 0 \), in the 3rd quadrant will be confined to the 3rd quadrant.

b) Similar reasoning will show that the convolution of two images with ROS's on adjacent quadrants and non-adjacent ones. Consider an example of the first case with sequences in the 1st and 2nd quadrants as shown in Figure 3. Here, the convolution will be zero when \( n < 0 \). Likewise consider the non-adjacent case with sequences in the 1st and 3rd quadrants as shown in Figure 4. Here, the convolution is non-zero for all \( m \) and \( n \).

Thus, if both sequences have their ROS's on the same quadrant the convolution has support on that quadrant. If both have ROS's on the same half-plane, the
Problem 5

a) \[ y(m,n) = \sum_{k} \sum_{l} x(k,l) h(m-k,n-l) \]
\[ = \sum_{k} a(k) b(l) c(m-k) d(n-l) \]
\[ = \sum_{k} a(k) c(m-k) \sum_{l} b(l) d(n-l) \]
\[ = y_1(m) y_2(n) \quad \text{i.e. separable} \]

b) \[ y(m,n) = (a(m) * c(m))(b(n) * d(n)) = y_1(m) y_2(n) \]

Separable where each term represents 1-D convolution along rows/columns.
Problem 6

Without loss in generality, we'll assume the spectrum of the original image is given by

\[ X(\omega_1, \omega_2) = \begin{cases} 1 & |\omega_1| < \omega_0, |\omega_2| < \omega_0 \\ 0 & \text{otherwise} \end{cases} \]

The spectrum of the image corrupted by white noise of power \( \sigma_n^2 \) is then given by

\[ Y(\omega_1, \omega_2) = \begin{cases} 1 + \sigma_n^2 & |\omega_1| < \omega_0, |\omega_2| < \omega_0 \\ \sigma_n^2 & \text{otherwise} \end{cases} \]

The top and side-views of this spectrum are given below.

The SNR of the original image is then given to be

\[ M_{\text{original}} = \frac{1}{\sigma_n^2} \]

Sampling the image at the Nyquist rates \( \omega_s = 2\omega_0 \) and \( \omega_{2s} = 2\omega_0 \) causes periodic extension of the spectrum as shown below.

Thus, the noise effects from the periodic replicas will fold over into the original spectrum forcing the SNR to zero, i.e.

\[ M_{\text{sampled}} = \lim_{n \to \infty} \frac{1}{\sigma_n^2} = 0 \]