ECE507 - Plasma Physics and Applications

Lecture 13

Colorado State University

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TRANSVERSE ELECTROMAGNETIC WAVES IN PLASMAS

Propagate with cutoff at $\omega = \omega_p$

- At high frequency ions are immobile

Maxwell Equations For transverse waves of weak intensity

$$\nabla \times \mathbf{H} = \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} - e n_o \mathbf{v}$$

$$\nabla \times \mathbf{E} = -\mu_o \frac{\partial \mathbf{H}}{\partial t}$$

- All non-linear terms can be neglected

Maxwell-Euler equations describe the waves

$$\frac{\partial n_e}{\partial t} + n_o \nabla \bullet \mathbf{v} = 0$$

- Only J_T contributes to wave

$$m \frac{\partial \mathbf{v}}{\partial t} = -\cancel{\frac{\gamma k T_e}{n_o}} \nabla n_o - e\mathbf{E}$$

- ($\nabla \rightarrow i\mathbf{k}$ longitudinal)

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{e\mathbf{E}}{m}$$

TRANSVERSE ELECTROMAGNETIC WAVES IN PLASMAS (continuation)

Develop a wave equation

$$\nabla \mathrm{x}\,\mathbf{H} = \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} - e n_o \mathbf{v}$$

$\downarrow$

differentiate

$\downarrow$

$$\nabla x \frac{\partial H}{\partial t} = \varepsilon_o \frac{\partial^2 E}{\partial t^2} - e n_o \frac{\partial v}{\partial t}$$

$$\nabla \mathrm{x}\,\mathbf{E} = -\mu_o \frac{\partial \mathbf{H}}{\partial t}$$

$\downarrow$

Take curl ∇

$\downarrow$

$$\nabla \mathrm{x}(\nabla \mathrm{x}\mathbf{E}) = -\mu_o \nabla \mathrm{x} \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \mathrm{x}(\nabla \mathrm{x}\mathbf{E}) = \nabla(\nabla \bullet \mathbf{E}) - \nabla^2 \mathbf{E}$$

$$\nabla(\nabla \bullet \mathbf{E}) - \nabla^2 \mathbf{E} = \mu_o \varepsilon_O \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_o e n_o \frac{\partial \mathbf{v}}{\partial t}$$

$\downarrow$

For $\nabla \bullet \mathbf{E} = 0$ and

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{e\mathbf{E}}{m}$$

$$\boxed{\frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{e n_o}{\varepsilon_o}\left(\frac{e\mathbf{E}}{m}\right) - \frac{1}{\varepsilon_o \mu_o}\nabla^2 \mathbf{E} = 0}$$

TRANSVERSE ELECTROMAGNETIC WAVES IN PLASMAS (continuation)

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{e n_o}{\varepsilon_o}\left(\frac{e\mathbf{E}}{m}\right) - \frac{1}{\varepsilon_o \mu_o}\nabla^2 \mathbf{E} = 0$$

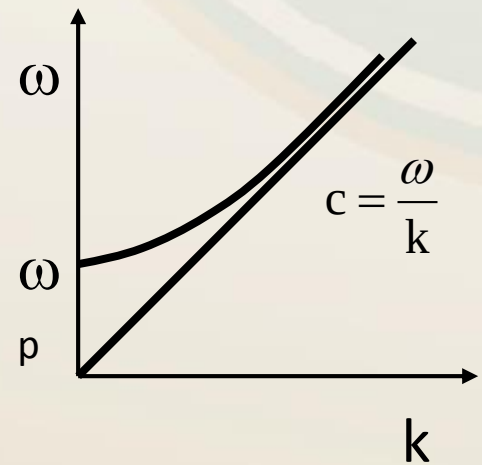
Defining

$$\omega_p{}^2 = \frac{e^2 n_o}{\varepsilon_o m} \qquad c^2 = \frac{1}{\varepsilon_o \mu_o}$$

Wave equation for transverse wave in a plasma

$$\boxed{\left(\frac{\partial^2}{\partial t^2} + \omega_p^2 - c^2\nabla^2\right)\mathbf{E}(\mathbf{r}, t) = 0}$$

For a plane wave $\mathbf{E}(r, t) = \mathbf{E}_o e^{-i(\omega t - \mathbf{k}.\mathbf{r})}$



Dispersion relation

$$\boxed{\omega^2 = \omega_P^2 + k^2 c^2}$$

TRANSVERSE ELECTROMAGNETIC WAVES IN PLASMAS (continuation)

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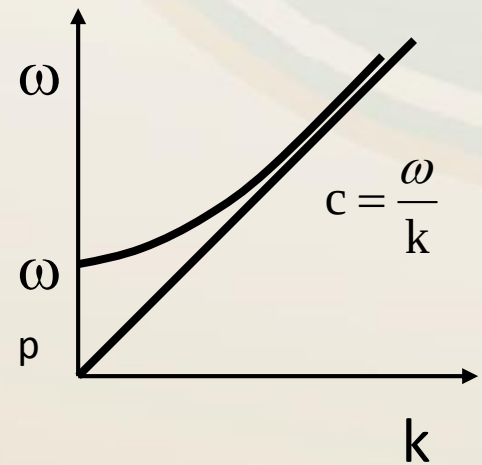


Dispersion relation

$$\boxed{\omega^2 = \omega_P^2 + k^2 c^2}$$

TRANSVERSE ELECTROMAGNETIC WAVES IN PLASMAS (continuation)

Dispersion relation in plasma

$$\omega^2 = \omega_P^2 + k^2 c^2 \longrightarrow k = \frac{\sqrt{\omega^2 - \omega_p^{\,2}}}{c}$$

Underdense plasma

$\omega > \omega_p$

k real

wave propagates with phase velocity:

$$v_\phi = \frac{\omega}{k} = \frac{c}{\sqrt{1-\omega_p^2/\omega^2}} = \frac{c}{\sqrt{1-n_e/n_c}}$$

refractive index $n \approx \frac{c}{v_\phi}$

$$n = \sqrt{1-\frac{\omega_P^2}{\omega^2}}$$

or

$$n = \sqrt{1-\frac{n_e}{n_c}}$$

Overdense plasma

$\omega < \omega p$

k imaginary

wave cannot propagate

For highly overdense

$\omega^2 << \omega^2_p$

$$k = i\frac{\omega_P}{c}$$

penetration depth

$l = c/\omega_p$

The plasma is dispersive

$$\text{phase velocity} \qquad v_\phi = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \omega_p^2 / \omega^2}} = \frac{c}{\sqrt{1 - n_e / n_c}}$$

- These velocities are not constant but change with ω_p/ω or n_e/n_c
- At low densities both v_φ, v_g approach c

TRANSVERSE ELECTROMAGNETIC WAVES IN PLASMAS (continuation)

- Critical plasma density : frequency for which $\omega = \omega_p$ is the critical frequency

$$n_c \equiv \frac{\varepsilon_o \omega^2 m}{e^2}$$

$$\boxed{n_c = \frac{1.11 \times 10^{21}\,\mathrm{cm}^{-3}}{\lambda^2(\mu\mathrm{m})}}$$

TABLE 6.1. Electron density, plasma frequency, critical photon energy for $\omega_c = \omega_p$, and critical wavelength for electromagnetic radiation.

$n_e(e/\mathrm{cm}^3)$	$\omega_p/2\pi$	$\hbar\omega_c$ (eV)	λ_c	Comments
1.00×10^6	8.94 MHz		33.5 m	Between AM and FM radio
1.00×10^{14}	89.4 GHz		3.35 mm	Microwaves
1.00×10^{19}			10.6 μm	CO_2 laser
1.00×10^{21}		1.17	1.06 μm	Nd laser
1.60×10^{22}		4.86	266 nm	4ω of Nd laser
4.60×10^{24}		80.0	15.5 nm	Ne-like Y laser

TRANSVERSE ELECTROMAGNETIC WAVES IN PLASMAS (continuation)

Collisional Absorption:

Now include collision between oscillating electrons and ions in momentum equation

$$m\frac{\partial v}{\partial t} = -e\mathbf{E} - m\upsilon_{ei}\mathbf{v}$$

Where ν_{ei} is the electron-ion collision frequency, mv is the electron momentum

For $\nu_{ei} << \omega$ the dispersion relation becomes:

$$\omega^2 = \omega_p^2\left(1 - i\frac{\nu_{ei}}{\omega}\right) + k^2c^2$$

$$\omega = \omega_r + i\ \omega_i$$

$$k = k_r + ik_i$$

$$\boxed{\begin{aligned} \omega_r^{\ 2} &= \omega_p^{\ 2} + k^2c^2 \\ \omega_i &\approx -\frac{\nu_{ei}\omega_p^2}{2\omega^2} = -\frac{n_e}{2n_c}\nu_{ei} \end{aligned}}$$

$$\boxed{\begin{aligned} k_r &= \frac{\sqrt{(\omega^2 + \omega_p^2)}}{c} \\ k_i &= \frac{\upsilon_e\omega_p^2}{2v_g\omega^2} \end{aligned}}$$

The negative sign indicates damping

Collisional Absorption:

$$I = E^2 = \left[E_0 e^{-i(\omega t - kx)}\right]^2$$

Then

$$l_{abs} = \frac{1}{2k_i} = \frac{\omega^2 v_g}{\omega_p^2 \upsilon_{ei}} = \frac{n_e v_g}{n_c \upsilon_{ei}}$$

Electron-ion collision frequency (Dawson et al)

$$\upsilon_{ei} = \frac{e^4 Z n_e \ln \Lambda}{3(2\pi)^{3/2} \varepsilon_o m^{1/2} (kT_e)^{3/2}}$$

Example: kT_e = 1keV, λ = 1.06 µm, Z = +14, n_e = ½ n_c → l_{abs} = 130 µm

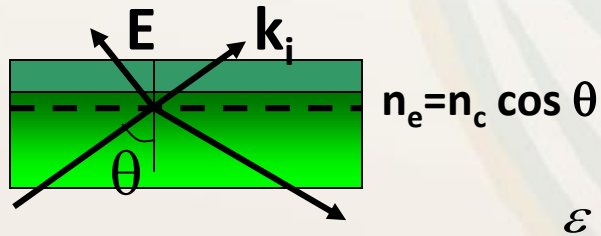
Resonance Absorption

It is possible to directly excite plasma waves through the resonance $\omega = \omega_p$ at the critical density

- Occurs for p-polarized radiation

$$\nabla \bullet (\varepsilon \mathrm{E}) = 0$$

for a plasma



E field has component in direction of Δn_e that tunnels into critical region)

$$\varepsilon = \varepsilon_o n^2 = \varepsilon_o \left(1 - \frac{n_e}{n_c}\right)$$

$$\varepsilon_o \nabla \bullet \left[1 - \frac{n_e}{n_c}\right] \mathbf{E} = 0$$

Absorption peaks at ≈50% at

$$(k_i \lambda)^{1/3} \sin\theta = 0.8$$

$\lambda =$ density scale length

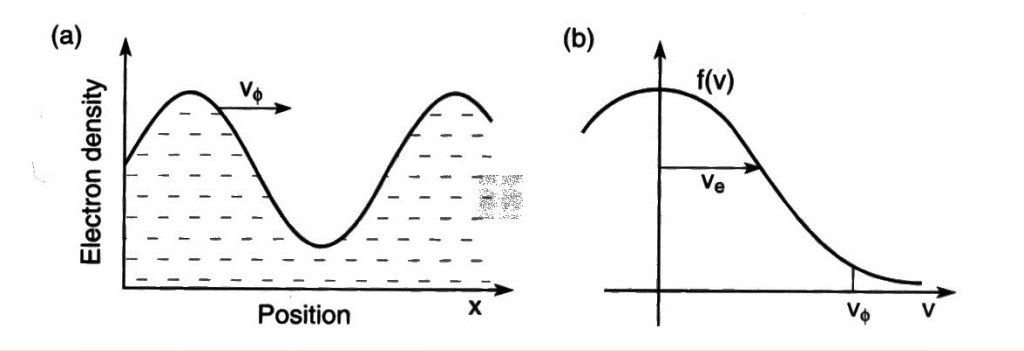
Tunneling field drives resonance response at the critical surface

- For normal incidence (θ=0) there is no axial component of **E** to drive the resonance
- For glancing incidence tunneling field is weak

Plasma Waves: Collective longitudinal waves that propagate naturally

- electron-acoustic waves (high frequency)
(ion too massive to participate)

- ion-acoustic waves (low frequency)



From Maxwell-Euler equations we obtain the longitudinal wave equation for n_e fluctuations

$$\left(\frac{\partial^2}{\partial t^2}+\omega_p^2-a_e{}^2\nabla^2\right)n_e(\mathbf{r},t)=0$$

Electron sound speed $$a_e=\left(\frac{\gamma kT_e}{m}\right)^{1/2}$$

Natural frequency of oscillation ω_p $$\omega_p=\left(\frac{e^2 n_o}{\varepsilon_o m}\right)^{1/2}$$

Plasma Waves

Assuming a solution $n_e(\mathbf{r},t) = n_e e^{-i(\omega t - \mathbf{k}.\mathbf{r})}$ for the electron wave equation

$$\boxed{\omega^2 = \omega_p^2 + k^2 a_e^{\ 2}}$$

a_e: electron sound speed

Dispersion diagram for naturally occurring waves in isotropic plasma

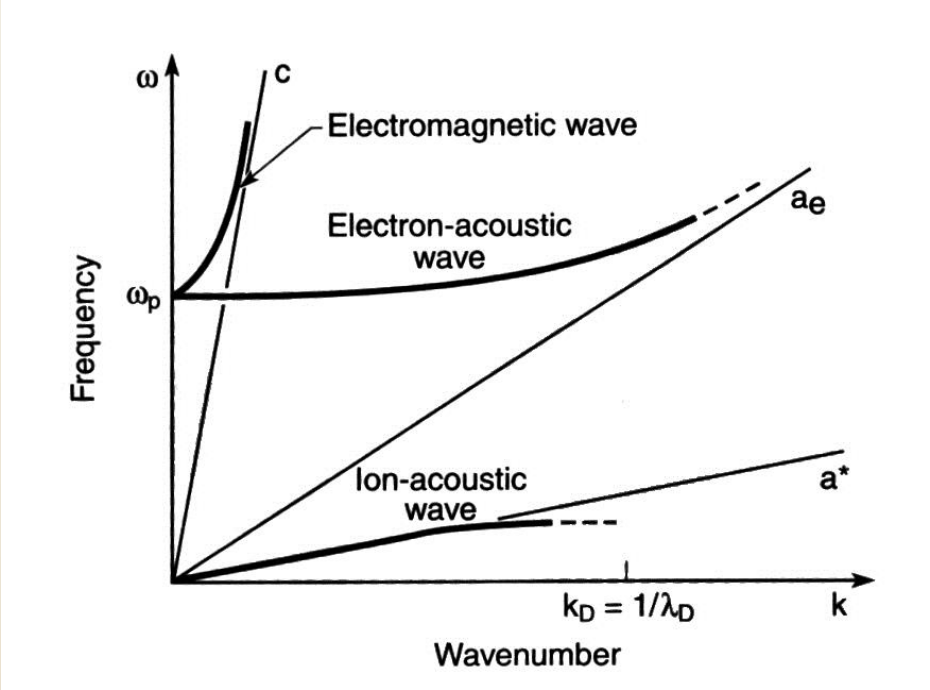


Fig. 6.7 Attwood

Non-Linear Processes in Plasmas

- In the Maxwell-Euler equations there are several terms involving the products of fields

$$\nabla \times \mathbf{H} = \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} + \sum_j \boxed{(nq\mathbf{v})_j}$$

$$\nabla \times \mathbf{E} = \mu_o \frac{\partial \mathbf{H}}{\partial t}$$

$$\frac{\partial n_j}{\partial t} + n_o \nabla \bullet \boxed{(n_j \mathbf{v})} = 0$$

$$m_j \boxed{n_j \left(\frac{\partial}{\partial t} + \boxed{\mathbf{v} \bullet \nabla}\right)\mathbf{v}} = -\nabla P_j + \boxed{\boxed{q_j n_j}(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

- These non-linear terms create possibility of non-linear growth and frequency mixing.

- Two processes involving intense incident radiation are:

  - Stimulated Brillouin scattering (SBS)

  - Stimulated Raman scattering (SRS)

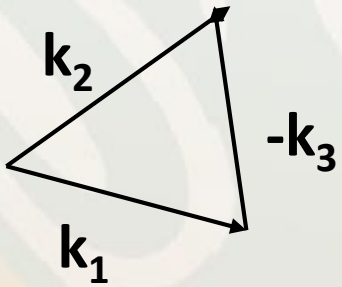
Non-Linear Processes in Plasmas

For example analyzing the electron current terms in:

$$\nabla \mathrm{x}\, \mathbf{H} = \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} + \sum_j (nq\mathbf{v})_j$$

$$\mathbf{J}(\mathbf{r},t) = -en_e(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)$$

$$Je^{-i(\omega_1 t - \mathbf{k_1}.\mathbf{r})} = -e\, n_e e^{-i(\omega_2 t - \mathbf{k_2}.\mathbf{r})} \mathbf{v}\, e^{-i(\omega_3 t - \mathbf{k_3}.\mathbf{r})}$$



A term to term match shows:

Conservation of energy

$$\omega_1 = \omega_2 \pm \omega_3$$

Conservation of momentum

$$\mathbf{k}_1 = \mathbf{k}_2 \pm \mathbf{k}_3$$

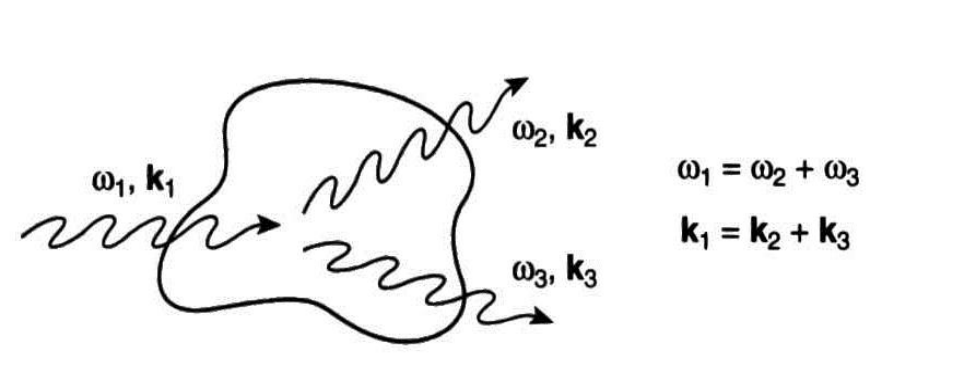


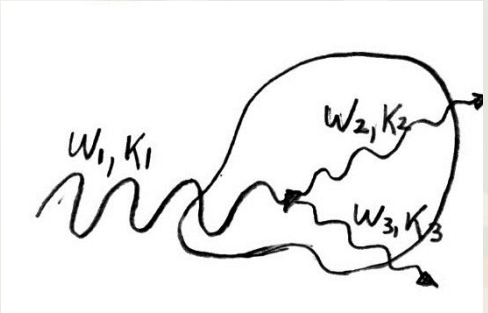
Fig. 6.10 Attwood

Non-linear Processes in Plasmas

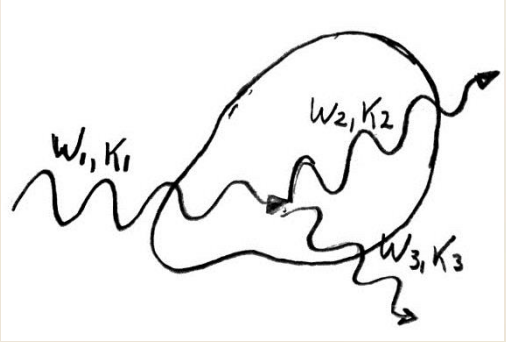
Stimulated Scattering:

- An intense incoming wave (ω_1, $\mathbf{k}_1$) drives an existing plasma wave (ω_2,$\mathbf{k}_2$) out of the noise and stimulates it to grow.

- (ω_1,$\mathbf{k}_1$) scatters from (ω_2, $\mathbf{k}_2$) producing (ω_3, $\mathbf{k}_3$)(not all combinations satisfy conservation equation)



- when matching occurs the new scattered wave (ω_3,k_3) interferes with (ω_1,$\mathbf{k}_1$) at ω_2=(ω_3-ω_1) $\mathbf{k}_2$= ($\mathbf{k}_3$-$\mathbf{k}_1$) causing the initial noise ω_2, $\mathbf{k}_2$ to grow in amplitude.



- The scattering and growth process continues.

Non-Linear Processes in Plasmas: Stimulated Scattering

Stimulated Brillouin scattering (SBS)

- Scattering from low-frequency ion-acoustic wave
- Scattered wave (ω_B, $\mathbf{k}_B$) at slightly shifted ω

Stimulated Raman Scattering (SRS)

- Scattering from high-frequency electron acoustic wave
- Scattered wave (ω_R. F_R) at a substantially shifted ω.

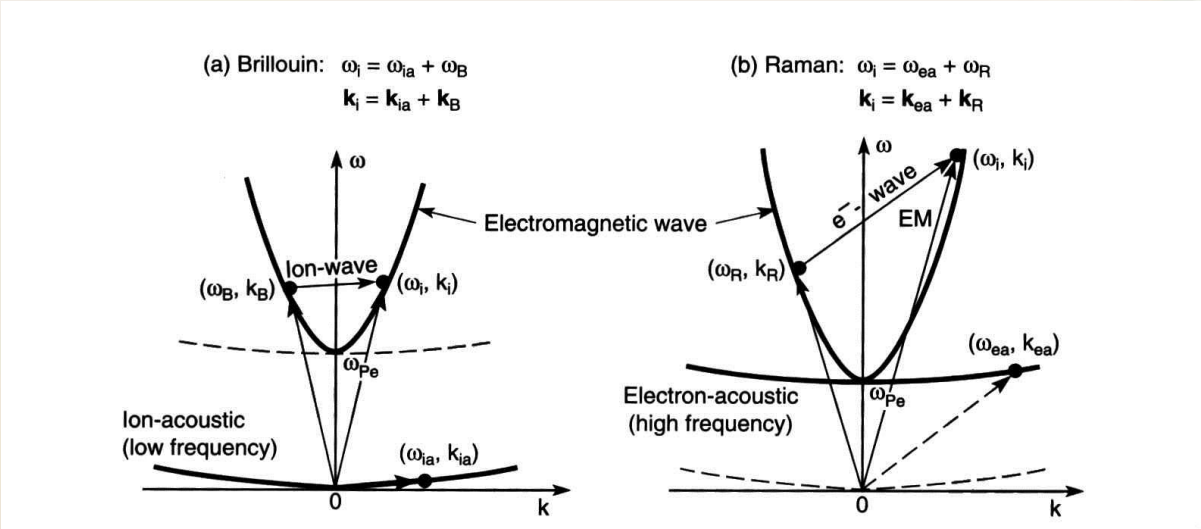


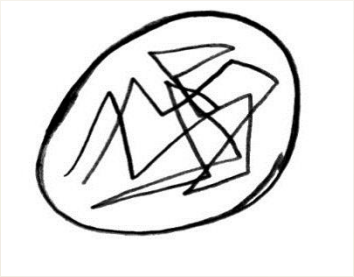
Fig. 6.11 - Attwood

- Negative **k** indicates backscattered waves.
- Matching occurs when the ω , **k** for all three waves lie on the dispersion curves.

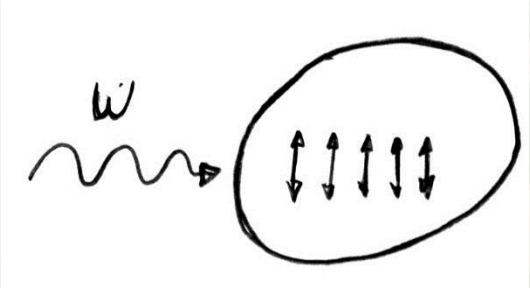
Both "stimulated" because the electron or ion plasma wave is caused to grow out of noise through stimulation at the beat frequency ($\omega_3 - \omega_1$),

Threshold for Non-Linear Processes in Plasmas

- **Nonlinear processes become significant when oscillation velocity v_{os} becomes important relative to random thermal velocity v_e**



Random motion



Induced coherent oscillations

Ratio of electron energy in coherent oscillations to that of random motion

$$\left|\frac{v_{os}}{v_e}\right|^2 = \frac{e^2E^2}{m\omega^2 kT_e} = \frac{1/c}{n_c kT_e}$$

Example: at $\lambda = 1.06\ \mu m$ (Nd-YAG laser), $n_c = 1 \times 10^{21}\ cm^{-3}$

$$\Rightarrow \left|\frac{v_{os}}{v_e}\right| = 1 \quad \text{at } I = 4.7 \times 10^{15}\ w/cm^2$$

To avoid non-linear processes in laser-created plasmas use:

- short wavelengths
- low intensities

Laser wavelength trends in Laser-Created Plasmas

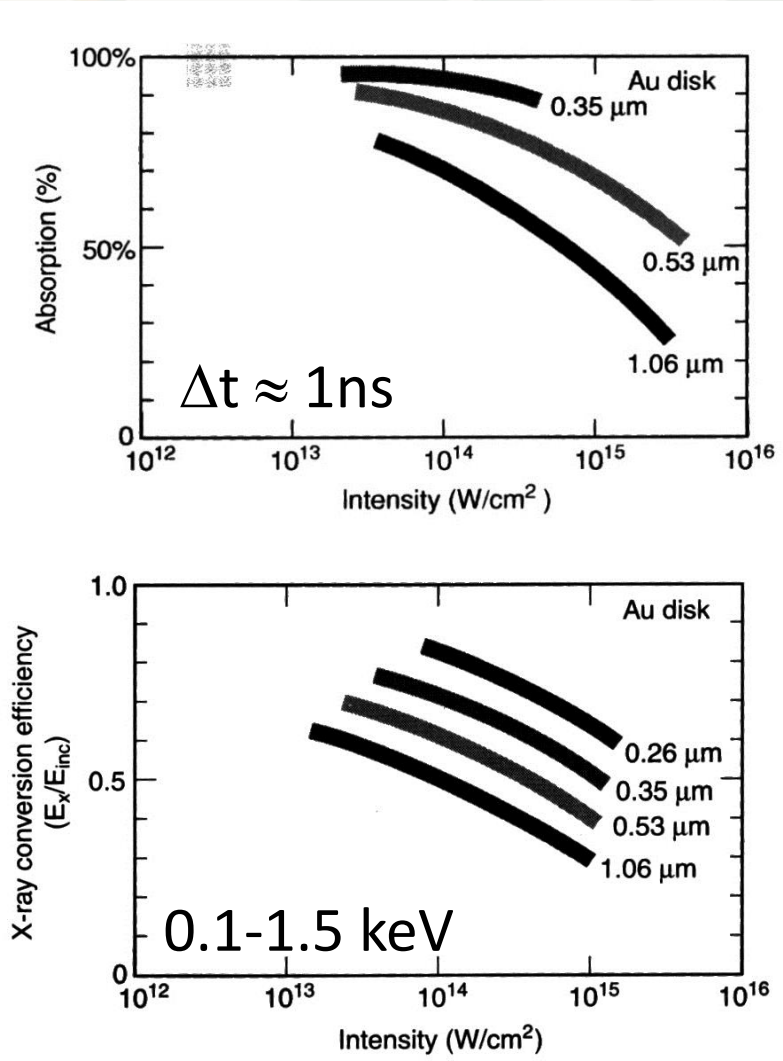
Plasma properties (n_e, T_e, ∇n_e, Z)

- Irradiation Wavelength
- Intensity
- Pulse length (throughout the absorption scale length *l*)

Collisional Absorption favors shorter λ

($\propto n_c$, $n_c \propto \frac{1}{\lambda^2}$)

- At high intensity 'radiation pressure' depresses sub-critical density, reducing absorption

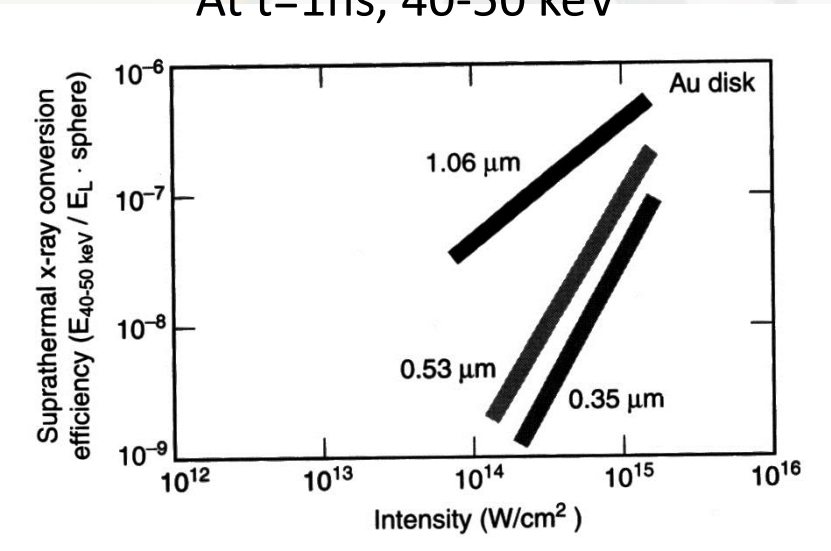


Attwood fig. 6.31 and 6.32

Laser wavelength trends in laser-created plasmas

- At high irradiation non-linear processes tend to dominate laser-plasma interaction
- Appearance of high energy suprathermal photons
- At very high intensities long λ results in increased stimulated scattering processes and in decrease absorption
- At very high intensities use of shorter wavelengths (2W, 3W) is preferred

At t=1ns, 40-50 keV



700ps

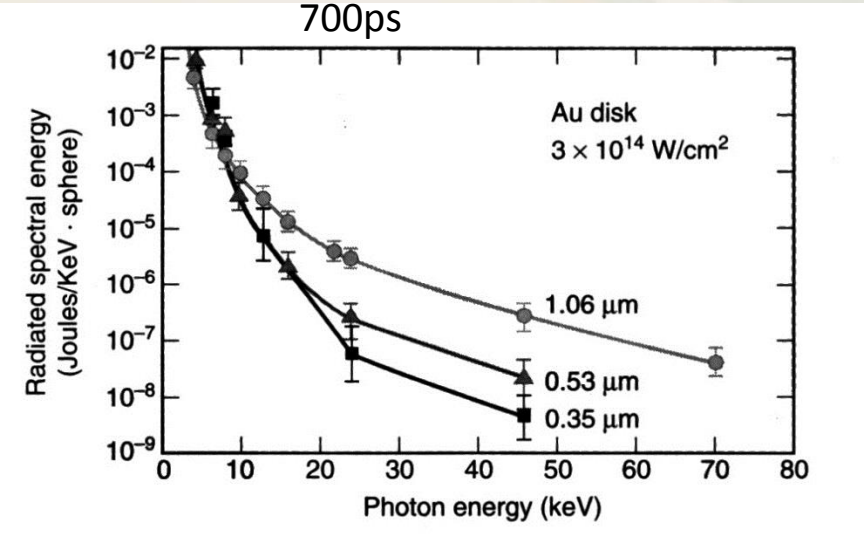


Figure 6.30 Attwood