

SIGNALS AND SYSTEMS LABORATORY 14:

The Rudiments of Antenna Design

INTRODUCTION

By and large, in our study of signals and systems we have been concerned with rational functions. These are functions of exponential or geometric type whose transforms are rational. However, for many applications in communications and array processing it is required to build or approximate non-rational systems. Perhaps the best examples are the finite time integrator in communications and the finite space aperture in antenna theory. When these continuous operators are approximated in discrete-time or discrete-space, the resulting systems are called FIR filters and antenna arrays.

Our purpose in this lab is to study these operators. We shall develop the close connection between FIR filters and antenna arrays, between complex frequency responses for filters and complex beampatterns for antennas, and so on.

THE FINITE-TIME INTEGRATOR AND THE FINITE-SPACE APERTURE

Time. Consider a system $H(s)$ whose impulse response is

$$(1) \quad h(t) = u(t) - u(t - \tau) \longleftrightarrow H(s) = \frac{1}{s} [1 - e^{-sT}]$$

If the input to this system is the signal $v(t)$, then the output is the convolution

$$(2) \quad y(t) = (h * v)(t) = \int_{t-\tau}^t v(\tau) d\tau \longleftrightarrow Y(s) = H(s)V(s) = \frac{V(s)}{s} - e^{-sT} \frac{V(s)}{s}$$

As indicated, the output is a finite-time integration of the input, and the integration time is T .

The transfer function $H(s)$ has canceling poles and zeros at $s = 0$, so the pole-zero plot for this, and all non-rational transforms is not informative. However, the complex frequency response is revealing:

$$(3) \quad H(j2\pi f) = \frac{1}{j2\pi f} [1 - e^{-j2\pi f T}] = T e^{-j\pi f T} \frac{\sin(\pi f T)}{(\pi f T)} = T e^{-j\pi f T} \text{sinc}(\pi f T)$$

There are several things to note about this complex frequency response: its value at $f = 0$ is T , which is the integral of $h(t)$, and it is zero at $f = n/T$, $n = \pm 1, 2, \dots$. This is illustrated in Figure One, using the function 'plotSINC.m'.

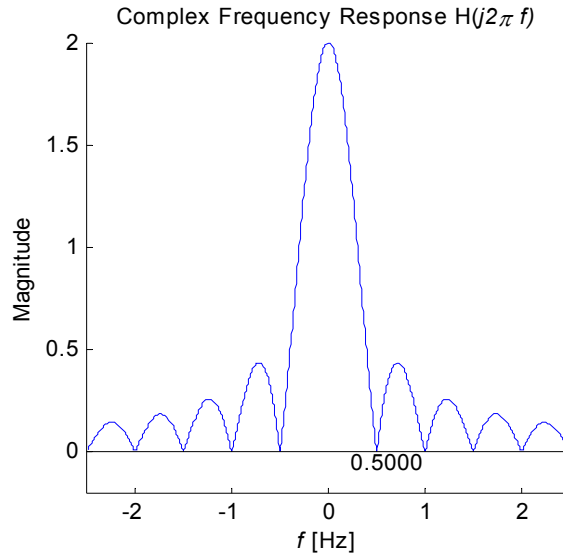


Figure One: Magnitude of $H(j2\pi f)$ for $T = 2$

The output of the finite-time integrator, when excited by the cosine $\cos(2\pi ft)$, whose frequency is f and whose period is $1/f$ is, of course,

$$(4) \quad y(t) = \text{Re}\{H(j2\pi f)e^{j2\pi ft}\} = T \text{sinc}(\pi f T) \cos[2\pi f(t - T/2)]$$

The output at time $t = T/2$ is $y(T/2) = T \text{sinc}(\pi f T)$, which is a real number that depends on the frequency f . In fact, $|y(T/2)| = |H(j2\pi f)|$, which is the magnitude of the complex frequency response at frequency f . By now, you should be able to explain why the complex frequency response should be zero at frequencies that are integer multiples of $1/T$ (that is, $f = k/T$, or periods that are integer fractions of T (that is, $1/f = T/k$).

Space. There is no reason why this simple idea cannot be extended to space, rather than time. To this end, we suggest that a finite integrating filter in space should have an impulse response

$$(5) \quad h(x) = u(x) - u(x - X) \longleftrightarrow H(s) = \frac{1}{s} [1 - e^{-sX}]$$

If the input to this system is the signal $v(x)$, then the output is the convolution

$$(6) \quad y(x) = (h * v)(x) = \int_{x-X}^x v(w) dw \longleftrightarrow Y(s) = H(s)V(s) = \frac{V(s)}{s} - e^{-sT} \frac{V(s)}{s}$$

As indicated, the output is a finite-space integration of the input, and the integration distance is X . The complex frequency response is revealing:

$$(7) \quad H(j2\pi/\lambda) = \frac{1}{j2\pi/\lambda} [1 - e^{-j2\pi X/\lambda}] = X e^{-j\pi X/\lambda} \frac{\sin(\pi X/\lambda)}{\pi X/\lambda} = X e^{-j\pi X/\lambda} \text{sinc}(\pi X/\lambda)$$

This complex wavelength response is identical in form to the finite-time integrator result. However the temporal frequency variable f , in units of cycles per second or Hz, has been replaced by the spatial

frequency variable $1/\lambda$, in units of cycles per meter, where λ is wavelength. If you think about it, it is cycles per meter that ought to be called Hz, in honor of Heinrich Hertz, who experimentally discovered the propagating waves predicted by James Clerk Maxwell. In any event, there are several things to note about this complex frequency response: its value at $\lambda = \infty$ is X , which is the integral of $h(x)$, and it is zero at $\lambda = X/k$, $k = \pm 1, 2, \dots$. This is illustrated in Figure Two, using 'plotSINC.m'.

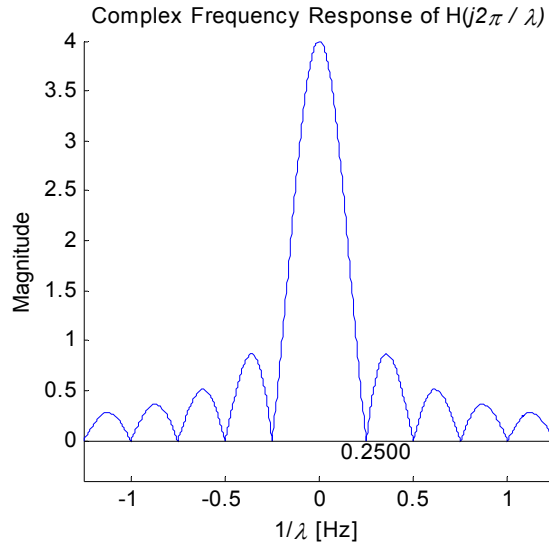


Figure Two: Magnitude of $H(j2\pi / \lambda)$ for $X = 4$

The output of this finite-space integrator, when excited by the cosine $\cos(2\pi x/\lambda)$, is, of course,

$$(8) \quad y(x) = \text{Re}\{H(j2\pi / \lambda)e^{j2\pi x/\lambda}\} = X \text{sinc}(\pi X / \lambda) \cos\left[\frac{2\pi}{\lambda}(x - X/2)\right]$$

The output at the so-called phase center of the aperture, namely $x = X/2$, is $y(X/2) = X \text{sinc}(\pi X / \lambda)$. In fact $|y(X/2)| = |H(j2\pi / \lambda)|$, which is the magnitude of the complex wavenumber response at wavelength λ . By now, you should be able to explain why the complex wavelength response should be zero at wavelengths that are integer fractions of X (that is $\lambda = X/k$) and spatial frequencies that are integer multiples of $1/X$ (that is, $1/\lambda = k/X$).

Planewaves. Now, a common way for wavelength to vary is for it to depend on the angle that a plane wave of fixed wavelength λ_0 makes with a linear aperture. This is illustrated in Figure Three, where the wavelength with respect to the aperture is

$$(9) \quad \lambda = \frac{\lambda_0}{\cos(\phi)}$$

Our convention, as illustrated in Figure Three, is that $\phi = 0$ is *endfire* and $\phi = \pi/2$ is *broadside*.

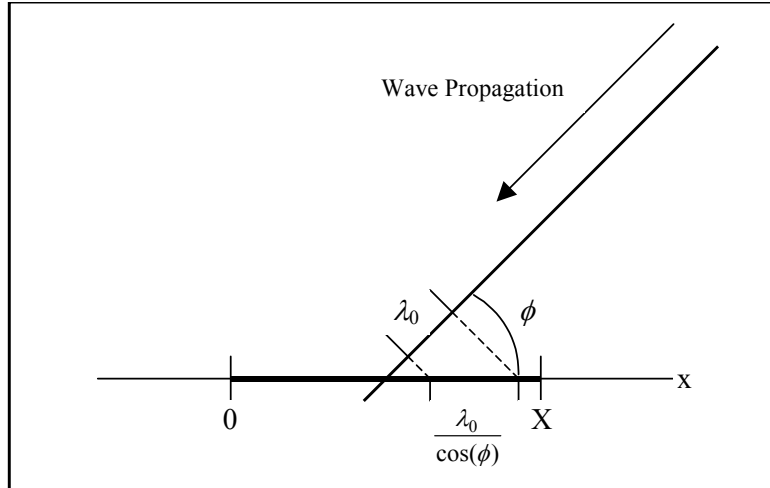


Figure Three: A wave propagating towards an aperture

When this result is inserted into the formula for the complex frequency response, the result is

$$(10) \quad H(j2\pi \cos(\phi) / \lambda_0) = X e^{-j\pi \frac{X}{\lambda_0} \cos(\phi)} \text{sinc}(\pi \frac{X}{\lambda_0} \cos(\phi)).$$

Typically, the aperture length X is chosen to be several times the wavelength λ_0 . Let's call the aperture length $X = N\lambda_0$, so that the normalized complex frequency response may be written as

$$(11) \quad B(\phi) = \frac{1}{\lambda_0} H(j2\pi \cos(\phi) / \lambda_0) = N e^{-j\pi N \cos(\phi)} \text{sinc}(\pi N \cos(\phi)).$$

We call this function the complex *beampattern* and note that it is identical in form to the complex frequency responses derived earlier. There is a fundamental point to be made here: this beampattern is just the original complex frequency response $H(2\pi f)$, evaluated at $f = \cos(\phi) / \lambda_0$, or equivalently the original transfer function $H(s)$ evaluated at $s = j2\pi \cos(\phi) / \lambda_0$. Of course in the original transfer function we have set the aperture length to $X = N\lambda_0$. In Figure Four we have used `plotSINC.m` to plot the beampattern $B(\phi)$ vs. $\cos(\phi)$. This plot illustrates the selectivity of the array to plane waves arriving from different angles. All we have really done is evaluate $H(2\pi f)$ at $f = \cos(\phi) / \lambda_0$ and *re-plot* against $\cos(\phi)$.

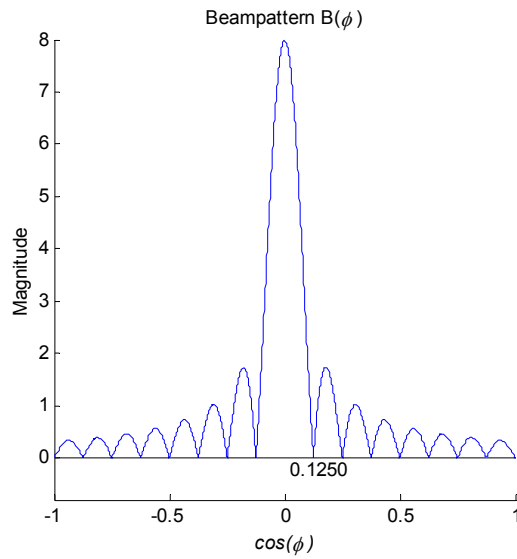


Figure Four: Magnitude of $B(\phi)$ for $N = 8$

It is common practice in antenna design to plot the magnitude of the beampattern $|B(\phi)|$ vs. ϕ , in polar coordinates. This illustrates the *resolving power* of the array and emphasizes the point that ϕ is, after all, a geometrical angle. How are we to do this? Here is a device for plotting a real function like $B(\phi)$ versus polar ϕ :

- construct the complex number $z(\phi) = |B(\phi)|e^{j\phi}$,
- plot the complex number $z(\phi)$ on the complex plane.

The tool '`plotBP_aper.m`' saves you all of this work by plotting, in a four-panel plot, the magnitude and phase of the complex beampattern $B(\phi)$ vs. ϕ , and the magnitude of $B(\phi)$ vs. ϕ in Cartesian and polar coordinates. Use `>type plotBP_aper` to see how polar plots are made. Figure Five illustrates the use of the tool '`plotBP_aper.m`'. You should try it for several values of N in order to see how the length of the array in multiples of wavelength determines the resolving power of the array.

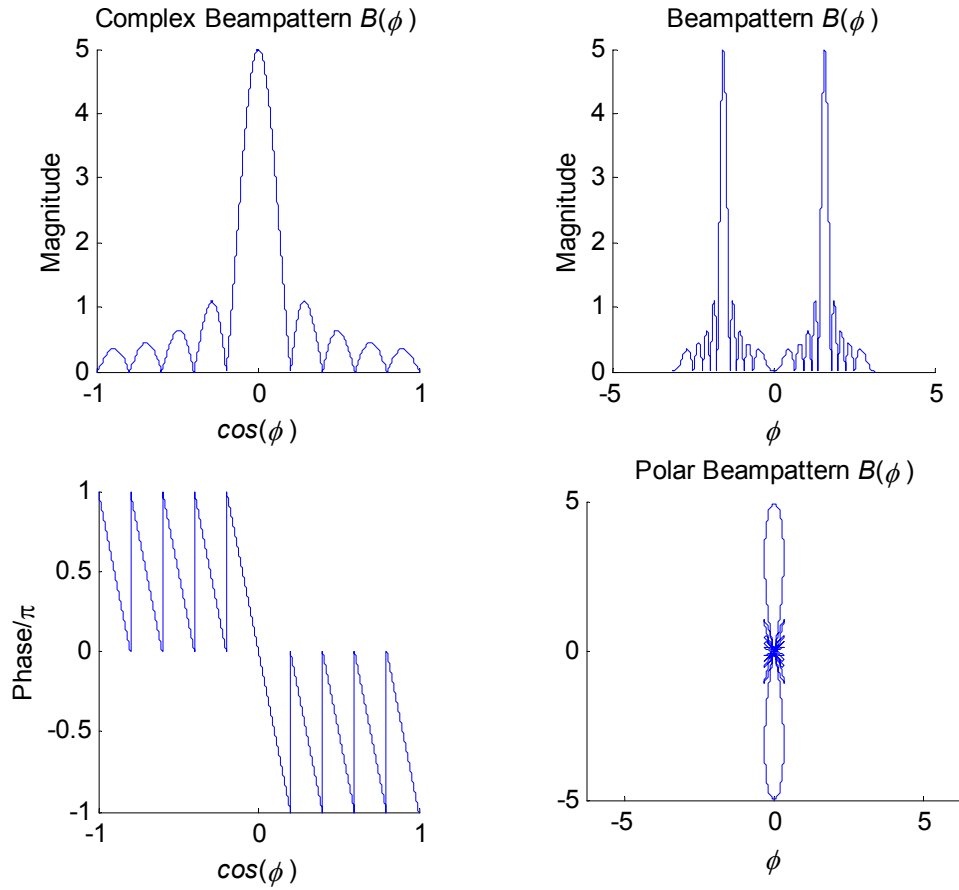


Figure Five: Beampatterns for an aperture, $N = 5$

THE FIR FILTER AND THE FINITE-SPACE ARRAY

Time. Let us now try to re-tell this story for the case of finite-time summers (or FIR filters) and finite-space arrays. These may be thought of as discrete or sampled versions of integrators and apertures. Consider a system $H(z)$ whose unit pulse response is

$$(12) \quad h[n] = u[n] - u[n - N] \longleftrightarrow H(z) = \frac{1}{1 - z^{-1}} [1 - z^{-N}]$$

If the input to this system is the signal $v[n]$, then the output is the convolution

$$(13) \quad y[n] = (h * v)[n] = \sum_{k=-N}^n v[k] \longleftrightarrow Y(z) = H(z)V(z) = \frac{V(z)}{1 - z^{-1}} - z^{-N} \frac{V(z)}{1 - z^{-N}}$$

As indicated, the output is a finite-time sum of the input, and the number of terms in the sum is N . The transfer function $H(z)$ has canceling poles and zeros at $z = 1$, but it has $N - 1$ other zeros which are revealing. In order to find them, we need to factor the polynomial $1 - z^{-N}$:

$$(14) \quad 1 - z^{-N} = \prod_{n=0}^{N-1} (1 - e^{j2\pi n/N} z^{-1})$$

These zeros are called the N roots of unity. Evidently, the transfer function $H(z)$ has $N - 1$ zeroes located at $z = e^{j2\pi n/N}$, for $n = 1, 2, \dots, N - 1$. Thus the transfer function $H(z)$ can now be written as

$$(15) \quad H(z) = \prod_{n=1}^{N-1} (1 - e^{j2\pi n/N} z^{-1})$$

The complex frequency response for $H(z)$ is revealing. We are thinking of the actual time at sample n as nt_0 , so we will evaluate the complex frequency response in the usual way:

$$(16) \quad H(e^{j2\pi f t_0}) = \frac{1}{1 - e^{-j2\pi f t_0}} [1 - e^{-jN2\pi f t_0}] = e^{-j(N-1)\pi f t_0} \frac{\sin(N\pi f t_0)}{\sin(\pi f t_0)}.$$

There are several things to note about this complex frequency response: its value at $f = 0$ is N , which is the sum of $h[n]$, and it is zero at $f = k/Nt_0, k = \pm 1, 2, \dots$. This is illustrated in Figure Six, using the function 'plotDSINC.m'.

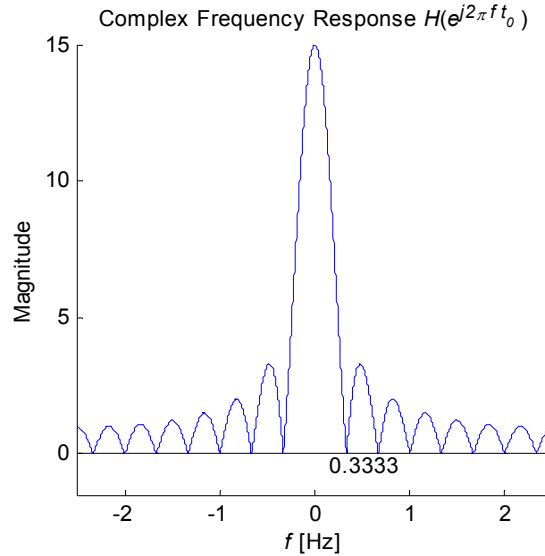


Figure Six: Magnitude of $H(e^{j2\pi f t_0})$ for $N = 15, t_0 = 0.2$

The output of this finite-time summer, when excited by the cosine $\cos(2\pi f n t_0)$, whose frequency is f , is

$$(17) \quad y[n] = \text{Re}\{H(e^{j2\pi f t_0})e^{j2\pi f n t_0}\} = \frac{\sin(N\pi f t_0)}{\sin(\pi f t_0)} \cos(2\pi f t_0 [n - \frac{N-1}{2}])$$

The output at time $n = (N - 1)/2$ is $y[(N - 1)/2] = \sin(N\pi f t_0) / \sin(\pi f t_0)$, which is a real number that depends on the frequency f . In fact, $|y[(N - 1)/2]| = |H(j2\pi f t_0)|$, which is the magnitude of the complex frequency response at frequency f . By now, you should be able to explain why the complex frequency response should be zero at frequencies that are integer multiples of $1/Nt_0$ (that is, $f = k/Nt_0$), or periods that are integer fractions of Nt_0 (that is, $1/f = Nt_0/k$).

Space. There is no reason why this simple idea cannot be extended to space, rather than time. To this end, we suggest that a finite integrating filter in space should have an impulse response and transfer function that is identical to the discrete-time case. Only the interpretation of the sampling index changes. In this case, the sampling index n corresponds to actual space point nx_0 . The complex frequency response is then

$$(18) \quad H(e^{j2\pi x_0 / \lambda}) = e^{-j(N-1)\pi x_0 / \lambda} \frac{\sin(N\pi x_0 / \lambda)}{\sin(\pi x_0 / \lambda)}$$

There are several things to note about this complex wavenumber response: its value at $1/\lambda = 0$ is N , which is the sum of $h[n]$, and it is zero at $\lambda = Nx_0 / k$, $k = \pm 1, 2, \dots$. This is illustrated in Figure Seven, using the function 'plotDSINC.m'.

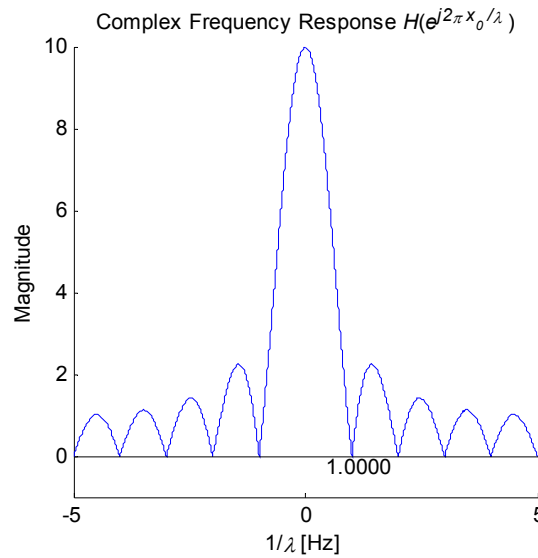


Figure Seven: Magnitude of $H(e^{j2\pi x_0 / \lambda})$ for $N = 10$, $x_0 = 0.1$

The output of this finite-space integrator, when excited by the cosine $\cos(2\pi nx_0 / \lambda)$, is, of course,

$$(19) \quad y[n] = \text{Re}\left\{H(e^{j2\pi x_0 / \lambda}) e^{j2\pi nx_0 / \lambda}\right\} = \frac{\sin(N\pi x_0 / \lambda)}{\sin(\pi x_0 / \lambda)} \cos\left(2\pi n \frac{x_0}{\lambda} \left[n - \frac{N-1}{2}\right]\right)$$

The magnitude of the output at the so-called phase center of the aperture, namely $n = (N-1)/2$, is $y[(N-1)/2] = \left|H(e^{j2\pi x_0 / \lambda})\right|$ which is the magnitude of the complex wavelength response at wavelength λ . By now, you should be able to explain why the complex wavenumber response should be zero at wavelengths that are integer fractions of Nx_0 and spatial frequencies that are integer multiples of $1/Nx_0$.

Now, a common way for wavelength to vary is for it to depend on the angle that a plane wave of fixed wavelength λ_0 makes with a linear aperture, as explained earlier. Thus, with λ replaced by $\lambda_0 / \cos(\phi)$, the complex beampattern is

$$(20) \quad B(\phi) = H\left(e^{j2\pi \frac{x_0}{\lambda_0} \cos(\phi)}\right) = e^{-j(N-1)\pi \frac{x_0}{\lambda_0} \cos(\phi)} \frac{\sin(N\pi \frac{x_0}{\lambda_0} \cos(\phi))}{\sin(\pi \frac{x_0}{\lambda_0} \cos(\phi))}$$

Typically x_0 , the spacing between array elements, is chosen to be a fraction of the wavelength λ_0 . Let's call this fraction $r = x_0/\lambda_0$ and rewrite the beampattern as

$$(21) \quad B(\phi) = H\left(e^{j2\pi r \cos(\phi)}\right) = e^{-j(N-1)\pi r \cos(\phi)} \frac{\sin(N\pi r \cos(\phi))}{\sin(\pi r \cos(\phi))}$$

This beampattern depends on two parameters, r and N , which are respectively the element spacing and array length, in multiples of the wavelength. So the pattern is normalized in the sense that it holds for all wavelengths. Of course, for a given pair (r, N) the actual element spacings will be $r\lambda_0$ and the actual array length will be $Nr\lambda_0$. The tool `'plotBP_arra.m'` simplifies the plotting of beampatterns by plotting, in a four-panel plot, the magnitude and phase of the beampattern vs. $\cos(\phi)$, and the magnitude of $B(\phi)$ vs. $\cos(\phi)$ in Cartesian and polar coordinates. This is illustrated in Figure Eight.

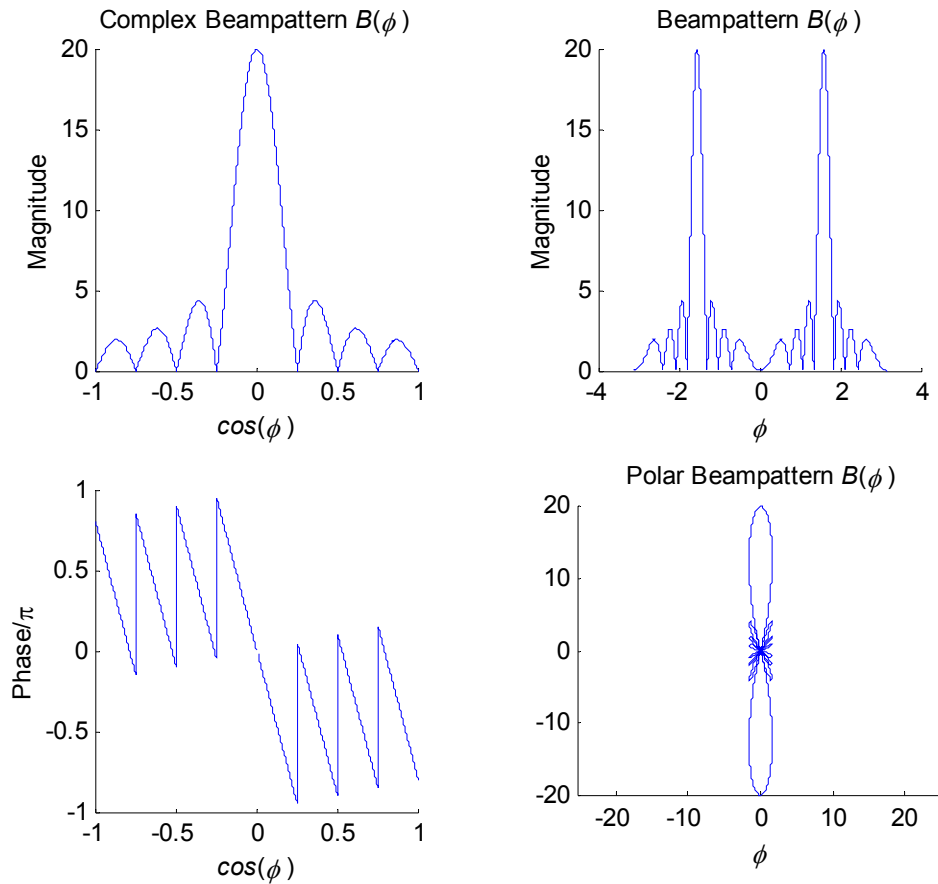


Figure Eight: Beampatterns for an array, $r = 0.2$, $N = 20$

Assignment

NOTE: For the following questions include explanations and printouts of results when applicable.

1. Run `pzd(b,a)`, with **b** and **a** chosen appropriately to compute and plot the poles and zeros for $H(z)$ given in equation (12). From the product representation for $H(z)$ in equation (15), argue that the complex frequency response should have zeros at $ft_0 = n/N$, or $f = n/Nt_0$ for $n = 1, 2, \dots, N-1$.
2. Rewrite the function '`plotDSINC.m`' to display the complex frequency response outside of the Nyquist Band. Explain what you see. Why are we only interested in the Nyquist Band?
3. Over plot '`plotBP_arra.m`' for several pairs of (N,r) to illustrate the dependence of a beampattern on the element spacing and the length of the array, in wavelengths. In particular, fix $r = 1/2$ and let $N = 4, 8, 16$ to see how resolving power depends on array length. Then fix $N = 8$, and let $r = 1/4, 1/2, 1, 3/2$ to see how the beampattern depends on element spacing. Perhaps you will observe an effect called *grating lobes*, which you will want to describe and explain.
4. You will note that the beampattern $B(\phi)$ is just the complex frequency response of a digital filter, namely $H(e^{j2\pi\cos(\phi)})$. This means you may design antenna arrays by designing digital filters, and then finding the beampattern for this array by evaluating the complex frequency response for your digital filter at $\theta = e^{j2\pi\cos(\phi)}$. Rewrite the tool '`plotBP_arra`' to create a new tool, '`plotBP.m`', that will accept the parameters (r,N,h) where **h** is an arbitrary unit pulse response $h = [h_0, h_1, \dots, h_{N-1}]$. The tool should make the four panel plot made by '`plotBP_arra.m`'.

(HINT: The function '`plotBP_arra.m`' uses the pre-derived complex frequency response $H(e^{j2\pi\cos(\phi)})$. In your new program, you will need to calculate this complex frequency response from the unit pulse response **h**. Type »`help freqz` in the command window.)
5. Reviewing your work in Lab 9, design lowpass and bandpass Chebyshev FIR filters that may be used for array shadings **h**. Use your tool `plotBP` to display the beampatterns for your designs.
6. Show how a bandpass Chebyshev filter produces a beampattern that is steered off broadside.
7. How would you modify a unit-pulse response **h** designed for a broadside beampattern to steer your array to $\pi/4$ off broadside? Run '`plotBP.m`' to illustrate your new beampattern. Why is the complex frequency response simply a shift, while the beampattern is less selective (or has less resolving power) at $\pi/4$ than at $\pi/2$?
8. Maybe you want a beampattern that has zeros at several specified arrival angles. Choose the zeros of an FIR filter $H(z)$ that will produce this effect, and use your tool '`plotBP.m`' to verify that you have achieved the desired effect.